Research Article

On the Compactness of the Commutator of the Parabolic Marcinkiewicz Integral with Variable Kernel

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The authors prove that the commutator of the parabolic Marcinkiewicz integral g_Ω with variable kernel is a compact operator on $L^p(\mathbb{R}^n)$ ($1 < p < \infty$) if and only if $b \in \text{CMO}(\mathbb{R}^n, \rho)$. The result is substantial improvement and extension of some known results.

1. Introduction and Main Results

Let $\mathbb{R}^n$ be the n -dimension Euclidean space. Let $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ be the unit sphere on $\mathbb{R}^n$ equipped with the Lebesgue measure $d\sigma(x')$. For $b \in L_{\text{loc}}(\mathbb{R}^n)$, denote by m_b the multiplication operator defined by $m_b f = bf$ for the measurable function f . Suppose that A is a linear operator on some measurable function space; then the commutator formed by A and m_b is defined by

$$[b, A] f(x) := m_b(Af)(x) - A(m_b f)(x) \\ = b(x) Af(x) - A(bf)(x). \quad (1)$$

A famous theorem of Coifman et al. [1] characterized the L^p -boundedness of $[b, R_j]$, where R_j ($j = 1, \dots, n$) are the Riesz transforms and $b \in \text{BMO}(\mathbb{R}^n)$. Using this characterization, the authors of [1] got a decomposition theorem of the real Hardy space $H^1(\mathbb{R}^n)$. Uchiyama [2] and Janson [3] showed that the Riesz transform R_j may be replaced by the Calderón-Zygmund singular integral operator T_Ω . Moreover, in 1978, Uchiyama [2] gave a characterization of the compactness for the commutator $[b, T_\Omega]$ defined by

$$[b, T_\Omega] f(x) = \text{p.v.} \int_{\mathbb{R}^n} [b(x) - b(y)] \frac{\Omega(x-y)}{|x-y|^n} f(y) dy, \quad (2)$$

where Ω is homogeneous function of degree zero on $\mathbb{R}^n \setminus \{0\}$, that is,

$$\Omega(tx) = \Omega(x) \quad \text{for any } t > 0, x \in \mathbb{R}^n \setminus \{0\}, \quad (3)$$

and Ω has mean zero on S^{n-1} , the unit sphere in $\mathbb{R}^n$, that is,

$$\int_{S^{n-1}} \Omega(x') d\sigma(x') = 0. \quad (4)$$

Theorem A (see [2]). Suppose that $\Omega \in \text{Lip}(S^{n-1})$ satisfies (3) and (4). Let $b \in L_{\text{loc}}(\mathbb{R}^n)$.

- (i) If $[b, T_\Omega]$ is a compact operator on $L^p(\mathbb{R}^n)$ for some p , $1 < p < \infty$, then $b \in \text{CMO}(\mathbb{R}^n)$.
- (ii) If $b \in \text{CMO}(\mathbb{R}^n)$, then $[b, T_\Omega]$ is a compact operator on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$, where $\text{CMO}(\mathbb{R}^n)$ is the BMO-closure of $C_c^1(\mathbb{R}^n)$ and $C_c^1(\mathbb{R}^n)$ denotes the set of C^1 -functions with compact support in $\mathbb{R}^n$.

The interest in the compactness of $[b, T_\Omega]$ in complex analysis is from the connection between the commutators and the Hankel-type operators. In fact, Beatrous and Li [4] proved the boundedness and compactness of $[b, T_\Omega]$ on L^p over some spaces of homogeneous type. Krantz and Li (see [5]) have applied the characterization of L^p -compactness of commutator to give a compactness of Hankel operators on

holomorphic Hardy spaces $H^2(D)$, where D is a bounded, strictly pseudoconvex domain in $\mathbb{C}^n$.

