Research Article

Optimal Couples of Rearrangement Invariant Spaces for Generalized Maximal Operators

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The optimal couples of rearrangement invariant spaces for boundedness of a generalized maximal operator, associated with a quasiconcave function, have been characterized in terms of certain indices connected with rearrangement invariant spaces and quasiconcave functions.

1. Introduction

Let L_{loc} be the space of all real-valued locally integrable functions on $\mathbf{R}^n$ with the Lebesgue measure. For any positive function ϕ on $(0, \infty)$, the generalized maximal operator M_ϕ is defined by

$$M_\phi f(x) = \sup_{B \ni x} \frac{1}{\phi(|B|)} \int_B |f(y)| dy, \quad f \in L_{\text{loc}}, \quad (1)$$

where the supremum is taken with respect to all balls B containing x , and $|\cdot|$ denotes the Lebesgue measure.

Note that for $\phi(t) = t$, M_ϕ is the classical Hardy-Littlewood maximal operator M , and for $\phi(t) = t^{1-s/n}$, $0 < s < n$, we get the fractional maximal operator M^s .

Let S_d and S_t be two classes of rearrangement invariant spaces (for definition, see Section 2). Take $E \in S_d$ and $G \in S_t$, and assume that $M_\phi : E \rightarrow G$, where the symbol $T : A \rightarrow B$ means that T is bounded from A into B . We say that E is an optimal domain space in the class S_d if $M_\phi : E' \rightarrow G$ for any rearrangement invariant space $E' \in S_d$, then it follows that $E' \hookrightarrow E$, where by $A \hookrightarrow B$, we mean that A is continuously embedded in B . We say that G is an optimal target space in the class S_t if $M_\phi : E \rightarrow G'$ implies that $G \hookrightarrow G'$ for any rearrangement invariant space $G' \in S_t$. Finally, the couple (E, G) is called optimal in the class (S_d, S_t) if E is an optimal

domain space in the class S_d and G is an optimal target space in the class S_t .

The optimal couples for the fractional maximal operator have been characterized in [1]. The characterizations are based on certain conditions on the Boyd indices associated with rearrangement invariant spaces. Earlier, Jan Vybíral had considered the problem of optimal domain space for the operator M_ϕ in [2].

Our aim in this paper is to extend some of the results in [1] by means of replacing the function $t \mapsto t^{1-s/n}$, $0 < s < n$, by an arbitrary quasiconcave function. By a quasiconcave function ϕ , we mean that ϕ is a positive function on $(0, \infty)$ such that ϕ is increasing and $t \mapsto \phi(t)/t$ is decreasing. To our end, we will impose certain conditions on rearrangement invariant spaces as well as on ϕ by means of the Boyd indices associated with them.

The paper is organized as follows. In Section 2, we give the necessary background material. In particular, we define the rearrangement invariant spaces. It is worthy of mention that our rearrangement invariant spaces are more general than those in [3]. Particularly, we do not use Fatou property and duality arguments. For convenience, we will say that a couple (E, G) is admissible if $M_\phi : E \rightarrow G$. The admissible and optimal couples are characterized in Sections 3 and 4, respectively.

2. Preliminaries

2.1. Rearrangement Invariant Spaces. Recall that the decreasing rearrangement of a measurable function f on $\mathbf{R}^n$, denoted by f^* , is defined as

$$f^*(t) = \inf \{ \lambda > 0 : \mu_f(\lambda) \leq t \}, \quad t \geq 0, \quad (2)$$

where μ_f is the distribution function of f given as

$$\mu_f(\lambda) = |\{x \in \mathbf{R}^n : |f(x)| > \lambda\}|, \quad \lambda \geq 0, \quad (3)$$

(see, e.g., [3]).

Let L be the space of all nonnegative locally integrable functions on $(0, \infty)$ with the Lebesgue measure. As usual, a quasinorm $\rho : L \rightarrow [0, \infty]$ satisfies the following conditions:

- (i) $\rho(g) = 0$ if and only if $g = 0$ a.e.;
- (ii) $\rho(ag) = a\rho(g)$, $a \geq 0$;
- (iii) $\rho(g + h) \leq c[\rho(g) + \rho(h)]$, $c \geq 1$.

