Research Article

The Identical Estimates of Spectral Norms for Circulant Matrices with Binomial Coefficients Combined with Fibonacci Numbers and Lucas Numbers Entries

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Improved estimates for spectral norms of circulant matrices are investigated, and the entries are binomial coefficients combined with either Fibonacci numbers or Lucas numbers. Employing the properties of given circulant matrices, this paper improves the inequalities for their spectral norms, and gets corresponding identities of spectral norms. Moreover, by some well-known identities, the explicit identities for spectral norms are obtained. Some numerical tests are listed to verify the results.

1. Introduction

Circulant matrices have connection to physics, signal and image processing, probability, statistics, numerical analysis, algebraic coding theory, and many other areas. There are lots of examples from statistical signal processing and information theory that illustrate the application of the circulant matrices, which emphasize how the asymptotic eigenvalue distribution theorem allows one to evaluate results for processes (for the details please refer to [1–3] and the reference therein). Meanwhile a real circulant stochastic process can be described with autocovariance matrices, which are subjected to a cyclical permutation. With the help of autocovariance circulant matrices, it is easy to provide derivations of some results that are central to the analysis of statistical periodograms and empirical spectral density functions (see [4]).

In past decades, the estimates for spectral norms of matrices have been investigated in lots of literatures. Moreover, the determinants and inverses of circulant matrices are stated in many articles. The norms of circulant matrices play an important role in analysing the process of statistics, numerical analysis, and many other problems (for more details, please refer to [3, 5–10] and the reference therein). Bryc and Sethuraman [11] investigated the maximum eigenvalue for circulant matrices. Solak [7] obtained lower and upper

bounds for the spectral norm of circulant matrices, where the entries are classical Fibonacci numbers. İpek [8] establishes spectral norms of circulant matrices with Fibonacci and Lucas numbers. Furthermore, circulant matrices take up an important status in stochastic calculus, Meckes [12, 13] gave some results on the spectral norm of a special random Toeplitz matrix and random circulant matrices, Mehta [14] made a deep discussion on random circulant matrices.

The outline of this paper is as follows. In Section 2, we state some preliminaries and recall some well-known results. In Section 3, we focus on the identities of estimations for spectral norms. In Section 4, we present various numerical examples to exhibit the accuracy and efficiency of our results. Finally, we summarise this paper and illustrate our future work.

2. Preliminaries

The Fibonacci and Lucas sequences $\{F_n\}$ and $\{L_n\}$ are defined by the recurrence relations:

$$F_n = F_{n-1} + F_{n-2}, \quad L_n = L_{n-1} + L_{n-2}, \quad (1)$$

with $F_0 = 0$, $F_1 = 1$, $L_0 = 2$, and $L_1 = 1$, respectively.

Obviously, the Fibonacci and Lucas sequences are listed in the following sequence:

$$\begin{array}{ccccccccccc} n & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \\ F_n & 0 & 1 & 1 & 2 & 3 & 5 & 8 & 13 & \cdots \\ L_n & 2 & 1 & 3 & 4 & 7 & 11 & 18 & 29 & \cdots, \end{array} \quad (2)$$

and their corresponding Binet forms are (see [15])

$$\begin{aligned} F_n &= \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \cos(\pi n) \left(\frac{1+\sqrt{5}}{2} \right)^{-n} \right], \\ L_n &= \left(\frac{1+\sqrt{5}}{2} \right)^n + \cos(\pi n) \left(\frac{1+\sqrt{5}}{2} \right)^{-n}. \end{aligned} \quad (3)$$

Now, we recall that, for

$$B = \begin{pmatrix} F_0 & F_1 & \cdots & F_{n-1} \\ F_{n-1} & F_0 & \cdots & F_{n-2} \\ F_{n-2} & F_{n-1} & \cdots & F_{n-3} \\ \vdots & \vdots & \ddots & \vdots \\ F_1 & F_2 & \cdots & F_0 \end{pmatrix}_{n \times n}, \quad (4)$$

there hold the following estimates:

$$\begin{aligned} & \sqrt{F_n F_{n-1} + 4 \left(F_{n-1}^2 + F_{n-1} F_{n-2} + \frac{1 - (-1)^n}{2} \right)} \leq \|B\|_2, \\ \|B\|_2 & \leq \sqrt{F_n F_{n-1} + 4 \left(F_{n-1}^2 + F_{n-1} F_{n-2} + \frac{1 - (-1)^n}{2} \right)} \\ & \times \sqrt{1 + F_n F_{n-1} + 4 \left(F_{n-1}^2 + F_{n-1} F_{n-2} - \frac{3 + (-1)^n}{2} \right)}. \end{aligned} \quad (5)$$

For the details please refer to [7].