In this paper, we will study the compactness of the commutator of the parabolic Marcinkiewicz integral with variable kernel. Let us recall the definition of parabolic metric. Let $\alpha_1, \dots, \alpha_n$ be fixed real numbers, $\alpha_i \geq 1$. For fixed $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, the function $F(x, \rho) = \sum_{i=1}^n (x_i^2 / \rho^{2\alpha_i})$ is a decreasing function in ρ ($\rho > 0$). We denote by $\rho(x)$ the unique solution of the equation $F(x, \rho) = 1$. In 1966, Fabes and Riviere [6] proved that $\rho(x)$ is a metric on $\mathbb{R}^n$. For $\mu > 0$ and $x \in \mathbb{R}^n$, define the dilation on $\mathbb{R}^n$ by

$$A_\mu : (x_1, x_2, \dots, x_n) \longrightarrow (\mu^{\alpha_1} x_1, \mu^{\alpha_2} x_2, \dots, \mu^{\alpha_n} x_n). \quad (5)$$

Then it is immediate to see that $\rho(A_\mu x) = \mu \rho(x)$ and $\rho(tx) \leq \rho(x)$ when $t \leq 1$. One has the polar decomposition $x = A_\rho x'$ with $x' = (x_1 / \rho(x)^{\alpha_1}, \dots, x_n / \rho(x)^{\alpha_n}) \in S^{n-1}$, $\rho = \rho(x)$ and $dx = \rho^{\alpha-1} J(x') d\rho d\sigma(x')$, where $\alpha = \sum_{i=1}^n \alpha_i$, $J(x')$ is a C^∞ function on S^{n-1} and bounded below uniformly by 1.

In 1974, Madych [7] gave the L^p ($1 < p < \infty$) boundedness of the parabolic Marcinkiewicz integral g_ψ with respect to the transform A_t , where

$$g_\psi(f)(x) = \left(\int_0^\infty |\psi_t * f(x)|^2 \frac{dt}{t} \right)^{1/2} \quad (6)$$

with $\psi \in S(\mathbb{R}^n)$ satisfying $\int_{\mathbb{R}^n} \psi(x) dx = 0$, and $\psi_t(x) = t^{-\alpha} \psi(A_{t^{-1}} x)$ for $t > 0$. Inspired by the works in [6, 7], recently, Xue et al. [8] improved the above result. More precisely, the authors in [8] proved that the parabolic Marcinkiewicz integral is still bounded on L^p if replacing $\psi(x)$ by a kernel function $\phi(x) = \Omega(x) \rho(x)^{-\alpha+1} \chi_{\{\rho(x) \leq 1\}}(x)$ with $\Omega \in L^q(S^{n-1})$ ($q > 1$) satisfying the following conditions:

$$\begin{aligned} \Omega(A_\mu z) &= \Omega(z), \quad \text{for any } z \in \mathbb{R}^n \setminus \{0\}, \quad \mu > 0, \\ \int_{S^{n-1}} \Omega(z') J(z') d\sigma(z') &= 0. \end{aligned} \quad (7)$$

Afterwards, many authors studied the properties of the parabolic Marcinkiewicz integral and its commutators which is defined by

$$g_{\psi;b}(f)(x) = \left(\int_0^\infty |[b, \psi_t] f(x)|^2 \frac{dt}{t} \right)^{1/2} \quad (8)$$

(see, e.g., [9–12]).

Notice that if $\alpha_1 = \dots = \alpha_n = 1$, then $\rho(x) = |x|$ and $(\mathbb{R}^n, \rho) = (\mathbb{R}^n, |\cdot|)$. In this case, g_ψ and $g_{\psi;b}$ are just the classical Marcinkiewicz integral and its commutators which play a very important role in the singular integral operators theory, function spaces theory, and PDE (see, e.g., [13–23]).

