Research Article

A Note on Starshaped Sets in 2-Dimensional Manifolds without Conjugate Points

Adem Kılıcman and Wedad Saleh

*Department of Mathematics and Institute for Mathematical Research, Universiti Putra Malaysia (UPM),
43400 Serdang, Selangor, Malaysia*

Correspondence should be addressed to Adem Kılıcman; akilic@upm.edu.my

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Let W^n be C^∞ complete, simply connected n -dimensional Riemannian manifolds without conjugate points. Assume that $S \subset W^2$ is starshaped where $\ker S \neq S$. For every point $x \in S \setminus \ker S$, define $A(x) = \{y : y \text{ lies on some geodesic segment in } S \text{ from } x \text{ to a point of } \ker S\}$. There is a finite collection $\mathcal{A}$ of all maximal A sets whose union is S . Further, $\ker S = \cap \{A : A \text{ in } \mathcal{A}\}$.

1. Introduction

We begin with some definitions from [1–3]. A subset B in a Riemannian manifold M is convex if, for each pair point $p, q \in B$, there is a unique minimal geodesic segment from p to q and this segment is in B . When dealing with a subset $B \subset W^n$, the word “a unique minimal geodesic segment” should be replaced by “the geodesic.” A subset S in a Riemannian manifold M is starshaped if there is a point $p \in S$ such that, for all $q \in S$, there is a unique minimal geodesic segment from p to q and this segment is in S . The subset of S consisting of all points like p is called the kernel of S ($\ker S$). In W^n , a subset S is starshaped if there is a point $p \in S$ such that, for all $q \in S$, the geodesic segment joining p and q is contained in S . The subset S is convex if and only if $\ker S = S$. Throughout the paper, $\text{Int}(B)$ and ∂B will denote the interior and the boundary of the subset B , respectively. Let $d(p, q)$ denote the distance between the two points $p, q \in W^n$. For the pair of points $p, q \in W^n$, γ_{pq} will denote the geodesic segment joining p, q . For $x \in S$ and $B \subset S$, $d(x, B) = \inf\{d(x, b) : b \in B\}$. The distance to $B \subset S$ is locally maximal at $x \in S$ if there is some neighborhood U of x such that $d(y, B) \leq d(x, B) \forall y \in U \cap S$. Finally, all manifolds, maps, fields, and so forth are discussions to make sense. All curves are parametrized by arc-length.

Readers may refer to [4] for discussions concerning visibility via geodesic segment and starshaped sets. The behavior

geodesic in W^n has been discussed by many geometrists as Eberlein [2] and Goto [5]. For more properties of convexity and starshapedness in W^n , see [6–9].

2. Main Results

Let W^2 be 2-dimensional manifolds without conjugate points, and assume that $S \subset W^2$ is starshaped. For every $x \in S \setminus \ker S$, we define $A(x) = \{y : y \in \gamma_{xk} \subset S, k \in \ker S\}$. Clearly $S = \cup\{A(x) : x \in S \setminus \ker S\}$.

Lemma 1. *Let S be starshaped in 2-dimensional manifolds without conjugate point. Then*

- the set $\ker S$ is convex,*
- if γ is any geodesic segment in S joining points of $\ker S$, then $\gamma \subset \ker S$,*
- for every point $x \in S$, there is a unique point k of $\ker S$ closest to x ; that is, there is a unique point k of $\ker S$ such that $d(x, k) = d(x, \ker S)$.*

Proof. We have the following.

- Since S is a starshaped subset of W^2 , by [10], $\ker S$ is convex.
- Since $\ker S$ is convex, clearly any geodesic segment in S joining points of $\ker S$ is in $\ker S$.