For each quasinorm ρ , there exists an equivalent quasinorm ρ_p satisfying the triangle inequality $\rho_p^p(g_1 + g_2) \leq \rho_p^p(g_1) + \rho_p^p(g_2)$, $g_1, g_2 \in L$, for some $p \in (0, 1]$ (see [4]). We will say that the quasinorm ρ satisfies Minkovski inequality if for the equivalent quasinorm ρ_p , we have

$$\rho_p^p\left(\sum g_j\right) \leq \sum \rho_p^p(g_j), \quad g_j \in L, \quad (4)$$

where the notation $a \leq b$ stands for the fact that a is bounded above by a multiple of b , the multiple being independent of any variables in a and b . Later we will use $a \approx b$ to indicate that both $a \leq b$ and $b \leq a$ hold.

A quasinorm ρ is said to be monotone if for $g_1, g_2 \in L$ with $g_1 \leq g_2$, we have $\rho(g_1) \leq \rho(g_2)$.

Let $\rho_E : L \rightarrow [0, \infty]$ be a quasinorm, and let E be the space consisting of all functions f in L_{loc} for which the quasinorm $\|f\|_E := \rho_E(f^*)$ is finite. The space E is a rearrangement invariant in the sense that if $f \in E$ and $f^* = g^*$ for $g \in L_{\text{loc}}$, then $g \in E$. We will say that E is the rearrangement invariant space generated by the quasinorm ρ_E .

The rearrangement invariant Banach function spaces considered in [3] are a particular example of our rearrangement invariant spaces according to the Luxemburg representation theorem (see [3, page 62]). More general examples are given by the Riesz-Fischer monotone spaces as in [3, page 305].

2.2. Boyd Indices. Let E be a rearrangement invariant space generated by a quasinorm ρ_E . The lower and upper Boyd indices of E , denoted by α_E and β_E , are defined similarly to [5]. Let

$$h_E(u) = \sup \left\{ \frac{\rho_E(g_u^*)}{\rho_E(g^*)} : g \in L \right\}, \quad g_u(t) := g\left(\frac{t}{u}\right) \quad (5)$$

be the dilation function generated by ρ_E . Suppose that h_E is finite. Then,

$$\alpha_E := \sup_{0 < t < 1} \frac{\log h_E(t)}{\log t}, \quad \beta_E := \inf_{1 < t < \infty} \frac{\log h_E(t)}{\log t}. \quad (6)$$

If ρ_E is monotone, then the function h_E is increasing. Hence, in view of $h_E(1) = 1$ and $h_E(u)h_E(1/u) \geq 1$, we obtain $0 \leq \alpha_E \leq \beta_E$. Furthermore, the submultiplicativity of h_E will imply (see [6, page 244]) that both indices are finite and given by the following limits:

$$\alpha_E = \lim_{t \rightarrow 0^+} \frac{\log h_E(t)}{\log t}, \quad \beta_E = \lim_{t \rightarrow \infty} \frac{\log h_E(t)}{\log t}. \quad (7)$$

Similarly, we can associate a pair of indices with a positive function ϕ on $(0, \infty)$ (see [7]). In this paper, we will need only one of them which we call as the lower Boyd index α_ϕ . It is defined by

$$\alpha_\phi = \sup_{0 < t < 1} \frac{\ln h_\phi(t)}{\ln t}, \quad (8)$$

where

$$h_\phi(u) = \sup_{0 < t < \infty} \frac{\phi(tu)}{\phi(t)}, \quad (9)$$

is the dilation function h_ϕ generated by ϕ . If the function ϕ is quasiconcave, then $0 \leq \alpha_\phi \leq 1$ (see [7, page 54]).

We will make use of the assertions of the following proposition (cf. [3, Lemma 5.9, page 147]) in the forthcoming sections.