Let us turn to the definitions of the parabolic Marcinkiewicz integral with variable kernel and its commutator. Suppose that $\Omega(x, z)$ satisfies the following conditions:

$$\begin{aligned} \Omega(x, A_\mu z) &= \Omega(x, z), \\ \text{for any } x \in \mathbb{R}^n, \quad z \in \mathbb{R}^n \setminus \{0\}, \quad \mu > 0, \end{aligned} \quad (9)$$

$$\int_{S^{n-1}} \Omega(x, z') J(z') d\sigma(z') = 0, \quad \text{for any } x \in \mathbb{R}^n. \quad (10)$$

Then the parabolic Marcinkiewicz integral with variable kernel is defined by

$$g_\Omega(f)(x) = \left\{ \int_0^\infty \left| \int_{\rho(x-y) \leq t} \frac{\Omega(x, x-y)}{\rho(x-y)^{\alpha-1}} f(y) dy \right|^2 \frac{dt}{t^3} \right\}^{1/2}. \quad (11)$$

Let us now give the definition of parabolic BMO space. Denote by $\mathcal{E}(x, r)$ the ellipsoid center at x and radius r . More precisely, $\mathcal{E}(x, r) = \{y : \rho(x-y) < r\}$. For $j > 0$, we denote $j\mathcal{E}(x, r)$ to be the j -times extension of $\mathcal{E}(x, r)$ with the same center. That is, $j\mathcal{E}(x, r) = \{y : \rho(x-y) < jr\}$. Moreover, $|\mathcal{E}(x, r)|$ stands the Lebesgue measure of $\mathcal{E}(x, r)$, which is comparable to r^α . $\mathcal{E}^c$ denotes the complement of $\mathcal{E}$.

For a measurable and locally integrable function $b : \mathbb{R}^n \rightarrow \mathbb{R}$ set

$$\eta_b(\theta) = \sup_{r \leq \theta} \frac{1}{|\mathcal{E}(r)|} \int_{\mathcal{E}(r)} |b(y) - b_{\mathcal{E}(r)}| dy \quad \text{for } \theta > 0, \quad (12)$$

where $\mathcal{E}(r)$ is any ellipsoid in $\mathbb{R}^n$ of radius r and $b_{\mathcal{E}(r)} = |\mathcal{E}(r)|^{-1} \int_{\mathcal{E}(r)} b(y) dy$ is the average of b on $\mathcal{E}(r)$. We say $b \in \text{BMO}(\mathbb{R}^n, \rho)$ if $\|b\|_{*,\rho} := \sup_{\theta} \eta_b(\theta) < \infty$ (see [24]).

Let $b \in L_{\text{loc}}(\mathbb{R}^n)$; the commutator of the parabolic Marcinkiewicz integral with variable kernel is defined by

$$\begin{aligned} g_{\Omega;b}f(x) &= \left\{ \int_0^\infty \left| \int_{\rho(x-y) \leq t} \frac{\Omega(x, x-y)}{\rho(x-y)^{\alpha-1}} (b(x) - b(y)) f(y) dy \right|^2 \frac{dt}{t^3} \right\}^{1/2}. \end{aligned} \quad (13)$$

Notice that if $\alpha_1 = \dots = \alpha_n = 1$, then $\rho(x) = |x|$; g_Ω and $g_{\Omega;b}$ are the classical Marcinkiewicz integral with variable kernel and the classical commutator of Marcinkiewicz integral with variable kernel which was studied by many authors (see, e.g., [25–28]).

In 2009, Chen and Ding [29] studied the parabolic Marcinkiewicz integral with variable kernel and its commutator on the Lebesgue space $L^p(\mathbb{R}^n)$, which is stated as follows.

Theorem B. *Let $1 < p < \infty$. Suppose that $\Omega(x, z)$ satisfies (9), (10) and for every multi-index β ,*

$$\sup_{\substack{x \in \mathbb{R}^n \\ \xi' \in S^{n-1}}} \left| D_{\xi'}^\beta \Omega(x, \xi') \right| \leq C(\beta). \quad (14)$$

Let $b \in \text{BMO}(\mathbb{R}^n, \rho)$. Then there exists a constant $C > 0$ such that, for any $f \in L^p(\mathbb{R}^n)$,

$$\|g_{\Omega;b}f\|_p \leq C \|b\|_{*,\rho} \|f\|_p. \quad (15)$$

In this paper, we will give a characterization of the commutator of the parabolic Marcinkiewicz integral with variable kernel which is compact operator on $L^p(\mathbb{R}^n)$ ($1 < p < \infty$). First, we will prove that the commutator of the parabolic Marcinkiewicz integral with variable kernel is bounded on $L^p(\mathbb{R}^n)$ ($1 < p < \infty$). To show our result, we give the following definition.

