Research Article

Arzela-Ascoli Theorem for Demi-Linear Mappings

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The Arzela-Ascoli theorem for demi-linear mappings is established. Some conclusions about preserving totally bounded sets are also obtained.

1. Introduction

In 2009, Li et al. introduced the definition of demi-linear mappings which is a meaningful extension of linear operators [1]. They also established a new equicontinuity theorem and a uniform boundedness principle for demi-linear mappings. After that, many related theories (such as demi-linear duality, demi-distributions, and matrix transformations) are obtained for demi-linear mappings which can be found in [2–4].

Arzela-Ascoli theorem has a wide range of applications in many fields of mathematics. Many modern formulations of Arzela-Ascoli theorem have been obtained [5–10].

In this paper, we will give some results on Arzela-Ascoli theorem for the space of demi-linear mappings.

2. Preliminary

Let X, Y be topological vector spaces over the scalar field $\mathbb{K}$ and $\mathcal{N}(X)$ the family of neighborhoods of $0 \in X$. Let

$$C(0) = \left\{ \gamma \in \mathbb{K}^{\mathbb{K}} : \lim_{t \rightarrow 0} \gamma(t) = \gamma(0) = 0, |\gamma(t)| \geq |t| \text{ if } |t| \leq 1 \right\}. \quad (1)$$

Definition 1 ([1, Definition 2.1]). A mapping $f : X \rightarrow Y$ is said to be demi-linear if $f(0) = 0$ and there exist a $\gamma \in C(0)$ and a $U \in \mathcal{N}(X)$ such that $x \in X, u \in U$, and $t \in \{t \in \mathbb{K} : |t| \leq 1\}$ yield $r, s \in \mathbb{K}$ for which $|r - 1| \leq |\gamma(t)|, |s| \leq |\gamma(t)|$, and $f(x + tu) = rf(x) + sf(u)$.

We denote by $\mathcal{L}_{\gamma, U}(X, Y)$ the family of demi-linear mappings related to $\gamma \in C(0)$ and $U \in \mathcal{N}(X)$ and by $\mathcal{K}_{\gamma, U}(X, Y)$ the subfamily of $\mathcal{L}_{\gamma, U}(X, Y)$ satisfying the following property: if $x \in X, u \in U$, and $|t| \leq 1$, then $f(x + tu) = f(x) + sf(u)$ for some s with $|s| \leq |\gamma(t)|$.

Note that, in general, $\mathcal{L}_{\gamma, U}(X, Y)$ is a large extension of $L(X, Y)$. For instance, if $\|\cdot\| : X \rightarrow [0, +\infty)$ is a norm, then $\|\cdot\| \in \mathcal{L}_{\gamma, X}(X, \mathbb{R})$ for every $\gamma \in C(0)$.

Definition 2. A family $\Gamma \subset Y^X$ is said to be equicontinuous at $x \in X$ if for every $W \in \mathcal{N}(Y)$, there is a $V \in \mathcal{N}(X)$ such that $f(x + V) \subset f(x) + W$ for all $f \in \Gamma$ and Γ is equicontinuous on X or, simply, equicontinuous if Γ is equicontinuous at each $x \in X$.

By the proof of [1, Theorem 3.1], the following conclusion can be easily obtained.

Theorem 3. Let $\Gamma \subset \mathcal{L}_{\gamma, U}(X, Y)$. Then, Γ is equicontinuous at 0 if and only if Γ is equicontinuous on X .

By the proof of [1, Theorem 3.2] and Definition 2, we also have the following result.

Theorem 4. Let $\{f_\alpha\} \subset \mathcal{L}_{\gamma, U}(X, Y)$ and for every $x \in X$ have $\lim f_\alpha(x) = f(x)$. Then, $f \in \mathcal{L}_{\gamma, U}(X, Y)$. If, in addition, $\{f_\alpha\}$ is equicontinuous, then f is continuous.

Definition 5. A family $\Gamma \subset Y^X$ is said to be uniformly continuous if for any neighborhood W of zero in Y , there exists a neighborhood V of zero in X such that $x_1 - x_2 \in V$ implies $f(x_1) - f(x_2) \in W$.

