Research Article

The Boundedness of the Hardy-Littlewood Maximal Operator and Multilinear Maximal Operator in Weighted Morrey Type Spaces

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Received 25 November 2013; Accepted 9 January 2014; Published 16 March 2014

Academic Editor: Vagif Guliyev

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The aim of this paper is to prove the boundedness of the Hardy-Littlewood maximal operator on weighted Morrey spaces and multilinear maximal operator on multiple weighted Morrey spaces. In particular, the result includes the Komori-Shirai theorem and the Iida-Sato-Sawano-Tanaka theorem for the Hardy-Littlewood maximal operator and multilinear maximal function.

1. Introduction

The aim of this paper is to investigate the boundedness of the Hardy-Littlewood maximal operator on weighted Morrey spaces and multilinear maximal operator on weighted Morrey type spaces. The Hardy-Littlewood maximal function Mf is given by

$$Mf(x) := \sup_{\substack{Q \ni x \\ Q: \text{cube}}} \int_Q |f(y)| dy, \quad (1)$$

where $x \in \mathbb{R}^n$, f is locally integrable function, and the description $\int_Q$ is an integral mean on Q :

$$\int_Q F(y) dy := \frac{1}{|Q|} \int_Q F(y) dy. \quad (2)$$

In this paper, for every cube $Q \subset \mathbb{R}^n$, we assume that the sides of Q are parallel to the coordinate axes and $|Q|$ describes the volume of Q . A weight is a nonnegative locally integrable function on $\mathbb{R}^n$ that takes values in $(0, \infty)$ almost everywhere. Given a weight w and a measurable set E , we use the notation

$$w(E) := \int_E w(x) dx \quad (3)$$

to denote the w -measure of the set E . We recall weighted Lebesgue spaces, Morrey spaces, weighted Morrey spaces, and multi-Morrey spaces.

Definition 1 (weighted Lebesgue spaces). Let $0 < p < \infty$ and let w be a weight. The weighted Lebesgue space is defined by the norm (or quasinorm)

$$\|f\|_{L^p(w)} := \left(\int_{\mathbb{R}^n} |f(x)|^p w(x) dx \right)^{1/p}. \quad (4)$$

Definition 2 (Morrey spaces). For $0 < p \leq p_0 < \infty$, the Morrey space $\mathcal{M}_p^{p_0}(\mathbb{R}^n)$ is defined by the norm (or quasinorm)

$$\|f\|_{\mathcal{M}_p^{p_0}(\mathbb{R}^n)} = \sup_{\substack{Q \subset \mathbb{R}^n \\ Q: \text{cube}}} |Q|^{1/p_0} \left(\int_Q |f(x)|^p dx \right)^{1/p}. \quad (5)$$

Definition 3 (weighted Morrey spaces). Let w_1 and w_2 be weights. For $0 < p \leq p_0 < \infty$, the weighted Morrey space is defined by the norm (or quasinorm)

$$\begin{aligned} & \|f\|_{\mathcal{M}_p^{p_0}(w_1, w_2)} \\ &:= \sup_{\substack{Q \subset \mathbb{R}^n \\ Q: \text{cube}}} w_1(Q)^{1/p_0} \left(\frac{1}{w_1(Q)} \int_Q |f(x)|^p w_2(x) dx \right)^{1/p}. \end{aligned} \quad (6)$$

Definition 4 (multi-Morrey space). Let $0 < p_1, \dots, p_m < \infty$ and $1/p_0 \leq 1/p_1 + \dots + 1/p_m$. The multi-Morrey space

is defined by the quantity for vector valued function $\vec{f} = (f_1, \dots, f_m)$:

$$\|\vec{f}\|_{\mathcal{M}_p^{p_0}(\mathbb{R}^n)} := \sup_{\substack{Q \subset \mathbb{R}^n; \\ Q: \text{cube}}} |Q|^{1/p_0} \prod_{j=1}^m \left(\int_Q |f_j(y_j)|^{p_j} dy_j \right)^{1/p_j}. \quad (7)$$

Hence it is natural to consider multiple weighted Morrey spaces.

Definition 5 (multiple weighted Morrey spaces). Let $0 < p_1, \dots, p_m < \infty$ and $1/p_0 \leq 1/p_1 + \dots + 1/p_m$. Let $\vec{w} = (w_1, \dots, w_m)$ be a multiple weight. Let W be a weight. The multiple weighted Morrey space is defined by the quantity for vector valued function $\vec{f} = (f_1, \dots, f_m)$:

$$\begin{aligned} \|\vec{f}\|_{\mathcal{M}_p^{p_0}(W, \vec{w})} \\ := \sup_{\substack{Q \subset \mathbb{R}^n; \\ Q: \text{cube}}} W(Q)^{1/p_0} \prod_{j=1}^m \left(\frac{1}{W(Q)} \int_Q |f_j(y_j)|^{p_j} w_j(y_j) dy_j \right)^{1/p_j}. \end{aligned} \quad (8)$$

In this framework, we investigate the well-known results. The following theorem which is elementary result was discovered by Muckenhoupt [1].

Theorem A. Let w be a weight and $1 < p < \infty$:

$$\|Mf\|_{L^p(w)} \leq C \|f\|_{L^p(w)} \quad (9)$$

if and only if

$$w \in A_p(\mathbb{R}^n), \quad (10)$$

where

$$[w]_{A_p} := \sup_{\substack{Q \subset \mathbb{R}^n; \\ Q: \text{cube}}} \left(\int_Q w(x) dx \right) \left(\int_Q w(x)^{1/(1-p)} dx \right)^{p-1} < \infty. \quad (11)$$

Komori and Shirai [2] introduced the weighted Morrey spaces and proved the following theorem. The following theorem gives us the boundedness of the Hardy-Littlewood maximal operator on weighted Morrey spaces.