Proposition 1. *Let E be a rearrangement invariant space generated by a monotone quasinorm ρ_E , and let ϕ be a quasiconcave function. Then, for any $\alpha > 0$,*

$$\int_0^1 v^{-\alpha} h_E^\alpha(v) h_\phi^\alpha(v) \frac{dv}{v} < \infty, \quad \text{if } \alpha_E + \alpha_\phi > 1, \quad (10)$$

$$\int_0^1 h_E^\alpha\left(\frac{1}{v}\right) h_\phi^\alpha(v) \frac{dv}{v} < \infty, \quad \text{if } \beta_E < \alpha_\phi. \quad (11)$$

Proof. We will derive only (10); the proof of (11) will be similar. Let ϵ be small enough such that $\alpha_E + \alpha_\phi - 2\epsilon > 1$. For this ϵ , there exists $0 < T_1 < 1$ such that

$$h_E(t) < t^{\alpha_E - \epsilon}, \quad 0 < t \leq T_1, \quad (12)$$

and there also exists $0 < T_2 < 1$ such that

$$h_\phi(t) < t^{\alpha_\phi - \epsilon}, \quad 0 < t \leq T_2. \quad (13)$$

Taking $T = \min(T_1, T_2)$, we get

$$h_E(t) h_\phi(t) < t^{\alpha_E + \alpha_\phi - 2\epsilon}, \quad 0 < t \leq T, \quad (14)$$

thus

$$\begin{aligned} & \int_0^1 v^{-\alpha} h_E^\alpha(v) h_\phi^\alpha(v) \frac{dv}{v} \\ & < \int_0^T v^{\alpha(-1 + \alpha_E + \alpha_\phi - 2\epsilon)} \frac{dv}{v} + \int_T^1 v^{-\alpha} h_E^\alpha(v) h_\phi^\alpha(v) \frac{dv}{v}. \end{aligned} \quad (15)$$

We note that first integral on right hand side is convergent since $\alpha_E + \alpha_\phi - 2\epsilon > 1$. The second integral is also convergent since it is estimated from above by a convergent integral $\int_T^1 v^{-\alpha} (dv/v)$; this completes the proof. $\square$

3. Admissible Couples

In what follows, ϕ will be a fixed quasiconcave function. We will need to work with the following classes of rearrangement invariant spaces:

S_0 consists of all rearrangement invariant spaces E generated by a monotone quasinorm ρ_E ;

$S_{d,1}$ consists of all rearrangement invariant spaces E generated by a quasinorm ρ_E which is monotone and satisfies Minkovski inequality along with $\alpha_E + \alpha_\phi > 1$.

In the formulation of our results, we will need also a subclass L_0 of L , formed of all decreasing functions in L .

We note here that the admissibility of a couple (E, G) (i.e., $M_\phi : E \rightarrow G$) is equivalent to the following estimate:

$$\rho_G((M_\phi f)^*) \lesssim \rho_E(f^*), \quad f \in E. \quad (16)$$

Our starting point is the following characterization of all admissible couples which is essentially a reformulation of the sharp rearrangement inequality which was proved in [2, Theorem 3.13].

Theorem 2. *Let $E, G \in S_0$; then the couple (E, G) is admissible if and only if*

$$\rho_G(Ng) \lesssim \rho_E(g), \quad g \in L_0, \quad (17)$$

where

$$Ng(t) := \sup_{u \geq t} \frac{1}{\phi(u)} \int_0^u g(u) du, \quad t > 0. \quad (18)$$

The one-dimensional operator N is rather complicated; however, we can replace it by a simpler one in the condition (17) by exploiting the next estimate.