There are lots of identities for Fibonacci numbers and Lucas numbers combined with Binomial coefficients (for more details please refer to [8, 16–18] and the reference therein). In this paper, we focus on the following identities:

$$\begin{aligned} \sum_{i=0}^n \binom{n}{i} F_i &= F_{2n}, & \sum_{i=0}^n \binom{n}{i} L_i &= L_{2n}, \\ \sum_{i=0}^n \binom{n}{i} F_{i+k} &= F_{2n+k}, & \sum_{i=0}^n \binom{n}{i} L_{i+k} &= L_{2n+k}. \end{aligned} \quad (6)$$

Furthermore, for all $k \in \mathbb{Z}^+$, there hold the following identities:

$$\sum_{i=0}^n \binom{n}{i} F_{2i+k} = \begin{cases} 5^{n/2} F_{n+k}, & (n \text{ even}), \\ 5^{(n-1)/2} L_{n+k}, & (n \text{ odd}), \end{cases} \quad (7)$$

$$\sum_{i=0}^n \binom{n}{i} L_{2i+k} = \begin{cases} 5^{n/2} L_{n+k}, & (n \text{ even}), \\ 5^{(n+1)/2} F_{n+k}, & (n \text{ odd}). \end{cases} \quad (8)$$

Definition 1 (see [19]). A circulant matrix is an $n \times n$ complex matrix with the following form:

$$A = \begin{pmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ a_{n-1} & a_0 & \cdots & a_{n-2} \\ a_{n-2} & a_{n-1} & \cdots & a_{n-3} \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \cdots & a_0 \end{pmatrix}_{n \times n}. \quad (9)$$

The first row of A is $(a_0, a_1, \dots, a_{n-1})$ and its $(j+1)$ th row is obtained by giving its j th row a right circular shift by one positions.

Definition 2 (see [3]). The spectral norm $\|\cdot\|_2$ of a matrix A with complex entries is the square root of the largest eigenvalue of the positive semidefinite matrix $A^* A$:

$$\|A\|_2 = \sqrt{\lambda_{\max}(A^* A)}, \quad (10)$$

where A^* denotes the conjugate transpose of A . Therefore if A is an $n \times n$ real symmetric matrix or A is a normal matrix, then

$$\|A\|_2 = \max_{1 \leq i \leq n} |\lambda_i|, \quad (11)$$

where $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues of A .

3. The Identities of Estimations for Spectral Norms

We give the main theorems of this paper in the following parts.

Theorem 3. Let B_1 be as the matrix in (9), and let the first row of B_1 be $((\binom{n}{0}) F_0, (\binom{n}{1}) F_1, \dots, (\binom{n}{n}) F_n)$. Then one has

$$\|B_1\|_2 = F_{2n}. \quad (12)$$

Proof. Combining with Definition 2, the spectral radius of B_1 is equal to its spectral norm, where we used the fact that B_1 is normal. Moreover, by the irreducible and entrywise nonnegative properties, we deduce that $\|B_1\|_2$ is equal to its Perron value. Denote by $v = (1, 1, \dots, 1)^T$ an n -dimensional column vector. There holds

$$B_1 v = \left(\sum_{i=0}^n \binom{n}{i} F_i \right) v. \quad (13)$$

Obviously, $\sum_{i=0}^n \binom{n}{i} F_i$ is an eigenvalue of B_1 associated with the positive eigenvector v , which is the Perron value of B_1 . Employing the first identity in (6), we have

$$\|B_1\|_2 = F_{2n}. \quad (14)$$

This completes the proof. $\square$

With the same approach, we obtain the following corollary.

TABLE 1: Spectral norms of B_i ($i = 1, 2, 3, 4$) and $k = 1$.

n	1	2	3	4	5	6	7	8
$\ B_1\ _2$	1	3	8	21	55	144	377	987
F_{2n}	1	3	8	21	55	144	377	987
$\ B_2\ _2$	3	7	18	47	123	322	843	2207
L_{2n}	3	7	18	47	123	322	843	2207
$\ B_3\ _2$	2	5	13	34	89	233	610	1597
F_{2n+1}	2	5	13	34	89	233	610	1597
$\ B_4\ _2$	4	11	29	76	199	521	1364	3571
L_{2n+1}	4	11	29	76	199	521	1364	3571

Corollary 4. Let B_2 be as the matrix in (9), and let the first row of B_2 be $((\binom{n}{0})L_0, (\binom{n}{1})L_1, \dots, (\binom{n}{n})L_n)$. Then one has the following identity:

$$\|B_2\|_2 = L_{2n}. \quad (15)$$

Theorem 5. Let B_3 be with the form as (9). For all $k \in \mathbb{Z}^+$, if the first row of B_3 is $((\binom{n}{0})F_k, (\binom{n}{1})F_{1+k}, \dots, (\binom{n}{n})F_{n+k})$, then one obtains

$$\|B_3\|_2 = F_{2n+k}. \quad (16)$$

Proof. Following the same techniques of the above theorem and combining with the fact that B_3 is irreducible and entrywise nonnegative, we declare that the spectral norm of B_3 is equal to its Perron value. Let $v^T = (1, 1, \dots, 1)_{1 \times n}$. Then

$$B_3 v = \left(\sum_{i=0}^n \binom{n}{i} F_{i+k} \right) v. \quad (17)$$

Obviously, we declare that $\sum_{i=0}^n \binom{n}{i} F_{i+k}$ is an eigenvalue of B_3 associated with v . With simple analysis, we obtain that $\sum_{i=0}^n \binom{n}{i} F_{i+k}$ is equal to the Perron value of B_3 . Combining with the third identity of binomial coefficients and Fibonacci numbers in (6), we obtain

$$\|B_3\|_2 = F_{2n+k}, \quad (18)$$

which completes the proof. $\square$

Similarly, there holds the following corollary.