Theorem B. Let w be a weight and $1 < p < p_0 < \infty$. If $w \in A_p(\mathbb{R}^n)$, then one has

$$\|Mf\|_{\mathcal{M}_p^{p_0}(w, w)} \leq C \|f\|_{\mathcal{M}_p^{p_0}(w, w)}. \quad (12)$$

On the other hand, in [3], the following theorem was proved. The following result gives also the boundedness of the Hardy-Littlewood maximal operator on weighted Morrey spaces.

Theorem C. Let w be a weight and $1 < p \leq p_0 < \infty$. If

$$\begin{aligned} \sup_{\substack{Q \subset Q'; \\ Q, Q': \text{cubes}}} \left(\frac{|Q|}{|Q'|} \right)^{1/p_0} \left(\int_Q w(x)^p dx \right)^{1/p} \left(\int_{Q'} w(x)^{-p'} dx \right)^{1/p'} \\ < \infty, \end{aligned} \quad (13)$$

then one has

$$\|Mf\|_{\mathcal{M}_p^{p_0}(1, w^p)} \leq C \|f\|_{\mathcal{M}_p^{p_0}(1, w^p)}. \quad (14)$$

Moreover we can extend Theorem C to the multilinear version (see [3, 4]).

Theorem D. Let $1 < p_1, \dots, p_m < \infty$ and $1/p_0 \leq 1/p_1 + \dots + 1/p_m$. Let $\vec{w} = (w_1, \dots, w_m)$ be a multiple weight. If

$$\begin{aligned} \sup_{\substack{Q \subset Q'; \\ Q, Q': \text{cubes}}} \left(\frac{|Q|}{|Q'|} \right)^{1/p_0} \left(\int_Q (w_1 \cdots w_m)(x)^p dx \right)^{1/p} \\ \times \prod_{j=1}^m \left(\int_{Q'} w_j(y_j)^{-p'} dy_j \right)^{1/p'} < \infty. \end{aligned} \quad (15)$$

Then one has

$$\|\mathcal{M}\vec{f}\|_{\mathcal{M}_p^{p_0}(1, (w_1 \cdots w_m)^p)} \leq C \|f\|_{\mathcal{M}_p^{p_0}(1, \vec{w}^{\vec{p}})}, \quad (16)$$

where $\vec{w}^{\vec{p}} = (w_1^{p_1}, \dots, w_m^{p_m})$ and

$$\mathcal{M}\vec{f}(x) := \sup_{Q \ni x} \prod_{j=1}^m \int_Q |f_j(y_j)| dy_j. \quad (17)$$

Because the ways of the evaluation by the weighted Morrey norm are different, Theorems B and C are independent. Therefore, it is natural to consider unifying Theorems B and C. The question is not settled yet. In this paper, we unify Theorems B–D.

2. Main Results

Theorem 6. Let $1 < p \leq p_0 < \infty$ and let w_1 and w_2 be weights. Additionally assume that w_1 satisfies the doubling condition: there exists $C > 0$ such that

$$w_1(2Q) \leq C w_1(Q). \quad (18)$$

Assume that the following condition holds:

$$\begin{aligned} [w_1, w_2]_{p_0, p} = \sup_{\substack{Q \subset Q'; \\ Q, Q': \text{cubes}}} \left(\frac{w_1(Q)}{w_1(Q')} \right)^{1/p_0 - 1/p} \left(\frac{|Q|}{|Q'|} \right)^{1/p} \\ \times \left(\int_Q w_2(x)^p dx \right)^{1/p} \left(\int_{Q'} w_2(x)^{-p'} dx \right)^{1/p'} \\ < \infty. \end{aligned} \quad (19)$$

Then, one has

$$\begin{aligned} & w_1(Q)^{1/p_0} \left(\frac{1}{w_1(Q)} \int_Q Mf(x)^p w_2(x)^p dx \right)^{1/p} \\ & \leq C[w_1, w_2]_{p_0, p} \sup_{\substack{Q \subset Q' \\ Q, Q': \text{cubes}}} w_1(Q')^{1/p_0} \\ & \quad \times \left(\frac{1}{w_1(Q')} \int_{Q'} |f(x)|^p w_2(x)^p dx \right)^{1/p}. \end{aligned} \quad (20)$$

Remark 7. If condition (19) holds, then $w_2^p \in A_p(\mathbb{R}^n)$. In fact, for every cube $Q \subset \mathbb{R}^n$,

$$\begin{aligned} & \left(\int_Q w_2(x)^p dx \right)^{1/p} \left(\int_Q w_2(x)^{-p'} dx \right)^{1/p'} \\ & = \left(\frac{w_1(Q)}{w_1(Q)} \right)^{1/p_0 - 1/p} \left(\frac{|Q|}{|Q|} \right)^{1/p} \left(\int_Q w_2(x)^p dx \right)^{1/p} \\ & \quad \times \left(\int_Q w_2(x)^{-p'} dx \right)^{1/p'} \\ & \leq [w_1, w_2]_{p_0, p} < \infty. \end{aligned} \quad (21)$$

Therefore, we obtain $w_2^p \in A_p(\mathbb{R}^n)$.