Definition 1 (see [30]). Let X and Y be Banach spaces and let U be a subset of X . Then $T : U \mapsto Y$ is compact if T is continuous and maps bounded subsets of U into strongly precompact subsets of Y .

Our first result can be stated as follows.

Theorem 2. Let $\Omega(x, z)$ satisfy (9), (10), (14) and for any multi-index β

$$\sup_{\xi' \in S^{n-1}} \left| D_{\xi}^{\beta} \Omega(x, \xi') - D_{\xi}^{\beta} \Omega(y, \xi') \right| \leq C \rho(x - y). \quad (16)$$

If $b \in \text{CMO}(\mathbb{R}^n, \rho)$, where $\text{CMO}(\mathbb{R}^n, \rho)$ is the $\text{BMO}(\mathbb{R}^n, \rho)$ -closure of $C_c^1(\mathbb{R}^n)$ and $C_c^1(\mathbb{R}^n)$ denotes the set of C^1 -functions with compact support in $\mathbb{R}^n$, then $g_{\Omega, b}$ is a compact operator on $L^p(\mathbb{R}^n)$ ($1 < p < \infty$).

On the other hand, we will give the converse part of the compactness of the commutator of the Marcinkiewicz integral with variable kernel.

Theorem 3. Suppose that Ω satisfies (9) and (10),

$$\sup_{z \in \mathbb{R}^n} \left| \Omega(z, x') - \Omega(z, y') \right| \leq \rho(x' - y'), \quad (17)$$

for $x', y' \in S^{n-1}$,

and there exists $\Theta \subset S^{n-1}$, $0 < \delta < 1$ such that, for any $x \in \mathbb{R}^n$,

$$\sigma(\Theta) = \sigma \left(\left\{ y' \in S^{n-1} : \inf_{x \in \mathbb{R}^n} \Omega(x, y') \geq 2\delta \right\} \right) > 0, \quad (18)$$

where σ is the measure on S^{n-1} which is induced from the Lebesgue measure on $\mathbb{R}^n$. If $[b, g_{\Omega}]$ is a bounded operator on $L^p(\mathbb{R}^n)$ for some $1 < p < \infty$, then $b \in \text{BMO}(\mathbb{R}^n, \rho)$.

Theorem 4. Let $\Omega(x, z)$ satisfy the same conditions as in Theorem 2. If $g_{\Omega, b}$ is a compact operator on $L^p(\mathbb{R}^n)$ for some p ($1 < p < \infty$), then $b \in \text{CMO}(\mathbb{R}^n, \rho)$.

Remark 5. Our results are still new even for the case $\rho(x) = |x|$.

This paper is organized as follows. First, in Section 2, we give some important notations and lemmas, which will be used in the proofs of the main results. In Section 3, we prove Theorem 2. The proof of Theorem 3 is given in Section 4. In Section 5, we will give the proof of Theorem 4.

The letter C in the paper denotes positive constants independent of essential variables. For $p \geq 1$, p' denotes the dual exponent of p , that is, $p' = p/(p-1)$.

2. Notations and Lemmas

For a given measurable function $f \in L_{\text{loc}}(\mathbb{R}^n)$, the Hardy-Littlewood maximal operator $\mathcal{M}$ and the sharp maximal operator $\mathcal{M}^{\#}$ related to ellipsoids are defined by

$$\begin{aligned} \mathcal{M}f(x) &= \sup_{\mathcal{E} \ni x} \frac{1}{|\mathcal{E}|} \int_{\mathcal{E}} |f(y)| dy, \\ \mathcal{M}^{\#}f(x) &= \sup_{\mathcal{E} \ni x} \frac{1}{|\mathcal{E}|} \int_{\mathcal{E}} |f(y) - f_{\mathcal{E}}| dy, \end{aligned} \quad (19)$$

respectively. Define also the operator $\mathcal{M}_q f(x) := (\mathcal{M}(|f|^q)(x))^{1/q}$ for $1 \leq q < \infty$.