Corollary 6. Let B_4 be as the matrix in (9). For all $k \in \mathbb{Z}^+$, the first row of B_4 is $((\binom{n}{0})L_k, (\binom{n}{1})L_{1+k}, \dots, (\binom{n}{n})L_{n+k})$; then

$$\|B_4\|_2 = L_{2n+k}. \quad (19)$$

Now, we are at the point to recall the following lemma to verify the identities of spectral norms with other approaches.

Lemma 7 (see [3]). Let A be a nonnegative matrix. If the column sums of A are equal, then

$$\rho(A) = \|A\|_1, \quad (20)$$

where $\rho(A) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of matrix } A\}$ and $\|\cdot\|_1$ denotes the maximum column sum matrix norm.

Theorem 8. Let B_5 be with the form as (9) and let the first row of B_5 be $((\binom{n}{0})F_k, (\binom{n}{1})F_{2+k}, \dots, (\binom{n}{n})F_{2n+k})$. Then one deduces the following identity:

$$\|B_5\|_2 = \begin{cases} 5^{n/2} F_{n+k} & (n \text{ even}), \\ 5^{(n-1)/2} L_{n+k} & (n \text{ odd}), \end{cases} \quad (21)$$

where $k \in \mathbb{Z}^+$.

Proof. Obviously, the circulant matrix B_5 is normal; with the results of Definition 2, we declare that the spectral radius of B_5 is equal to $\rho(B_5)$; that is, $\|B_5\|_2 = \rho(B_5)$. Furthermore, applying entrywise nonnegative properties and column sum of B_5 are certain constant K_{col} , which is described in (7). By Lemma 7, we obtain

$$\rho(B_5) = K_{\text{col}}. \quad (22)$$

Employing the identities of Fibonacci numbers and Binomial coefficients in (7), we have

$$\|B_5\|_2 = \begin{cases} 5^{n/2} F_{n+k} & (n \text{ even}), \\ 5^{(n-1)/2} L_{n+k} & (n \text{ odd}). \end{cases} \quad (23)$$

This completes the proof. $\square$

Furthermore, we give the following corollary without proofs, which can be proved with the same approaches as the above theorem.

Corollary 9. Let B_6 be as the matrix in (9). For all $k \in \mathbb{Z}^+$, the first row of B_6 is $((\binom{n}{0})L_k, (\binom{n}{1})L_{2+k}, \dots, (\binom{n}{n})L_{2n+k})$; then we have the following identity:

$$\|B_6\|_2 = \begin{cases} 5^{n/2} L_{n+k} & (n \text{ even}), \\ 5^{(n+1)/2} F_{n+k} & (n \text{ odd}). \end{cases} \quad (24)$$

4. Numerical Examples

In this section, we give some examples to verify our identities in the above theorems and corollaries.

Example 10. In this example, we give the numerical results for B_i ($i = 1, 2, 3, 4$) in Table 1.

TABLE 2: Spectral norms of B_5 and B_6 .

n	1	2	3	4	5	6	7	8
$\ B_5\ _2$	3	10	35	125	450	1625	5875	21250
$5^{n/2}F_{n+1}$	\	10	\	125	\	1625	\	21250
$5^{(n-1)/2}L_{n+1}$	3	\	35	\	450	\	5875	\
$\ B_6\ _2$	5	20	100	275	1000	3625	13125	47500
$5^{(n+1)/2}F_{n+1}$	5	\	100	\	1000	\	13125	\
$5^{n/2}L_{n+1}$	\	20	\	275	\	3625	\	47500

Example 11. For simplicity, let $k = 1$. We give the numerical results for B_5 and B_6 in Table 2.

With the data in Tables 1 and 2, we declare that the identity for the spectral norm of B_i ($i = 1, \dots, 6$) holds.

5. Conclusion

This paper had discussed the identical estimates of spectral norms for some circulant matrices, which are listed by explicit formulations. In the future, we are going to investigate the determinants, inverses of circulant matrices with certain entries, and, inspired by [6], we will investigate the properties of g -circulant matrices. Particularly worth mentioning is the fact that, for the g -circulant matrix, we had some numerical results to prove the fact that the same identical estimates hold precisely, and we will concern on the theoretical confirmation in part of the future work.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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