Theorem 8. Let w_1 and w_2 be weights. Let $1 < p \leq p_0 < \infty$. Additionally assume that w_1 satisfies the doubling condition. If $[w_1, w_2]_{p_0, p} < \infty$, then one has

$$\|Mf\|_{\mathcal{M}_p^{p_0}(w_1, w_2^p)} \leq C[w_1, w_2]_{p_0, p} \|f\|_{\mathcal{M}_p^{p_0}(w_1, w_2^p)}. \quad (22)$$

Remark 9. Theorem 8 includes Theorems B and C. In Theorem 8, if $w_1 = 1$, then Theorem 8 corresponds with Theorem C. In Theorem 8, if $w_1 = w_2^p = w$, then Theorem 8 corresponds with Theorem B. In fact, condition (19) corresponds with $w \in A_p(\mathbb{R}^n)$. We state the detail:

$$\begin{aligned} & [w, w^{1/p}]_{p_0, p} \\ & = \sup_{\substack{Q \subset Q' \\ Q, Q': \text{cubes}}} \left(\frac{w(Q)}{w(Q')} \right)^{1/p_0 - 1/p} \left(\frac{|Q|}{|Q'|} \right)^{1/p} \\ & \quad \times \left(\int_Q w(x) dx \right)^{1/p} \left(\int_{Q'} w(x)^{-p'/p} dx \right)^{1/p'} \end{aligned}$$

$$\begin{aligned} & = \sup_{\substack{Q \subset Q' \\ Q, Q': \text{cubes}}} \left(\frac{w(Q)}{w(Q')} \right)^{1/p_0} \left(\frac{w(Q')}{|Q'|} \right)^{1/p} \\ & \quad \times \left(\int_{Q'} w(x)^{-p'/p} dx \right)^{1/p'} \\ & = \sup_{\substack{Q \subset Q' \\ Q, Q': \text{cubes}}} \left(\frac{w(Q)}{w(Q')} \right)^{1/p_0} \\ & \quad \times \left[\left(\int_{Q'} w(x) dx \right) \left(\int_{Q'} w(x)^{1/(1-p)} dx \right)^{p-1} \right]^{1/p} \\ & \leq [w]_{A_p}^{1/p} \cdot \sup_{\substack{Q \subset Q' \\ Q, Q': \text{cubes}}} \left(\frac{w(Q)}{w(Q')} \right)^{1/p_0} \\ & \leq [w]_{A_p}^{1/p} < \infty. \end{aligned} \quad (23)$$

Theorem 10. Let w_1, w_2 , and v be weights. Additionally assume that w_1 satisfies the doubling condition. Let $1 < p \leq p_0 < \infty$, $a > 1$, and

$$\begin{aligned} & [w_1, w_2, v]_{p_0, p, p/a} \\ & := \sup_{\substack{Q \subset Q' \\ Q, Q': \text{cubes}}} \left(\frac{w_1(Q)}{w_1(Q')} \right)^{1/p_0 - 1/p} \left(\frac{|Q|}{|Q'|} \right)^{1/p} \\ & \quad \times \left(\int_Q v(y)^p dy \right)^{1/p} \left(\int_{Q'} w_2(y)^{-(p/a)'} dy \right)^{1/(p/a)'} \\ & < \infty. \end{aligned} \quad (24)$$

Then one has

$$\|Mf\|_{\mathcal{M}_p^{p_0}(w_1, v^p)} \leq C[w_1, w_2, v]_{p_0, p, p/a} \|f\|_{\mathcal{M}_p^{p_0}(w_1, w_2^p)}. \quad (25)$$

We can extend Theorem 10 as follows.

Theorem 11. Let w_1, w_2 , and v be weights. Additionally assume that w_1 satisfies the doubling condition. Let $1 < p \leq p_0 \leq q_0 < \infty$, $a > 1$, and

$$\begin{aligned} & [w_1, w_2, v]_{p_0, q_0, p, p/a} \\ & := \sup_{\substack{Q \subset Q' \\ Q, Q': \text{cubes}}} \frac{w_1(Q)^{1/q_0}}{w_1(Q')^{1/p_0}} \left(\frac{w_1(Q')}{w_1(Q)} \right)^{1/p} \left(\frac{|Q|}{|Q'|} \right)^{1/p} \\ & \quad \times \left(\int_Q v(y)^p dy \right)^{1/p} \left(\int_{Q'} w_2(y)^{-(p/a)'} dy \right)^{1/(p/a)'} \\ & < \infty. \end{aligned} \quad (26)$$

Then one has

$$\|Mf\|_{\mathcal{M}_p^{q_0}(w_1, v^p)} \leq C[w_1, w_2, v]_{p_0, q_0, p, p/a} \|f\|_{\mathcal{M}_p^{p_0}(w_1, w_2^p)}. \quad (27)$$

By considering the multilinear version, we obtain the following theorems.