Lemma 3. *Assume that $E \in S_{d,1}$. Then,*

$$\rho_E(t^{-1}\phi(t)Qg(t)) \lesssim \rho_E(t^{-1}\phi(t)g(t)), \quad g \in L_0, \quad (19)$$

where

$$Qg(t) := \int_t^\infty g(s) \frac{ds}{s}, \quad t > 0. \quad (20)$$

Proof. We start off with a simple change of variables to have

$$Qg(t) = \int_0^1 g\left(\frac{t}{v}\right) \frac{dv}{v}. \quad (21)$$

Since the function $v \mapsto g(t/v)$ is increasing in v , therefore

$$\begin{aligned} Qg(t) &= \sum_{k=-\infty}^{-1} \int_{2^k}^{2^{k+1}} g\left(\frac{t}{v}\right) \frac{dv}{v} \\ &\leq \sum_{k=-\infty}^{-1} g\left(\frac{t}{2^{k+1}}\right) \int_{2^k}^{2^{k+1}} \frac{dv}{v} \\ &\approx \sum_{k=-\infty}^{-1} g\left(\frac{t}{2^{k+1}}\right). \end{aligned} \quad (22)$$

As ρ_E is monotone and satisfies Minkowski's inequality, we get

$$\rho_E^p(t^{-1}\phi(t)Qg(t)) \lesssim \sum_{k=-\infty}^{-1} \rho_E^p\left(t^{-1}\phi(t)g\left(\frac{t}{2^{k+1}}\right)\right). \quad (23)$$

Now we observe that the function $v \mapsto \rho_E^p(t^{-1}\phi(t)g(t/2v))$ is increasing in v ; thus,

$$\begin{aligned} &\rho_E^p(t^{-1}\phi(t)Qg(t)) \\ &\leq \sum_{k=-\infty}^{-1} \int_{2^k}^{2^{k+1}} \rho_E^p\left(t^{-1}\phi(t)g\left(\frac{t}{2v}\right)\right) \frac{dv}{v} \\ &\leq \int_0^1 \rho_E^p\left(t^{-1}\phi(t)g\left(\frac{t}{2v}\right)\right) \frac{dv}{v}. \end{aligned} \quad (24)$$

Finally, using the definitions of h_ϕ and h_E along with the quasiconcavity of ϕ , we obtain

$$\begin{aligned} &\rho_E^p(t^{-1}\phi(t)Qg(t)) \\ &\leq \int_0^1 h_\phi^p(v) \rho_E^p\left(t^{-1}\phi\left(\frac{t}{v}\right)g\left(\frac{t}{2v}\right)\right) \frac{dv}{v} \\ &\approx \int_0^1 h_\phi^p(v) \rho_E^p\left(t^{-1}\phi\left(\frac{t}{2v}\right)g\left(\frac{t}{2v}\right)\right) \frac{dv}{v} \\ &\leq \rho_E^p(t^{-1}\phi(t)g(t)) \int_0^1 h_\phi^p(2v) v^{-p} h_E^p(2v) \frac{dv}{v} \\ &\approx \rho_E^p(t^{-1}\phi(t)g(t)) \int_0^2 v^{-p} h_\phi^p(v) h_E^p(v) \frac{dv}{v}, \end{aligned} \quad (25)$$

and the required estimate follows thanks to (10). $\square$

Theorem 4. *Let $E \in S_{d,1}$ and $G \in S_0$. Then, the couple (E, G) is admissible if and only if*

$$\rho_G(Tg) \lesssim \rho_E(g), \quad g \in L_0, \quad (26)$$

where

$$Tg(t) := \frac{1}{\phi(t)} \int_0^t g(u) du, \quad t > 0. \quad (27)$$

Remark 5. Note that if ϕ is an identity function, then the condition $\alpha_E + \alpha_\phi > 1$, which turns into $\alpha_E > 0$, is not needed since $Ng = Tg$ if $g \in L_0$.

Proof. Assume first that the couple (E, G) is admissible. Then, in view of the condition (17), $T \leq N$, and the monotonicity of ρ_G , the condition (26) follows immediately.

Conversely, assume that the condition (26) holds, and fix $g \in L_0$. Denoting

$$T'g(t) := \sup_{u \geq t} \frac{1}{\phi(u)} \int_t^u g(v) dv, \quad (28)$$

we note that $N \approx T + T'$ as ϕ is increasing. Then, by Theorem 2, it will suffice to establish that $\rho_G(T'g) \lesssim \rho_E(g)$.