Denote by H_m the space of all n -dimensional spherical harmonic of degree m with its dimension $D_m = \dim H_m$. By [24, 31], we have

$$D_m \leq C(n) m^{n-2}, \quad m \geq 1. \quad (20)$$

Furthermore, let $\{Y_{m,s}(x')\}_{s=1}^{D_m}$ be an orthonormal base of H_m . Then $\{Y_{m,s}(x')\}_{s=1}^{D_m}$ ($m = 0, 1, \dots$) is a complete orthonormal system in $L^2(S^{n-1})$ and for any multi-index β (see [24, 31]),

$$\sup_{x' \in S^{n-1}} \left| D_{x'}^{\beta} Y_{m,s}(x') \right| \leq C(n) m^{|\beta| + (n-2)/2}, \quad m = 1, 2, \dots \quad (21)$$

If, for instance, $\phi \in C^{\infty}(S^{n-1})$, then $\sum_m \sum_s a_{m,s} Y_{m,s}(x')$ is the Fourier series expansion of $\phi(x')$ with respect to $\{Y_{m,s}(x')\}_{m,s}$, where

$$a_{m,s} = \int_{S^{n-1}} \phi(y') Y_{m,s}(y') d\sigma(y'), \quad (22)$$

and (see [24, 31])

$$\begin{aligned} a_{m,s} &= (-1)^l m^{-l} (m+n-2)^{-l} \\ &\times \int_{S^{n-1}} L^l(\phi) Y_{m,s}(x') d\sigma(x'), \quad m \geq 1, \end{aligned} \quad (23)$$

where $L(\phi) = |x|^2 \Delta \phi(x)$, $L^l(\phi) = L(L^{l-1}(\phi))$ for any integer l . In particular, the expansion of ϕ into spherical harmonics converges uniformly to ϕ . The next results will be employed in the forthcoming considerations.

Lemma 6 (see [29]). Suppose that $\mathcal{E} = \mathcal{E}(x_0, r)$. Let $\gamma > 0$. Then

$$\begin{aligned} &\left| \frac{Y_{m,s}((x-y)')}{J((x-y)') \rho(x-y)^{\gamma}} - \frac{Y_{m,s}((x_0-y)')}{J((x_0-y)') \rho(x_0-y)^{\gamma}} \right| \\ &\leq C m^{n/2} \frac{\rho(x-x_0)}{\rho(y-x_0)^{\gamma+1}}, \end{aligned} \quad (24)$$

for each $x \in \mathcal{E}$ and $y \in (2\mathcal{E})^c$, where $x' = (x_1/\rho(x)^{\alpha_1}, \dots, x_n/\rho(x)^{\alpha_n})$.

Lemma 7 (Frechet-Kolmogorov, see [32, 33]). *A subset G of $L^p(\mathbb{R}^n)$, $1 < p < \infty$, is strongly precompact if and only if it satisfies the conditions*

$$\sup_{f \in G} \|f\|_p < \infty; \quad (25)$$

$$\lim_{\rho(y) \rightarrow 0} \|f(\cdot + y) - f(\cdot)\|_p = 0 \quad \text{uniformly in } f \in G; \quad (26)$$

$$\lim_{\beta \rightarrow \infty} \|f\chi_{E_\beta}\|_p = 0 \quad \text{uniformly in } f \in G, \quad (27)$$

$$\text{where } E_\beta = \{x \in \mathbb{R}^n : \rho(x) > \beta\}.$$

Lemma 8 (see [29]). *Suppose that $\mathcal{E} = \mathcal{E}(x_0, r)$. Let $\gamma > 0$. Then*

$$\left| \frac{1}{\rho(x-y)^\gamma} - \frac{1}{\rho(x_0-y)^\gamma} \right| \leq 2(2^\gamma - 1) \frac{\rho(x-x_0)}{\rho(y-x_0)^{\gamma+1}}, \quad (28)$$

for each $x \in \mathcal{E}$ and $y \in (2\mathcal{E})^c$. (Here and below, for $a > 0$ and $\mathcal{E}(x_0, r)$, $a\mathcal{E}$ denotes the a -times extension of $\mathcal{E}$ with the same center, that is, $a\mathcal{E} = \{y : \rho(y-x_0) < ar\}$.)