Theorem 12. Let $1 < p_1, \dots, p_m < \infty$, $1/p = 1/p_1 + \dots + 1/p_m$, and $a > 1$. Let v and W be weights. Let $\vec{w} = (w_1, \dots, w_m)$ be a multiple weight. Additionally assume that W satisfies the doubling condition and

$$\begin{aligned} [v, W, \vec{w}]_{p_0, p, \vec{p}/a} &:= \sup_{\substack{Q \subset Q' \\ Q, Q' : \text{cubes}}} \left(\frac{W(Q)}{W(Q')} \right)^{1/p_0 - 1/p} \left(\frac{|Q|}{|Q'|} \right)^{1/p} \\ &\quad \times \left(\int_Q v(x)^p dx \right)^{1/p} \\ &\quad \times \prod_{j=1}^m \left(\int_{Q'} w_j(y_j)^{-(p_j/a)'} dy_j \right)^{1/(p_j/a)'} \\ &< \infty. \end{aligned} \quad (28)$$

Then one has

$$\|\mathcal{M}\vec{f}\|_{\mathcal{M}_p^{p_0}(W, v^p)} \leq C\|\vec{f}\|_{\mathcal{M}_{\vec{p}}^{p_0}(W, \vec{w}^{\vec{p}})}, \quad (29)$$

where $\vec{w}^{\vec{p}} = (w_1^{p_1}, \dots, w_m^{p_m})$.

We can extend Theorem 12 as follows.

Theorem 13. Let $1 < p_1, \dots, p_m < \infty$, $1/p = 1/p_1 + \dots + 1/p_m$, $0 < p \leq p_0 \leq q_0 < \infty$, and $a > 1$. Let v and W be weights. Let $\vec{w} = (w_1, \dots, w_m)$ be a multiple weight. Additionally assume that W satisfies the doubling condition and

$$\begin{aligned} [v, W, \vec{w}]_{p_0, q_0, p, \vec{p}/a} &:= \sup_{\substack{Q \subset Q' \\ Q, Q' : \text{cubes}}} \frac{W(Q)^{1/q_0}}{W(Q')^{1/p_0}} \left(\frac{W(Q')}{W(Q)} \right)^{1/p} \\ &\quad \times \left(\frac{|Q|}{|Q'|} \right)^{1/p} \left(\int_Q v(x)^p dx \right)^{1/p} \\ &\quad \times \prod_{j=1}^m \left(\int_{Q'} w_j(y_j)^{-(p_j/a)'} dy_j \right)^{1/(p_j/a)'} \\ &< \infty. \end{aligned} \quad (30)$$

Then one has

$$\|\mathcal{M}\vec{f}\|_{\mathcal{M}_p^{q_0}(W, v^p)} \leq C\|\vec{f}\|_{\mathcal{M}_{\vec{p}}^{p_0}(W, \vec{w}^{\vec{p}})}, \quad (31)$$

where $\vec{w}^{\vec{p}} = (w_1^{p_1}, \dots, w_m^{p_m})$.

3. Some Lemmas

We use the following lemma (see [3, 5–9]).

Lemma 14. Let $0 < p < \infty$. Let v be a weight. For every cube $Q_0 \subset \mathbb{R}^n$, one has

$$\|M(f\chi_{3Q_0})v\|_{L^p(Q_0)} \leq C\|\widetilde{M}_p^{Q_0}(f, v)\|_{L^p(Q_0)}, \quad (32)$$

where

$$\begin{aligned} \widetilde{M}_p^{Q_0}(f, v)(x) &:= \sup_{Q_0 \supset Q \ni x} \left(\int_{3Q} |f(y)| dy \right) \left(\int_Q v(y)^p dy \right)^{1/p}. \end{aligned} \quad (33)$$

Moreover we can extend Lemma 14 to the multilinear maximal function.

Lemma 15. Let $0 < p < \infty$. Let v be a weight. For every cube $Q_0 \subset \mathbb{R}^n$, one has

$$\|\mathcal{M}(f_1\chi_{3Q_0}, \dots, f_m\chi_{3Q_0})v\|_{L^p(Q_0)} \leq C\|\widetilde{M}_p^{Q_0}(\vec{f}, v)\|_{L^p(Q_0)}, \quad (34)$$

where

$$\begin{aligned} \widetilde{M}_p^{Q_0}(\vec{f}, v)(x) &:= \sup_{Q_0 \supset Q \ni x} \prod_{j=1}^m \left(\int_{3Q} |f_j(y_j)| dy_j \right) \left(\int_Q v(y)^p dy \right)^{1/p}. \end{aligned} \quad (35)$$

Remark 16. In [3], there is not the restriction $Q \subset Q_0$ in the maximal function $\widetilde{M}_p^{Q_0}(\vec{f}, v)$. However, reexamining the argument of the proof in [3], we can remove the restriction. Hence the norm inequalities in Lemmas 14 and 15 are sharper than the norm inequalities in [3].

4. Proof of Theorems

4.1. Onelinear Version. We prove Theorem 11. We omit the proof of Theorems 6, 8, and 10.

Proof of Theorem 11. For every cube $Q_0 \subset \mathbb{R}^n$, by sublinearity, $Mf(x) \leq M(f\chi_{3Q_0})(x) + M(f\chi_{(3Q_0)^c})(x)$. Let $f_0 = f\chi_{3Q_0}$ and $f_\infty = f\chi_{(3Q_0)^c}$. By Lemma 14,

$$\|(Mf_0)v\|_{L^p(Q_0)} \leq C\|\widetilde{M}_p^{Q_0}(f, v)\|_{L^p(Q_0)}. \quad (36)$$