- (c) Let x be an arbitrary point in S . If $x \in \ker S$, then the result is trivial. $x \in S \setminus \ker S$. Let there be two points k_1 and k_2 in $\ker S$ closest to x , that is, $d(x, k_1) = d(x, k_2) = d(x, \ker S)$, to show that $k_1 = k_2$. Suppose on the contrary that $k_1 \neq k_2$. Since k_1 and k_2 are in $\ker S$, then $\gamma_{k_1 k_2} \subset \ker S$. Let $k \in \gamma_{k_1 k_2} \subset \ker S$, such that $k \neq k_1$ and $k \neq k_2$. However, k is closer to x than k_1 and k_2 , contradicting the assumption that k_1 and k_2 are closest to x ; then $k_1 = k_2$. $\square$

Proposition 2. Let S be starshaped in 2-dimensional manifolds without conjugate points. For every point $x \in S \setminus \ker S$, one defines $A(x) = \{y : y \in \gamma_{xk} \subset S, k \in \ker S\}$. For any pair of points $x, z \in S \setminus \ker S$, if $x \in A(z)$, then $A(x) \subseteq A(z)$.

Proof. Let $s \in A(x)$; then $s \in \gamma_{xk}$ where $k \in \ker S$. Since $x \in A(z)$, then $x \in \gamma_{zk}$ where $k \in \ker S$. This implies that $\gamma_{xk} \subset \gamma_{zk}$; then $s \in \gamma_{zk}$. We conclude that $s \in A(z)$ and the proof is complete. $\square$

Proposition 3. Let $S \subset W^2$ be a starshaped set. Let $x \in S \setminus \ker S$. If $x \notin \partial S$, then there is some point $z \in \partial S$ such that $x \in A(z)$ and hence $A(x) \subset A(z)$. Moreover, $d(x, \ker S)$ is not locally maximal at x .

Proof. Let $z \in \partial S$ and $x \notin \partial S$. Let $\gamma_{xk} \subset S$ be a geodesic segment from x to k where $k \in \ker S$. Then $\gamma_{zx} \cup \gamma_{xk}$ is a geodesic segment in S from z to x to k . Hence $x \in A(z)$. Then, by Proposition 2, we have $A(x) \subseteq A(z)$. Moreover, $z \in A(z) \setminus A(x)$, because $z \notin A(x)$. Then $A(z) \not\subseteq A(x)$, and $A(x) \subset A(z)$. Finally, it is clear that, for all points $s \in \gamma_{zx}$, $d(s, \ker S) > d(x, \ker S)$. Then the distance to $\ker S$ is not locally maximal at x . $\square$

Proposition 4. Let $x \in S \setminus \ker S$. If $A(x)$ is a maximal A set, then $d(x, \ker S)$ is locally maximal at x .

Proof. Assume on the contrary that every neighborhood of x contains some point z for which $d(x, \ker S) < d(z, \ker S)$. By Proposition 3, let $z \in \partial S$. Then $x \in \gamma_{zx} \cup \gamma_{xk}$ where $k \in \ker S$. Then $x \in \gamma_{zx}$, so $x \in A(z)$. By Proposition 2, $A(x) \subset A(z)$. Since $A(x)$ is maximal, then $A(z) \subset A(x)$ which means that $z \in \gamma_{xk}$ where $k \in \ker S$, which contradicts $d(x, \ker S) < d(z, \ker S)$. $\square$

Proposition 5. Let $S \subset W^2$ be a starshaped set. Let $x \in S$ such that $d(x, \ker S)$ is locally maximal at x . Then

- (a) the point x is a vertex of S ,
- (b) for every point z near x , $z \neq x$, $d(z, \ker S)$ is not locally maximal at z ,
- (c) $A(x) \not\subseteq A(w)$ for any point $w \neq x$.

Proof. We have the following.

- (a) Since $d(x, \ker S)$ is locally maximal at x , then $x \in S \setminus \ker S$ and every neighborhood of x contains some point z such that $d(x, \ker S) > d(z, \ker S)$. By Proposition 3, $x \in \partial S$. Then x must be a vertex of S .

- (b) It is direct from (a).

- (c) $x \in A(x) \setminus A(w)$, because $x \notin A(w)$ for any $w \neq x$. Then $A(x) \not\subseteq A(w)$. $\square$

Theorem 6. Let $S \subset W^2$ be a starshaped set. For every point $x \in S$, there is a finite subcollection $\mathcal{A}$ of all maximal A . Then, the following conditions hold:

- (a) $S = \cup\{A : A \text{ in } \mathcal{A}\}$,
- (b) for every point $z \in S \setminus \ker S$, $A(z)$ in $\mathcal{A}$ if and only if $A(z)$ is a maximal A set.