Since both the functions $t \mapsto t^{-1}\phi(t)$ and $t \mapsto T'g(t)$ are decreasing in t , we have

$$T'g \leq Tg_1, \quad (29)$$

where $g_1(t) := t^{-1}\phi(t)T'g(t)$. Observe that $g_1 \in L_0$. Therefore, using the monotonicity of ρ_G and applying the condition (26), we obtain $\rho_G(T'g) \leq \rho_E(g_1)$. The proof will thus be complete if we show that $\rho_E(g_1) \leq \rho_E(g)$. To this end, we set $g_2(t) := tg(t)/\phi(t)$. Then, by definition of Q , we have

$$Qg_2(t) \geq \int_t^u \frac{g(v)}{\phi(v)} dv, \quad (30)$$

whenever $u \geq t$. As ϕ is increasing, we further have

$$Qg_2(t) \geq \frac{1}{\phi(v)} \int_t^u g(v) dv, \quad (31)$$

whenever $u \geq t$, whence we get

$$Qg_2(t) \geq T'g(t), \quad (32)$$

by taking supremum over all $u \geq t$. Finally, multiplying both sides by $t^{-1}\phi(t)$, using monotonicity of ρ_E , and applying Lemma 3, we arrive at

$$\rho_E(g_1) \leq \rho_E(g), \quad (33)$$

as desired. The proof is complete. $\square$

Remark 6. By Theorem 1.3 in [8], the conditions (17) and (26) are also equivalent if G is a rearrangement invariant space as in [3]. The key ingredient of the proof of the above mentioned theorem is the Hardy-Littlewood-Pólya principle (see [3, Theorem 4.6, page 61]) which is not guaranteed in our rearrangement invariant spaces.

4. Optimal Couples

We introduce two more classes of rearrangement invariant spaces as follows:

$S_{d,2}$ consists of all rearrangement invariant spaces E generated by a quasinorm ρ_E which is monotone and satisfies Minkovski inequality along with $\alpha_E + \alpha_\phi > 1$ and $\beta_E < 1$;

$S_{t,1}$ consists of all rearrangement invariant spaces G generated by a quasinorm ρ_G which is monotone and satisfies Minkovski inequality along with $\beta_G < \alpha_\phi$.

First we construct optimal couples with the aid of Theorem 2. We will need the next estimate which can be proved by using Minkovski inequality and (11) as in Lemma 3.

Lemma 7. *Let $G \in S_{t,1}$; then,*

$$\rho_G\left(\frac{t}{\phi(t)}Pg(t)\right) \leq \rho_G\left(\frac{t}{\phi(t)}g(t)\right), \quad g \in L_0, \quad (34)$$

where

$$Pg(t) := \frac{1}{t} \int_0^t g(s) ds, \quad t > 0. \quad (35)$$

Remark 8. In fact, the above estimate is two sided because the reverse estimate holds trivially since $Pg \geq g$ if $g \in L_0$.

Theorem 9. *Let $G \in S_{t,1}$ be a given rearrangement invariant space. Let E be the rearrangement invariant space generated by the quasinorm as follows:*

$$\rho_E(g) = \rho_G(Ng^*), \quad g \in L. \quad (36)$$

Then, the couple (E, G) is optimal in the class $(S_0, S_{t,1})$.

Proof. Clearly, the couple (E, G) is admissible by Theorem 2. To see that E is the optimal domain space in the class S_0 , let $E' \in S_0$ be another rearrangement invariant space such that the couple (E', G) is admissible. Take $f \in E'$ and apply Theorem 2 to obtain

$$\rho_E(f^*) = \rho_G(Nf^*) \leq \rho_{E'}(f^*), \quad (37)$$

where we get $E' \hookrightarrow E$ as desired.