Lemma 9 (see [32]). *If $\rho(x) \geq 4\rho(y)$, then*

$$\rho((x-y)' - x') \leq 3 \frac{\rho(y)}{\rho(x)}, \quad (29)$$

where $x' = (x_1/\rho(x)^{\alpha_1}, \dots, x_n/\rho(x)^{\alpha_n}) \in S^{n-1}$.

Lemma 10 (see [2, 32]). *If $b \in BMO(\mathbb{R}^n, \rho)$, $C_5 > C_4 > 2$, $\mathcal{E}$ is an ellipsoid centered at x_0 and radius r , then there exist positive constants C_6, C_7 , and C_8 (depending on C_4, C_5 , and b), such that*

$$\begin{aligned} & \left| \{C_4 r < \rho(x-x_0) < C_5 r : |b(x) - b_{\mathcal{E}}| > \nu + C_6\} \right| \\ & \leq C_7 |\mathcal{E}| e^{-C_8 \nu}, \quad (0 < \nu < \infty). \end{aligned} \quad (30)$$

Lemma 11 (see [34]). *Suppose that $f(x)$ is a measurable function, $\lambda_f(w) = |\{x \in \mathbb{R}^n : |f(x)| > w > 0\}|$, and E is a measurable set. Define $f^*(t) = \inf\{w : \lambda_f(w) \leq t\}$ for $t > 0$; then*

$$\int_E |f(x)|^p dx \leq \int_0^{|E|} |f^*(t)|^p dt, \quad 1 \leq p < \infty. \quad (31)$$

Lemma 12 (see [2, 32]). *Let $b \in BMO(\mathbb{R}^n, \rho)$. Then $b \in CMO(\mathbb{R}^n, \rho)$ if and only if b satisfies the following three conditions:*

- (i) $\lim_{a \rightarrow 0} \sup_{|\mathcal{E}|=a} M(b, \mathcal{E}) = 0$,
- (ii) $\lim_{a \rightarrow \infty} \sup_{|\mathcal{E}|=a} M(b, \mathcal{E}) = 0$,
- (iii) $\lim_{x \rightarrow \infty} M(b, \mathcal{E}+x) = 0$, for each $\mathcal{E}$, where $M(b, \mathcal{E}) = (1/|\mathcal{E}|) \int_{\mathcal{E}} |b(y) - b_{\mathcal{E}}| dy$.

It is easy to get the following lemma.

Lemma 13. *If Ω satisfies conditions (9) and (17), then for $\beta > 0$,*

$$\left| \frac{\Omega(z, x-y)}{\rho(x-y)^\beta} - \frac{\Omega(z, x)}{\rho(x)^\beta} \right| \leq C \frac{\rho(y)}{\rho(x)^{\beta+1}} \quad (32)$$

for $\rho(x) \geq 4\rho(y)$.

3. Proof of Theorem 2

Suppose that $b \in CMO(\mathbb{R}^n, \rho)$. Then, by Theorem A, we can get

$$\|g_{\Omega; b} f\|_p \leq C \|b\|_{*, \rho} \|f\|_p, \quad \text{for any } f \in L^p(\mathbb{R}^n). \quad (33)$$

Then we can easily get $g_{\Omega; b}$ is continuous on $L^p(\mathbb{R}^n)$, so by Definition 1, it suffices to prove that, for any bounded set F in $L^p(\mathbb{R}^n)$, $G = \{g_{\Omega; b} f : f \in F\}$ is strongly precompact in $L^p(\mathbb{R}^n)$. Since for any $\varepsilon > 0$ there exists $b^\varepsilon \in C_0^\infty$ such that $\|b - b^\varepsilon\|_{*, \rho} < \varepsilon$, then by (33),

$$\|g_{\Omega; b} - g_{\Omega; b^\varepsilon}\|_{L^p \rightarrow L^p} \leq \|g_{\Omega; b-b^\varepsilon}\|_{L^p \rightarrow L^p} < C\varepsilon. \quad (34)$$

Thus, it suffices to prove that G is strongly precompact in $L^p(\mathbb{R}^n)$ for $b \in C_c^\infty(\mathbb{R}^n)$. By Lemma 7, we need only to verify that (25)–(27) hold uniformly in G .