Definition 6. Let E be a nonempty set and X a topological vector space, $F \subset X^E$. A topology τ on E is said to be (E, F) -topology if there is an $\mathcal{F} \subset 2^F$ such that $\cup \mathcal{F} = F$ and τ is just the topology of uniform convergence on each $M \in \mathcal{F}$; that is, $u_\alpha \rightarrow u$ in (E, τ) if and only if for every $M \in \mathcal{F}$, $f(u_\alpha) \rightarrow f(u)$ uniformly with respect to $f \in M$.

Let $\sigma(E, F)$ be the weakest (E, F) -topology; that is, $u_\alpha \rightarrow u$ in $(E, \sigma(E, F))$ if and only if $f(u_\alpha) \rightarrow f(u)$, for all $f \in F$.

Let E, X , and F be as in Definition 6. For $x \in E$, define $x : X^E \rightarrow X$ by $x(f) = f(x)$. Then, $E \subset X^F$ and by Definition 6 we have the (F, E) -topology on F . Also, we denote the weakest (F, E) -topology by $\sigma(F, E)$, namely, the topology of pointwise convergence. That is,

$$f_\alpha \xrightarrow{\sigma(F, E)} f \quad \text{in } F \text{ iff } \forall x \in X, \quad \lim_\alpha f_\alpha(x) = f(x). \quad (2)$$

3. Main Results

Throughout this section, let X and Y be topological vector spaces on $\mathbb{C}$. Denote by $\mathcal{L}_{\gamma, U_0}(X, Y)$ the family of demi-linear mappings related to $\gamma \in C(0)$ and $U_0 \in \mathcal{N}(X)$. For $U_0 \in \mathcal{N}(X)$, there is a balanced closed $U \in \mathcal{N}(X)$ such that $U \subset U_0$. Then, $\mathcal{L}_{\gamma, U_0}(X, Y) \subset \mathcal{L}_{\gamma, U}(X, Y)$. Without generality, we will assume that U_0 is balanced closed in $\mathcal{L}_{\gamma, U_0}(X, Y)$.

Recall that a set S in a topological vector space is called totally bounded if for each neighborhood U of zero there is a finite set F with $S \subset F + U$.

Theorem 7. Let $f \in \mathcal{L}_{\gamma, U_0}(X, Y)$. If f is continuous at zero, then f preserves totally bounded sets.

Proof. Suppose that B is a totally bounded subset of X . Since Y is a topological vector space, for every balanced closed neighborhood $V \in \mathcal{N}(Y)$, there exists a balanced closed neighborhood $V_1 \in \mathcal{N}(Y)$ such that $V_1 + V_1 \subset V$. For V_1 , there is a $U \in \mathcal{N}(X)$ with $U \subset U_0$ such that

$$f(U) \subset \frac{1}{1 + |\gamma(1)|} V_1 \quad (3)$$

by the continuity of f at zero. For U , there exists a finite subset $\{x_1, x_2, \dots, x_m\}$ of B such that $B \subset \bigcup_{i=1}^m (x_i + U)$; that is, for every $x \in B$ there exists an i_0 with $1 \leq i_0 \leq m$ and $u \in U$ such that $x = x_{i_0} + u$. Then,

$$f(x) = f(x_{i_0} + u) = r_x f(x_{i_0}) + s_x f(u), \quad (4)$$

where $|r_x - 1| \leq |\gamma(1)|$ and $|s_x| \leq |\gamma(1)|$. Let $A = \{f(x_i) : i = 1, \dots, m\}$ and $\tilde{A}$ be the balanced hull of A . Obviously, the finite set A is totally bounded and hence $\tilde{A}$ is totally bounded. From

$$\begin{aligned} M &= \{r_x f(x_i) : i = 1, \dots, m, x \in B\} \\ &= (1 + |\gamma(1)|) \left\{ \frac{r_x}{1 + |\gamma(1)|} f(x_i) : i = 1, \dots, m, x \in B \right\} \\ &\subset (1 + |\gamma(1)|) \tilde{A}, \end{aligned} \quad (5)$$

it follows that M is totally bounded and hence there exists a finite subset $\{y_1, y_2, \dots, y_n\}$ of Y such that

$$M \subset \{y_1, y_2, \dots, y_n\} + V_1. \quad (6)$$

Then, from (3), (4), and (6), we have

$$\begin{aligned} f(x) &\in M + s_x f(U) \\ &\subset \{y_1, y_2, \dots, y_n\} + V_1 + \frac{s_x}{1 + |\gamma(1)|} V_1 \\ &\subset \{y_1, y_2, \dots, y_n\} + V_1 + V_1 \\ &\subset \{y_1, y_2, \dots, y_n\} + V \end{aligned} \quad (7)$$

for all $x \in B$. Thus, $f(B)$ is totally bounded. $\square$

Recall that a mapping with uniform continuity is also equicontinuous. The following theorem shows that the converse holds true for mappings in $\mathcal{K}_{\gamma, U_0}(X, Y)$.