By Hölder's inequality, we have

$$\begin{aligned}
& \widetilde{M}_p^{Q_0}(f, v)(x) \\
& \leq \sup_{x \in Q \subset Q_0} \left(\int_{3Q} |f(y) w_2(y)|^{p/a} dy \right)^{a/p} \\
& \quad \times \left(\int_{3Q} w_2(y)^{-(p/a)'} dy \right)^{1/(p/a)'} \left(\int_Q v(y)^p dy \right)^{1/p} \\
& = \sup_{x \in Q \subset Q_0} \frac{w_1(Q)^{1/q_0}}{w_1(3Q)^{1/p_0}} \left(\frac{w_1(3Q)}{w_1(Q)} \right)^{1/p} \left(\frac{|Q|}{|3Q|} \right)^{1/p} \\
& \quad \times \left(\int_Q v(y)^p dy \right)^{1/p} \left(\int_{3Q} w_2(y)^{-(p/a)'} dy \right)^{1/(p/a)'} \\
& \quad \times \frac{w_1(Q)^{-1/q_0}}{w_1(3Q)^{-1/p_0}} \left(\frac{w_1(3Q)}{w_1(Q)} \right)^{-1/p} \left(\frac{|3Q|}{|Q|} \right)^{1/p} \\
& \quad \times \left(\int_{3Q} |f(y) w_2(y)|^{p/a} dy \right)^{a/p}. \tag{37}
\end{aligned}$$

By condition (26), we have

$$\begin{aligned}
& \widetilde{M}_p^{Q_0}(f, v)(x) \\
& \leq C[w_1, w_2, v]_{p_0, q_0, p, p/a} \sup_{x \in Q \subset Q_0} \frac{w_1(Q)^{-1/q_0}}{w_1(3Q)^{-1/p_0}} \\
& \quad \times \left(\frac{w_1(3Q)}{w_1(Q)} \right)^{-1/p} \left(\int_{3Q} |f(y) w_2(y)|^{p/a} dy \right)^{a/p}. \tag{38}
\end{aligned}$$

Since $w_1(Q) \leq w_1(3Q)$, we have

$$\begin{aligned}
& \sup_{x \in Q \subset Q_0} \frac{w_1(Q)^{-1/q_0}}{w_1(3Q)^{-1/p_0}} \left(\frac{w_1(3Q)}{w_1(Q)} \right)^{-1/p} \\
& \quad \times \left(\int_{3Q} |f(y) w_2(y)|^{p/a} dy \right)^{a/p} \\
& = \sup_{x \in Q \subset Q_0} \frac{w_1(3Q)^{1/p_0}}{w_1(Q)^{1/q_0}} \left(\frac{w_1(Q)}{w_1(3Q)} \right)^{1/p} \\
& \quad \times \left(\int_{3Q} |f(y) w_2(y)|^{p/a} dy \right)^{a/p} \\
& \leq \sup_{x \in Q \subset Q_0} \frac{w_1(3Q)^{1/p_0}}{w_1(Q)^{1/q_0}} \left(\int_{3Q} |f(y) w_2(y)|^{p/a} dy \right)^{a/p}. \tag{39}
\end{aligned}$$

Since w_1 satisfies the doubling condition, we obtain the following inequality; for every cube $Q \subset \mathbb{R}^n$,

$$\frac{w_1(3Q)^{1/p_0}}{w_1(Q)^{1/q_0}} \leq C^{1/q_0} \frac{w_1(3Q)^{1/p_0}}{w_1(3Q)^{1/q_0}} = C^{1/q_0} w_1(3Q)^{1/p_0-1/q_0}. \tag{40}$$

Since $p_0 \leq q_0$, for pair of cubes $Q \subset Q_0$, we have

$$w_1(3Q)^{(1/p_0)-(1/q_0)} \leq w_1(3Q_0)^{(1/p_0)-(1/q_0)}. \tag{41}$$

Hence we have

$$\begin{aligned}
& \sup_{x \in Q \subset Q_0} \frac{w_1(Q)^{-1/q_0}}{w_1(3Q)^{-1/p_0}} \left(\frac{w_1(3Q)}{w_1(Q)} \right)^{-1/p} \\
& \quad \times \left(\int_{3Q} |f(y) w_2(y)|^{p/a} dy \right)^{a/p} \\
& \leq C^{1/q_0} w_1(3Q_0)^{1/p_0-1/q_0} M[|fw_2|^{p/a} \chi_{3Q_0}](x)^{a/p}. \tag{42}
\end{aligned}$$

This implies that

$$\begin{aligned}
& \left(\int_{Q_0} Mf_0(x)^p v(x)^p dx \right)^{1/p} \\
& \leq C[w_1, w_2, v]_{p_0, q_0, p, p/a} \cdot w_1(3Q_0)^{1/p_0-1/q_0} \\
& \quad \times \left(\int_{Q_0} M[|fw_2|^{p/a} \cdot \chi_{3Q_0}](x)^{(a/p) \cdot p} dx \right)^{1/p} \\
& = C[w_1, w_2, v]_{p_0, q_0, p, p/a} \cdot w_1(3Q_0)^{1/p_0-1/q_0} \\
& \quad \times \left(\int_{Q_0} M[|fw_2|^{p/a} \chi_{3Q_0}](x)^a dx \right)^{1/p}. \tag{43}
\end{aligned}$$

By the boundedness of $M : L^a(\mathbb{R}^n) \rightarrow L^a(\mathbb{R}^n)$, we obtain

$$\begin{aligned}
& \left(\int_{Q_0} M[|fw_2|^{p/a} \chi_{3Q_0}](x)^a dx \right)^{1/p} \\
& \leq C \left(\int_{3Q_0} |f(x) w_2(x)|^p dx \right)^{1/p}. \tag{44}
\end{aligned}$$