It now remains to establish that G is the optimal target space in the class $S_{t,1}$. Let $G' \in S_{t,1}$ be another rearrangement space such that the couple (E, G') is admissible, and take $f \in G$. Set

$$f_1(t) := t^{-1}\phi(t)f^*(t), \quad (38)$$

to observe that $f^* \leq Nf_1$ because $f_1 \in L_0$. Consequently,

$$\begin{aligned} \rho_{G'}(f^*) &\leq \rho_{G'}(Nf_1) \\ &\leq \rho_E(f_1), \end{aligned} \quad (39)$$

again by Theorem 2. It follows, by definition of ρ_E , that $\rho_{G'}(f^*) \leq \rho_G(Nf_1)$. In view of $N \approx T + T'$, we can write the following:

$$\rho_{G'}(f^*) \leq \rho_G(Tf_1) + \rho_G(T'f_1). \quad (40)$$

Since $Tf_1(t) = (t/\phi(t))Pf_1(t)$, an application of Lemma 7 yields

$$\rho_{G'}(f^*) \leq \rho_G(f^*) + \rho_G(T'f_1). \quad (41)$$

Hence, the required embedding $G \hookrightarrow G'$ will follow if we show that $\rho_G(T'f_1) \leq \rho_G(f^*)$. For this, let us note that

$$\begin{aligned} T'f_1(t) &= \sup_{u \geq t} \frac{1}{\phi(u)} \int_t^u \phi(v) f^*(v) \frac{dv}{v} \\ &\leq f^*(t) \sup_{u \geq t} \frac{1}{\phi(u)} \int_t^u \phi(v) \frac{dv}{v} \\ &\leq f^*(t) \sup_{u \geq t} \frac{1}{\phi(u)} \int_0^u \phi(v) \frac{dv}{v}. \end{aligned} \quad (42)$$

Since $0 \leq \beta_G < \alpha_\phi$, thus $\alpha_\phi > 0$. Therefore, by [7, Corollary 3, page 57], we get $T'f_1 \leq f^*$. This gives $\rho_{G'}(T'f_1) \leq \rho_G(f^*)$ as desired. The proof is complete. $\square$

Now we apply Theorem 4 to construct optimal couples.

Theorem 10. Let $E \in S_{d,2}$ be a given rearrangement invariant space. Let G be the rearrangement invariant space generated by the quasinorm as follows:

$$\rho_G(g) = \rho_E(t^{-1}\phi(t)g(t)), \quad g \in L. \quad (43)$$

Then, the couple (E, G) is optimal in the class $(S_{d,2}, S_0)$.

Proof. Let $g \in L_0$. Then, by the definition of ρ_G ,

$$\rho_G(Tg) = \rho_E\left(\frac{1}{t} \int_0^t g(v) dv\right). \quad (44)$$

Since $\beta_E < 1$, we can apply Lemma 7 to $\phi(t) = t$ to obtain $\rho_G(Tg) \leq \rho_E(g)$. Hence, the couple (E, G) is admissible by Theorem 4.

To see that G is the optimal target space in the class S_0 , let $G' \in S_0$ be another rearrangement space such that the couple (E, G') is admissible. Given that $f \in G$, consider again the function $f_1(t) = t^{-1}\phi(t)f^*(t) \in L_0$. Then, $f^* \leq Tf_1$. Thus,

$$\rho_{G'}(f^*) \leq \rho_{G'}(Tf_1) \leq \rho_E(f_1), \quad (45)$$

by Theorem 4. Therefore,

$$\rho_{G'}(f^*) \leq \rho_E(t^{-1}\phi(t)f^*(t)) = \rho_G(f^*), \quad (46)$$

whence we get the desired embedding $G \hookrightarrow G'$.

Our remaining task is to show that E is the optimal domain space in the class $S_{d,2}$. To do so, let $E' \in S_{d,2}$ be another rearrangement invariant space such that the couple (E', G) is admissible. Take $f \in E'$ and note that $f^*(t) \leq t^{-1}\phi(t)Tf^*(t)$. Thus,

$$\rho_E(f^*) \leq \rho_E(t^{-1}\phi(t)Tf^*) = \rho_G(Tf^*), \quad (47)$$

which gives $\rho_E(f^*) \leq \rho_{E'}(f^*)$ by Theorem 4. It follows that $E' \hookrightarrow E$ as required. The proof is complete. $\square$

Remark 11. The statements of Theorems 9 and 10 generalize those of Theorems 2.10 and 3.1 in [1], respectively.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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