Theorem 8. Let $\Gamma \subset \mathcal{L}_{\gamma, U_0}(X, Y)$. Then, Γ is uniformly continuous if and only if Γ is equicontinuous at zero.

Proof. We only need to prove the “if” part. For every $W \in \mathcal{N}(Y)$, there exists a balanced closed $W_1 \in \mathcal{N}(Y)$ with $W_1 \subset W$. For $(1/|\gamma(1)|)W_1 \in \mathcal{N}(Y)$, since Γ is equicontinuous at zero, there is a $V \in \mathcal{N}(X)$ such that

$$f(V) \subset \left(\frac{1}{|\gamma(1)|} \right) W_1 \quad (8)$$

for all $f \in \Gamma$. For arbitrary $x_1, x_2 \in X$ with $x_1 - x_2 \in V$, we have

$$f(x_1) = f(x_2 + x_1 - x_2) = f(x_2) + s_f f(x_1 - x_2), \quad (9)$$

where $|s_f| \leq |\gamma(1)|$ and $f \in \Gamma$. It follows that

$$\begin{aligned} f(x_1) - f(x_2) &= s_f f(x_1 - x_2) \\ &\in s_f f(V) \\ &\subset s_f \left(\frac{1}{|\gamma(1)|} \right) W_1 \\ &\subset |\gamma(1)| \left(\frac{1}{|\gamma(1)|} \right) W_1 \\ &= W_1 \subset W \end{aligned} \quad (10)$$

for all $f \in \Gamma$, which shows that Γ is uniformly continuous. $\square$

Theorem 9 (Ascoli Theorem). Let X, Y be topological vector spaces, Y be Hausdorff, and $\mathcal{E}$ be the family of all compact subsets of X . Let $\tau_{\mathcal{E}}$ be the topology on $\mathcal{L}_{\gamma, U_0}(X, Y)$ of uniform convergence on each $K \in \mathcal{E}$. If $\Gamma \subset \mathcal{L}_{\gamma, U_0}(X, Y)$ is equicontinuous and $\bar{\Gamma}^{\sigma(\mathcal{L}_{\gamma, U_0}(X, Y), X)}$ is compact, then $\bar{\Gamma}^{\tau_{\mathcal{E}}}$ is equicontinuous and compact.

Proof. Since Γ is equicontinuous, for every balanced closed $V \in \mathcal{N}(Y)$, there exists a balanced closed $U \in \mathcal{N}(X)$ such

that $f(U) \subset V$ for all $f \in \Gamma$. For every $f \in \bar{\Gamma}^{\tau_{\mathcal{E}}}$, there is a net $\{f_\alpha\} \subset \Gamma$ such that $f_\alpha \xrightarrow{\tau_{\mathcal{E}}} f$. Since the singleton is compact, that is, $\{x\} \in \mathcal{E}$ for all $x \in X$, we have $\lim_\alpha f_\alpha(x) = f(x)$ for all $x \in X$. Then, by the balanced closed property of V , we have

$$f(x) = \lim_\alpha f_\alpha(x) \in V, \quad \forall x \in U; \quad (11)$$

that is, $f(U) \subset V$. As f runs over the set $\bar{\Gamma}^{\tau_{\mathcal{E}}}$, we have that $\bar{\Gamma}^{\tau_{\mathcal{E}}}$ is equicontinuous at zero and hence $\bar{\Gamma}^{\tau_{\mathcal{E}}}$ is equicontinuous on X by Theorem 3.