Since $p \leq q_0$, by the doubling condition of w_1 , we have

$$\begin{aligned}
& w_1(Q_0)^{1/q_0-1/p} \left(\int_{Q_0} Mf_0(x)^p v(x)^p dx \right)^{1/p} \\
& \leq C[w_1, w_2, v]_{p_0, q_0, p, p/a} \\
& \quad \cdot w_1(Q_0)^{1/q_0-1/p} w_1(3Q_0)^{1/p_0-1/q_0} \\
& \quad \times \left(\int_{3Q_0} |f(x) w_2(x)|^p dx \right)^{1/p} \\
& \leq C[w_1, w_2, v]_{p_0, q_0, p, p/a} \\
& \quad \cdot w_1(3Q_0)^{1/p_0-1/p} \left(\int_{3Q_0} |f(x) w_2(x)|^p dx \right)^{1/p} \\
& \leq C[w_1, w_2, v]_{p_0, q_0, p, p/a} \\
& \quad \cdot w_1(3Q_0)^{1/p_0-1/p} \left(\int_{3Q_0} |f(x) w_2(x)|^p dx \right)^{1/p}. \tag{45}
\end{aligned}$$

Taking the weighted Morrey norm, we obtain the following inequality:

$$\|Mf_0\|_{\mathcal{M}_p^{q_0}(w_1, v^p)} \leq C \|f\|_{\mathcal{M}_p^{p_0}(w_1, w_2^p)}. \quad (46)$$

Next, we estimate $Mf_\infty(x)$. By routine geometric observation, for $x \in Q_0$, we have

$$Mf_\infty(x) \leq C \sup_{Q \supset Q_0} \int_Q |f(y)| dy. \quad (47)$$

For pair of cubes $Q \supset Q_0$, by Hölder's inequality, we obtain

$$\begin{aligned} Mf_\infty(x) &\leq C \sup_{Q \supset Q_0} \left(\int_Q |f(y)|^{p/a} w_2(y)^{p/a} dy \right)^{a/p} \\ &\quad \times \left(\int_Q w_2(y)^{-(p/a)'} dy \right)^{1/(p/a)'} \\ &\leq C \sup_{Q \supset Q_0} w_1(Q)^{1/p_0 - 1/p} \\ &\quad \times \left(\int_Q |f(y) w_2(y)|^p dy \right)^{1/p} \frac{1}{|Q|^{1/p}} \\ &\quad \times \left(\int_Q w_2(y)^{-(p/a)'} dy \right)^{1/(p/a)'} w_1(Q)^{1/p - 1/p_0} \\ &\leq C \|f\|_{\mathcal{M}_p^{p_0}(w_1, w_2^p)} \cdot \sup_{Q \supset Q_0} \frac{1}{|Q|^{1/p}} \\ &\quad \times \left(\int_Q w_2(y)^{-(p/a)'} dy \right)^{1/(p/a)'} w_1(Q)^{1/p - 1/p_0}. \end{aligned} \quad (48)$$

Therefore we have

$$\begin{aligned} w_1(Q_0)^{1/q_0 - 1/p} \left(\int_{Q_0} Mf_\infty(x)^p v(x)^p dx \right)^{1/p} \\ \leq C \|f\|_{\mathcal{M}_p^{p_0}(w_1, w_2^p)} \sup_{Q \supset Q_0} \frac{w_1(Q_0)^{1/q_0}}{w_1(Q)^{1/p_0}} \left(\frac{w_1(Q)}{w_1(Q_0)} \right)^{1/p} \\ \times \left(\frac{|Q_0|}{|Q|} \right)^{1/p} \left(\int_{Q_0} v(x)^p dx \right)^{1/p} \\ \times \left(\int_Q w_2(y)^{-(p/a)'} dy \right)^{1/(p/a)'}. \end{aligned} \quad (49)$$

By condition (26), we have

$$\begin{aligned} \sup_{\substack{Q \supset Q_0 \\ Q, Q_0: \text{cubes}}} \frac{w_1(Q_0)^{1/q_0}}{w_1(Q)^{1/p_0}} \left(\frac{w_1(Q)}{w_1(Q_0)} \right)^{1/p} \left(\frac{|Q_0|}{|Q|} \right)^{1/p} \\ \times \left(\int_{Q_0} v(x)^p dx \right)^{1/p} \left(\int_Q w_2(y)^{-(p/a)'} dy \right)^{1/(p/a)'} \\ \leq [v, w_1, w_2]_{p_0, q_0, p, p/a}. \end{aligned} \quad (50)$$

This implies that

$$\begin{aligned} \|Mf_\infty\|_{\mathcal{M}_p^{q_0}(w_1, v^p)} \\ \leq C [v, w_1, w_2]_{p_0, q_0, p, p/a} \|f\|_{\mathcal{M}_p^{p_0}(w_1, w_2^p)}. \end{aligned} \quad (51)$$

Therefore we obtain the desired result. $\square$

4.2. Multilinear Version. We prove Theorem 13.

Proof of Theorem 13. For every cube $Q \subset \mathbb{R}^n$, we have $f_j(y_j) = f_j(y_j) \chi_{3Q_0}(y_j) + f_j(y_j) \chi_{(3Q_0)^c}(y_j) = f_j^0(y_j) + f_j^\infty(y_j)$. Let $\vec{f}_0 = (f_1^0, \dots, f_m^0)$, $\vec{f}_l = (f_1^l, \dots, f_m^l)$, and $\vec{l} = (l_1, \dots, l_m)$:

$$\mathcal{M}(\vec{f})(x) \leq \mathcal{M}(\vec{f}_0)(x) + \sum_{\vec{l} \neq \vec{0}} \mathcal{M}(\vec{f}_l)(x). \quad (52)$$