Suppose that $\|f\|_p \leq D$ for every $f \in F$. Notice that $b \in C_c^\infty(\mathbb{R}^n)$ and applying (33), we have

$$\sup_{f \in F} \|g_{\Omega; b} f\|_p \leq C \|b\|_{*, \rho} \sup_{f \in F} \|f\|_p \leq C' D < \infty. \quad (35)$$

On the other hand, for any $\varepsilon > 0$, we may take $B > 0$ large enough such that

$$\left(\int_{\rho(x) > B} \rho(x)^{-\alpha p} dx \right)^{1/p} < \varepsilon. \quad (36)$$

Assume $\text{supp}(b) \subset \{y : \rho(y) < r\}$ for some $r > 0$. Thus, for any x satisfying $\rho(x) > \max\{B, 4r\}$ and every $f \in F$, then by $\Omega \in L^\infty(\mathbb{R}^n) \times L^\infty(S^{n-1})$ and $\rho(x-y) \sim \rho(x)$, we have

$$\begin{aligned} & |g_{\Omega; b} f(x)| \\ &= \left\{ \int_0^\infty \left| \int_{\rho(x-y) \leq t} \frac{\Omega(x, x-y)}{\rho(x-y)^{\alpha-1}} (b(x) - b(y)) f(y) dy \right|^2 \frac{dt}{t^3} \right\}^{1/2} \\ &\leq C \int_{\text{supp } b} \frac{|b(y)|}{\rho(x-y)^{\alpha-1}} |f(y)| \left\{ \int_{\rho(x-y) \leq t} \frac{dt}{t^3} \right\}^{1/2} dy \\ &\leq C \rho(x)^{-\alpha} \|f\|_p. \end{aligned} \quad (37)$$

Applying (36), we have

$$\left(\int_{\rho(x) > B} |g_{\Omega; b} f(x)|^p dx \right)^{1/p} \leq C\varepsilon. \quad (38)$$

Equation (38) shows that (27) holds uniformly in G . Finally, to finish the proof of Theorem 2, it remains to show (26) holds uniformly in G . We need to prove that, for any $\varepsilon > 0$, if $\rho(z)$ is sufficiently small, then for every $f \in F$,

$$\|g_{\Omega,b}f(\cdot) - g_{\Omega,b}f(\cdot + z)\|_p \leq C\varepsilon. \quad (39)$$

To do this, for $x, y \in \mathbb{R}^n$, by (10), we have

$$J(y')\Omega(x, y') = \sum_{m=1}^{\infty} \sum_{s=1}^{D_m} a_{m,s}(x) Y_{m,s}(y'), \quad (40)$$

where

$$a_{m,s}(x) = \int_{S^{n-1}} \Omega(x, y') Y_{m,s}(y') d\sigma(y'). \quad (41)$$

So we get

$$\Omega(x, y') = \sum_{m=1}^{\infty} \sum_{s=1}^{D_m} a_{m,s}(x) \frac{Y_{m,s}(y')}{J(y')}. \quad (42)$$

In this way, (14) and (23) imply for any $x \in \mathbb{R}^n$

$$|a_{m,s}(x)| \leq C(n, l) m^{-2l}, \quad (43)$$

for any integer $l > 1$. We fix hereafter $l = n$. Let

$$\begin{aligned} g_{m,s}(f)(x) &= \left\{ \int_0^\infty \left| \int_{\rho(x-y) \leq t} \frac{Y_{m,s}((x-y)')}{J((x-y)') \rho(x-y)^{\alpha-1}} f(y) dy \right|^2 \frac{dt}{t^3} \right\}^{1/2}, \\ g_{m,s,b}f(x) &= \left\{ \int_0^\infty \left| \int_{\rho(x-y) \leq t} \frac{Y_{m,s}((x-y)')}{J((x-y)') \rho(x-y)^{\alpha-1}} \right. \right. \\ &\quad \left. \left. \times (b(x) - b(y)) f(y) dy \right|^2 \frac{dt}{t^3} \right\}^{1/2}. \end{aligned} \quad (44)$$