Let $\{f_\alpha : D\}$ be a net in $\bar{\Gamma}^{\tau_{\mathcal{E}}}$. Since the topology $\sigma(\mathcal{L}_{Y,U_0}(X,Y),X)$ is weaker than $\tau_{\mathcal{E}}$, we have $\bar{\Gamma}^{\tau_{\mathcal{E}}} \subset \bar{\Gamma}^{\sigma(\mathcal{L}_{Y,U_0}(X,Y),X)}$ and hence $\{f_\alpha : D\}$ is also a net in $\bar{\Gamma}^{\sigma(\mathcal{L}_{Y,U_0}(X,Y),X)}$. Since $\bar{\Gamma}^{\sigma(\mathcal{L}_{Y,U_0}(X,Y),X)}$ is compact, the net $\{f_\alpha : D\}$ has a convergent subnet of $\{f_\alpha : D\}$, which we also denote by $\{f_\alpha : D\}$. Let f_0 be the limit of $\{f_\alpha : D\}$ in $\bar{\Gamma}^{\sigma(\mathcal{L}_{Y,U_0}(X,Y),X)}$; that is,

$$\lim_\alpha f_\alpha(x) = f_0(x), \quad \forall x \in X. \quad (12)$$

Then, we have $f_0 \in \mathcal{L}_{Y,U_0}(X,Y)$ and f_0 is continuous by Theorem 4.

To prove $\bar{\Gamma}^{\tau_{\mathcal{E}}}$ is compact, we only need to prove $f_\alpha \xrightarrow{\tau_{\mathcal{E}}} f_0$. Assume this is not true; then, there exists a $K \in \mathcal{E}$ such that $f_\alpha(x) \rightarrow f_0(x)$ is not uniform for $x \in K$. Then, there are $V \in \mathcal{N}(Y)$, increasing sequence $\{\alpha_k\} \subset D$, and $\{x_k\} \subset K$ such that

$$f_{\alpha_k}(x_k) - f_0(x_k) \notin V. \quad (13)$$

For V , there exists a balanced $V_1 \in \mathcal{N}(Y)$ such that $V_1 + V_1 \subset V$. Since K is compact, the sequence $\{x_n\}$ has a convergent subnet $\{x_{n_\delta}\}$ with $x_{n_\delta} \rightarrow x_0 \in K$. For V_1 , there exists a $U \in \mathcal{N}(X)$ such that

$$f(x_0 + U) \subset f(x_0) + V_1, \quad \forall f \in \bar{\Gamma}^{\tau_{\mathcal{E}}} \quad (14)$$

by the equicontinuity of $\bar{\Gamma}^{\tau_{\mathcal{E}}}$. Then,

$$f_\alpha(x_0 + U) \subset f_\alpha(x_0) + V_1, \quad \forall \alpha \in D. \quad (15)$$

For U , there exists a δ_0 such that $x_{n_\delta} \in x_0 + U$, for all $\delta \geq \delta_0$. Then,

$$f_\alpha(x_{n_\delta}) \in f_\alpha(x_0) + V_1, \quad \forall \alpha \in D, \quad \forall \delta \geq \delta_0. \quad (16)$$

Since f_0 is continuous, there is a $\delta_1 \geq \delta_0$ such that

$$f_0(x_{n_\delta}) \in f_0(x_0) + V_1, \quad \forall \delta \geq \delta_0. \quad (17)$$

Since $f_\alpha(x_0) \rightarrow f_0(x_0)$, there exists an α_0 such that

$$f_\alpha(x_0) \in f_0(x_0) + V_1, \quad \forall \alpha \geq \alpha_0. \quad (18)$$

Then, (16) and (18) imply that

$$\begin{aligned} f_\alpha(x_{n_\delta}) &\in f_\alpha(x_0) + V_1 \subset f_\alpha(x_0) + V_1 + V_1, \\ \forall \delta &\geq \delta_0, \quad \forall \alpha \geq \alpha_0. \end{aligned} \quad (19)$$

It follows from (17) that

$$\begin{aligned} f_\alpha(x_{n_\delta}) - f_0(x_{n_\delta}) &\in f_0(x_0) + V_1 + V_1 - f_0(x_0) - V_1 \\ &= V_1 + V_1 - V_1 \subset V, \quad \forall \delta \geq \delta_1, \quad \forall \alpha \geq \alpha_0. \end{aligned} \quad (20)$$

This contracts (13) and hence $f_\alpha \xrightarrow{\tau_{\mathcal{E}}} f_0$, which yields the desired result. $\square$

In fact, the converse of Theorem 9 is also true. The following theorem gives the Arzela Theorem for demi-linear mappings.