Firstly, we estimate $\mathcal{M}(\vec{f}_0)(x)$. By Lemma 15, we have

$$\|\mathcal{M}(\vec{f}_0) v\|_{L^p(Q_0)} \leq C \|\widetilde{M}_p^{Q_0}(\vec{f}, v)\|_{L^p(Q_0)}, \quad (53)$$

where

$$\begin{aligned} \widetilde{M}_p^{Q_0}(\vec{f}, v)(x) \\ := \sup_{Q_0 \supset Q \ni x} \prod_{j=1}^m \left(\int_{3Q} |f_j(y_j)| dy_j \right) \left(\int_Q v(y)^p dy \right)^{1/p}. \end{aligned} \quad (54)$$

By Hölder's inequality, we obtain

$$\begin{aligned} \widetilde{M}_p^{Q_0}(\vec{f}, v)(x) \\ \leq \sup_{Q_0 \supset Q \ni x} \prod_{j=1}^m \left(\int_{3Q} |f_j(y_j)|^{p_j/a} w_j(y_j)^{p_j/a} dy_j \right)^{a/p_j} \\ \times \left(\int_{3Q} w_j(y_j)^{-(p_j/a)'} dy_j \right)^{1/(p_j/a)'} \left(\int_Q v(y)^p dy \right)^{1/p}. \end{aligned} \quad (55)$$

By condition (30), we obtain

$$\begin{aligned} \widetilde{M}_p^{Q_0}(\vec{f}, v)(x) \\ \leq C [v, W, \vec{w}]_{p_0, q_0, p, \vec{p}/a} \\ \times \sup_{x \in Q \subset Q_0} \prod_{j=1}^m \left(\int_{3Q} |f_j(y_j)|^{p_j/a} w_j(y_j)^{p_j/a} dy_j \right)^{a/p_j} \\ \times \frac{W(3Q)^{1/p_0}}{W(Q)^{1/q_0}} \left(\frac{W(Q)}{W(3Q)} \right)^{1/p}. \end{aligned} \quad (56)$$

Since $W(Q) \leq W(3Q)$, we have

$$\left(\frac{W(Q)}{W(3Q)} \right)^{1/p} \leq 1. \quad (57)$$

Since W satisfies the doubling condition, we have

$$\frac{W(3Q)^{1/p_0}}{W(Q)^{1/q_0}} \leq CW(3Q)^{1/p_0-1/q_0}. \quad (58)$$

Since $p_0 \leq q_0$, for pair of cubes $Q \subset Q_0$, we have

$$W(3Q)^{1/p_0-1/q_0} \leq W(3Q_0)^{1/p_0-1/q_0}. \quad (59)$$

Therefore we obtain

$$\begin{aligned} & \widetilde{M}_p^{Q_0}(\vec{f}, v)(x) \\ & \leq C[v, W, \vec{w}]_{p_0, q_0, p, \vec{p}/a} \cdot W(3Q_0)^{1/p_0-1/q_0} \\ & \quad \cdot \sup_{x \in Q \subset Q_0} \prod_{j=1}^m \left(\int_{3Q} |f_j(y_j)|^{p_j/a} w_j(y_j)^{p_j/a} dy_j \right)^{a/p_j} \\ & \leq C[v, W, \vec{w}]_{p_0, q_0, p, \vec{p}/a} \cdot W(3Q_0)^{1/p_0-1/q_0} \\ & \quad \cdot \prod_{j=1}^m M \left[|f_j w_j|^{p_j/a} \chi_{3Q_0} \right] (x)^{a/p_j}. \end{aligned} \quad (60)$$

This implies that

$$\begin{aligned} & \left(\int_{Q_0} \widetilde{M}_p^{Q_0}(\vec{f}, v)(x)^p dx \right)^{1/p} \\ & \leq C[v, W, \vec{w}]_{p_0, q_0, p, \vec{p}/a} \cdot W(3Q_0)^{1/p_0-1/q_0} \\ & \quad \times \left(\int_{Q_0} \prod_{j=1}^m M \left[|f_j w_j|^{p_j/a} \chi_{3Q_0} \right] (x)^{(a/p_j)p} dx \right)^{1/p}. \end{aligned} \quad (61)$$

Since $1/p = 1/p_1 + \dots + 1/p_m$, by Hölder's inequality, we have

$$\begin{aligned} & \left(\int_{Q_0} \widetilde{M}_p^{Q_0}(\vec{f}, v)(x)^p dx \right)^{1/p} \\ & \leq C[v, W, \vec{w}]_{p_0, q_0, p, \vec{p}/a} \cdot W(3Q_0)^{1/p_0-1/q_0} \\ & \quad \times \prod_{j=1}^m \left(\int_{Q_0} M \left[|f_j w_j|^{p_j/a} \chi_{3Q_0} \right] (x)^a dx \right)^{1/p_j}. \end{aligned} \quad (62)$$