Proof. Assume that $\mathcal{A} = \{A_i(x) : x \in S \setminus \ker S, A_i(x) \not\subseteq A_j(x) \forall 1 \leq i, j \leq n\}$.

- (a) For $z \in S$, z belongs to at least one set A_i . Either A_i is in $\mathcal{A}$ or $A_i \subset A_j$ where A_j is in $\mathcal{A}$. This implies that $z \in \cup\{A : A \text{ in } \mathcal{A}\}$; then $S \subseteq \cup\{A : A \text{ in } \mathcal{A}\}$. The reverse inclusion is immediate, so the sets are equal.
- (b) $z \in S \setminus \ker S$. Let $A(z)$ be in $\mathcal{A}$; then $A(z)$ is a maximal set. To prove that $A(z)$ is in $\mathcal{A}$, assume on the contrary that $A(z)$ is not in $\mathcal{A}$. Since $z \in S \setminus \ker S$, by (a), there is $A_i(w)$ in $\mathcal{A}$ such that $A(z) \subset A_i(w)$, but $A(z)$ is a maximal A set; this implies that $A(z) = A_i(w)$. Then, $A(z)$ is not subset of any set A_j , so $A(z)$ is in $\mathcal{A}$. $\square$

Lemma 7. Let $S \subset W^2$ be a starshaped set. Each point $x \in S \setminus \ker S$. Then

- (a) the set $A(x) = \{y : y \in \gamma_{xy} \subset S, k \in \ker S\}$ is convex,
- (b) if γ is any geodesic segment in S joining points of $A(x)$, then $\gamma \subset A(x)$.

Proof. We have the following.

- (a) Let $k_1, k_2 \in \ker S$; then $\gamma_{k_1 k_2} \subset \ker S$, because $\ker S$ is convex. Moreover $\gamma_{xk_1} \cup \gamma_{xk_2} \cup \gamma_{k_1 k_2} \subset S$. Let $z_1, z_2 \in A(x)$, such that $z_i \in \gamma_{xk_i}$, $i = 1, 2$, respectively. We will prove that $\gamma_{z_1 z_2} \subset A(x)$. Assume on the contrary that $\gamma_{z_1 z_2}$ is not in $A(x)$; then there is $z \in \gamma_{z_1 z_2}$ and $z \in S \setminus A(x)$. Let k be an arbitrary point of $\ker S$ such that $z \in \gamma_{xk}$, but $z \notin A(x)$ which means that γ_{xk} is not in $A(x)$, which is a contradiction.
- (b) It is clear that any geodesic segment in S joining points of $A(x)$ is in $A(x)$, because $A(x)$ is convex. $\square$

Theorem 8. Let $S \subset W^2$ be a starshaped set. Then $\ker S = \cap\{A : A \text{ in } \mathcal{A}\} = \cap\{A(x) : x \in S \setminus \ker S\}$.

Proof. Clearly $\ker S \subseteq \cap\{A(x) : x \in S \setminus \ker S\} \subseteq \cap\{A : A \text{ in } \mathcal{A}\}$. Now, we will prove that $\cap\{A : A \text{ in } \mathcal{A}\} \subseteq \ker S$. Let $z_1 \notin \ker S$; then there is a point $z_2 \in S$ such that z_1 does not see z_2 via S . Since $z_2 \in S$, by Theorem 6, there is $A_i(w)$ in $\mathcal{A}$ such that $z_2 \in A_i(w)$; by Lemma 7, $A_i(w)$ is convex; then $z_1 \notin A_i(w)$; this implies that $z_1 \notin \cap\{A : A \text{ in } \mathcal{A}\}$; then $\cap\{A : A \text{ in } \mathcal{A}\} \subseteq \ker S$ and the proof is complete. $\square$

3. Conclusion

All results of the present work are valid in Euclidean space E^n as manifolds without conjugate points [5], but the generalization of these results to W^n , $n \geq 3$, is more difficult and is left as open problem.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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