Theorem 10 (Arzela Theorem). *Let X, Y be topological vector spaces, Y be Hausdorff, and $\mathcal{E}$ be the family of all compact subsets of X . Let $\tau_{\mathcal{E}}$ be the topology on $\mathcal{L}_{Y,U_0}(X,Y)$ of uniform convergence on each $K \in \mathcal{E}$. If Γ is a family of continuous mappings in $\mathcal{L}_{Y,U_0}(X,Y)$ and $\bar{\Gamma}^{\tau_{\mathcal{E}}}$ is compact, then*

- (I) Γ is equicontinuous;
- (II) $\bar{\Gamma}^{\sigma(\mathcal{L}_{Y,U_0}(X,Y),X)}$ is compact.

Proof. For conclusion (I), we just need to prove that Γ is equicontinuous at zero. Suppose that Γ is not equicontinuous at zero; that is, there exists a $V \in \mathcal{N}(Y)$ such that for every $U \in \mathcal{N}(X)$ there are $x_U \in U$ and $f_U \in \Gamma$ such that

$$f_U(x_U) \notin V. \quad (21)$$

Since $\mathcal{N}(X)$ is directed by the relation “ $\subset$,” we have that $\{x_U : \mathcal{N}(X)\}$ and $\{f_U : \mathcal{N}(X)\}$ are nets, respectively, in X and $\mathcal{L}_{Y,U_0}(X,Y)$. Since $\{f_U : U \in \mathcal{N}(X)\} \subset \Gamma$ and $\bar{\Gamma}^{\tau_{\mathcal{E}}}$ is compact, the net $\{f_U : \mathcal{N}(X)\}$ has a convergent subnet, which we also denote by $\{f_U : \mathcal{N}(X)\}$. Let f_0 be the limit of $\{f_U : \mathcal{N}(X)\}$ in $\bar{\Gamma}^{\tau_{\mathcal{E}}}$; that is, for each $K \in \mathcal{E}$ we have $f_U(x) \rightarrow f_0(x)$ uniformly for $x \in K$. Obviously,

$$\lim_U x_U = 0 \quad (22)$$

and hence $\{x_U : U \in \mathcal{N}(X)\}$ is compact; that is, $\{x_U : U \in \mathcal{N}(X)\} \in \mathcal{E}$. Therefore, we have $f_U(x) \rightarrow f_0(x)$ uniformly for $\{x_U : U \in \mathcal{N}(X)\}$. For V , there exists a $V_1 \in \mathcal{N}(Y)$ such that

$$V_1 + V_1 \subset V. \quad (23)$$

Then for V_1 , there exists a $U_1 \in \mathcal{N}(X)$ such that

$$f_U(x) - f_0(x) \in V_1, \quad \forall U \subset U_1, \quad \forall x \in \{x_V : V \in \mathcal{N}(X)\}. \quad (24)$$

It follows that

$$f_U(x_U) - f_0(x_U) \in V_1, \quad \forall U \subset U_1. \quad (25)$$

By (22) and the continuity of f_0 , for V_1 , there is a $U_2 \in \mathcal{N}(X)$ such that

$$f_0(x_U) \in V_1, \quad \forall U \subset U_2. \quad (26)$$

From (23), (25), and (26), we have

$$f_U(x_U) = (f_U(x_U) - f_0(x_U)) + f_0(x_U) \in V_1 + V_1 \subset V \quad (27)$$

which is in contradiction with (21) and hence the desired result holds.

For conclusion (II), we will prove that $\bar{\Gamma}^{\sigma(\mathcal{L}_{Y,U_0}(X,Y),X)} = \bar{\Gamma}^{\tau_g}$. And we just need to show that $\bar{\Gamma}^{\sigma(\mathcal{L}_{Y,U_0}(X,Y),X)} \subset \bar{\Gamma}^{\tau_g}$. For each $f \in \bar{\Gamma}^{\sigma(\mathcal{L}_{Y,U_0}(X,Y),X)}$, there exists a net $\{f_\alpha\}$ in Γ such that $\lim_\alpha f_\alpha(x) = f(x)$, for all $x \in X$. We only need to prove that $f_\alpha \xrightarrow{\tau_g} f$. If this is not true, then there is a $K \in \mathcal{G}$ such that $f_\alpha(x) \rightarrow f(x)$ is not uniform for $x \in K$. Then, there are $V \in \mathcal{N}(Y)$, increasing sequence $\{\alpha_k\}$, and $\{x_k\} \subset K$ such that