By the boundedness of M on $L^a(\mathbb{R}^n) \rightarrow L^a(\mathbb{R}^n)$, we have

$$\begin{aligned} & \left(\int_{Q_0} \widetilde{M}_p^{Q_0}(\vec{f}, v)(x)^p dx \right)^{1/p} \\ & \leq C[v, W, \vec{w}]_{p_0, q_0, p, \vec{p}/a} \cdot W(3Q_0)^{1/p_0-1/q_0} \\ & \quad \times \prod_{j=1}^m \left(\int_{3Q_0} |f_j(x)|^{p_j} w_j(x)^{p_j} dx \right)^{1/p_j}. \end{aligned} \quad (63)$$

Since $p \leq q_0$, by the doubling condition of W , we have

$$\begin{aligned} & W(Q_0)^{1/q_0} \left(\frac{1}{W(Q_0)} \int_{Q_0} \mathcal{M}(\vec{f}_0)(x)^p v(x)^p dx \right)^{1/p} \\ & \leq C[v, W, \vec{w}]_{p_0, q_0, p, \vec{p}/a} \cdot W(3Q_0)^{1/p_0-1/p} \\ & \quad \times \prod_{j=1}^m \left(\int_{3Q_0} |f_j(x)|^{p_j} w_j(x)^{p_j} dx \right)^{1/p_j}. \end{aligned} \quad (64)$$

Taking the multiple weighted Morrey quantity, we have

$$\begin{aligned} & W(Q_0)^{1/p_0} \left(\frac{1}{W(Q_0)} \int_{Q_0} \mathcal{M}(\vec{f}_0)(x)^p v(x)^p dx \right)^{1/p} \\ & \leq C[v, W, \vec{w}]_{p_0, q_0, p, \vec{p}/a} \cdot \|\vec{f}\|_{\mathcal{M}_p^{p_0}(W, \vec{w}^p)}. \end{aligned} \quad (65)$$

Next, we estimate $\mathcal{M}(\vec{f}_i)(x)$. By routine geometric observation, for $x \in Q_0$, we have

$$\mathcal{M}(\vec{f}_i)(x) \leq C \sup_{Q \supset Q_0} \prod_{j=1}^m \int_Q |f_j(y_j)| dy_j. \quad (66)$$

By Hölder's inequality, we have

$$\begin{aligned} & \sup_{Q \supset Q_0} \prod_{j=1}^m \int_Q |f_j(y_j)| dy_j \\ & \leq \sup_{Q \supset Q_0} \prod_{j=1}^m \left(\int_Q |f_j(y_j)|^{p_j/a} w_j(y_j)^{p_j/a} dy_j \right)^{a/p_j} \\ & \quad \cdot \left(\int_Q w_j(y_j)^{-(p_j/a)'} dy_j \right)^{1/(p_j/a)'} \\ & \leq \sup_{Q \supset Q_0} \prod_{j=1}^m \left(\int_Q |f_j(y_j)|^{p_j} w_j(y_j)^{p_j} dy_j \right)^{1/p_j} \\ & \quad \cdot \left(\int_Q w_j(y_j)^{-(p_j/a)'} dy_j \right)^{1/(p_j/a)'}. \end{aligned} \quad (67)$$

Taking the multiple weighted Morrey quantity, we obtain

$$\begin{aligned} & \sup_{Q \supset Q_0} \prod_{j=1}^m \left(\int_Q |f_j(y_j)|^{p_j} w_j(y_j)^{p_j} dy_j \right)^{1/p_j} \\ & \quad \cdot \left(\int_Q w_j(y_j)^{-(p_j/a)'} dy_j \right)^{1/(p_j/a)'} \\ & \leq C \|\vec{f}\|_{\mathcal{M}_p^{p_0}(W, \vec{w}^p)} \cdot \sup_{Q \supset Q_0} \frac{1}{|Q|^{1/p}} W(Q)^{1/p-1/p_0} \\ & \quad \times \prod_{j=1}^m \left(\int_Q w_j(y_j)^{-(p_j/a)'} dy_j \right)^{1/(p_j/a)'}. \end{aligned} \quad (68)$$

This implies that

$$\begin{aligned}
 & W(Q_0)^{1/q_0} \left(\frac{1}{W(Q_0)} \int_{Q_0} \mathcal{M}(\vec{f}_I)(x)^p v(x)^p dx \right)^{1/p} \\
 & \leq C \|\vec{f}\|_{\mathcal{M}_p^{p_0}(W, \vec{w}^{\vec{P}})} \sup_{Q \supset Q_0} \frac{W(Q_0)^{1/q_0}}{W(Q)^{1/p_0}} \left(\frac{W(Q)}{W(Q_0)} \right)^{1/p} \\
 & \quad \times \left(\frac{|Q_0|}{|Q|} \right)^{1/p} \left(\int_{Q_0} v(x)^p dx \right)^{1/p} \\
 & \quad \times \prod_{j=1}^m \left(\int_Q w_j(y_j)^{-(p_j/a)'} dy_j \right)^{1/(p_j/a)'}.
 \end{aligned} \tag{69}$$

By condition (28), we obtain

$$\begin{aligned}
 & W(Q_0)^{1/p_0} \left(\frac{1}{W(Q_0)} \int_{Q_0} \mathcal{M}(\vec{f}_I)(x)^p v(x)^p dx \right)^{1/p} \\
 & \leq C[v, W, \vec{w}]_{p_0, p, \vec{P}/a} \|\vec{f}\|_{\mathcal{M}_p^{p_0}(W, \vec{w}^{\vec{P}})}.
 \end{aligned} \tag{70}$$

Therefore we obtain the desired result. $\square$

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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