$$f_{\alpha_k}(x_k) - f(x_k) \notin V. \quad (28)$$

Since K is compact, there exist an $x_0 \in K$ and a convergent subnet $\{x_{k_\delta}\}$ of $\{x_k\}$ such that $\lim_\delta x_{k_\delta} = x_0$. By the equicontinuity of Γ , the continuity of f , and (28), we have

$$f_{k_\delta}(x_0) - f(x_0) \notin V. \quad (29)$$

However,

$$\lim_\delta f_{k_\delta}(x_0) = \lim_\alpha f_\alpha(x_0) = f(x_0) \quad (30)$$

which leads a contradiction to (29) and hence the desired result holds. $\square$

Theorem 11. *Let X, Y be topological vector spaces and X be of second category. If Γ is a family of continuous mappings in $\mathcal{L}_{Y,U_0}(X, Y)$ and for every $x \in X$, the set $\{f(x) : f \in \Gamma\}$ is totally bounded, then for every totally bounded subset B of X , we have that $\{f(x) : x \in B, f \in \Gamma\}$ is totally bounded.*

Proof. Since totally bounded set is bounded, it follows that Γ is equicontinuous on X by the equicontinuity theorem [1, Theorem 3.1]. For every balanced closed $V \in \mathcal{N}(Y)$, there exists a balanced closed $V_1 \in \mathcal{N}(Y)$ such that $V_1 + V_1 \subset V$. For V_1 , there is a $U \in \mathcal{N}(X)$ with $U \subset U_0$ such that

$$f(U) \subset \frac{1}{1 + |\gamma(1)|} V_1, \quad \forall f \in \Gamma. \quad (31)$$

For U , there exist $x_1, x_2, \dots, x_n \in B$ such that $B \subset \bigcup_{i=1}^n (x_i + U)$; that is, for every $x \in B$, there exists an i_0 with $1 \leq i_0 \leq n$ and $u \in U$ such that $x = x_{i_0} + u$. Then,

$$f(x) = f(x_{i_0} + u) = r_f f(x_{i_0}) + s_f f(u), \quad (32)$$

where $|r_f - 1| \leq |\gamma(1)|$, $|s_f| \leq |\gamma(1)|$, and $f \in \Gamma$. Let $A = \{f(x_i) : f \in \Gamma, i = 1, 2, \dots, n\}$. Since for every $x \in X$, the set $\{f(x) : f \in \Gamma\}$ is totally bounded, we see that A is totally bounded. Let $\tilde{A}$ be the balanced hull of A ; that is, $\tilde{A} = \bigcup_{|t| \leq 1} tA$. Then, $\tilde{A}$ is totally bounded. It follows from

$$\begin{aligned} \{r_f A : f \in \Gamma\} &= (1 + |\gamma(1)|) \left\{ \frac{r_f}{1 + |\gamma(1)|} A : f \in \Gamma \right\} \\ &\subset (1 + |\gamma(1)|) \tilde{A} \end{aligned} \quad (33)$$

that $\{r_f A : f \in \Gamma\}$ is totally bounded. Hence, there exist $y_1, y_2, \dots, y_m \in Y$ such that

$$\{r_f A : f \in \Gamma\} \subset \{y_1, y_2, \dots, y_m\} + V_1, \quad \forall f \in \Gamma. \quad (34)$$

Then, (31), (32), and (34) imply that

$$\begin{aligned} f(x) &= r_f f(x_{i_0}) + s_f f(u) \\ &\in r_f A + \frac{s_f}{1 + |\gamma(1)|} V_1 \\ &\subset \{y_1, y_2, \dots, y_m\} + V_1 + V_1 \\ &\subset \{y_1, y_2, \dots, y_m\} + V \end{aligned} \quad (35)$$

for all $x \in B$ and $f \in \Gamma$. Thus, $\{f(x) : x \in B, f \in \Gamma\}$ is totally bounded. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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