It is easy to see that $\Omega_{m,s}(x) = (Y_{m,s}/J)(x')$ has the following properties:

$$\begin{aligned} \int_{S^{n-1}} \Omega_{m,s}(x') J(x') d\sigma(x') &= 0, \\ \Omega_{m,s}(A_\lambda x) &= \Omega_{m,s}(x), \quad \forall \lambda > 0, x \in \mathbb{R}^n \setminus \{0\}. \end{aligned} \quad (45)$$

Moreover, $\Omega_{m,s} \in L^2(S^{n-1}) \subset L(\log^+ L)^2(S^{n-1}) \subset L \log^+ L(S^{n-1})$. Then by the result in [11, 12], we know for $1 < p < \infty$,

$$\|g_{m,s}f\|_p \leq C \|f\|_p, \quad (46)$$

$$\|g_{m,s,b}f\|_p \leq C \|b\|_{*,p} \|f\|_p, \quad (47)$$

where C is independent of f, m, s .

For any $z \in \mathbb{R}^n$, by the Minkowski inequality, the Hölder inequality, we get

$$\begin{aligned} &|g_{\Omega,b}f(x) - g_{\Omega,b}f(x+z)| \\ &\leq \left\{ \int_0^\infty \left| \sum_{m=1}^{\infty} \sum_{s=1}^{D_m} a_{m,s}(x) \right. \right. \\ &\quad \times \left(\int_{\rho(x-y) \leq t} \frac{Y_{m,s}((x-y)')}{J((x-y)') \rho(x-y)^{\alpha-1}} \right. \\ &\quad \times (b(x) - b(y)) f(y) dy \\ &\quad \left. \left. - \int_{\rho(x+z-y) \leq t} \left(Y_{m,s}((x+z-y)') \right) \right. \right. \\ &\quad \times \left(J((x+z-y)') \right. \\ &\quad \times \rho(x+z-y)^{\alpha-1} \Big)^{-1} \\ &\quad \times (b(x+z) - b(y)) f(y) dy \Big)^2 \frac{dt}{t^3} \Big\}^{1/2} \\ &+ \left\{ \int_0^\infty \left| \sum_{m=1}^{\infty} \sum_{s=1}^{D_m} (a_{m,s}(x) - a_{m,s}(x+z)) \right. \right. \\ &\quad \times \int_{\rho(x+z-y) \leq t} \frac{Y_{m,s}((x+z-y)')}{J((x+z-y)') \rho(x+z-y)^{\alpha-1}} \\ &\quad \times (b(x+z) - b(y)) f(y) dy \Big|^2 \frac{dt}{t^3} \Big\}^{1/2} \\ &= \left\{ \int_0^\infty \left| \sum_{m=1}^{\infty} \sum_{s=1}^{D_m} a_{m,s}(x) I_{m,s}(x, t, z) \right|^2 \frac{dt}{t^3} \right\}^{1/2} \\ &+ \left\{ \int_0^\infty \left| \sum_{m=1}^{\infty} \sum_{s=1}^{D_m} (a_{m,s}(x) - a_{m,s}(x+z)) \right. \right. \\ &\quad \times II_{m,s}(x, t, z) \Big|^2 \frac{dt}{t^3} \Big\}^{1/2} \\ &\leq \left\{ \int_0^\infty \sum_{m=1}^{\infty} m^{-2} \sum_{m=1}^{\infty} m^2 \left| \sum_{s=1}^{D_m} a_{m,s}(x) I_{m,s} \right. \right. \\ &\quad \times (x, t, z) \Big|^2 \frac{dt}{t^3} \Big\}^{1/2} \\ &+ \left\{ \int_0^\infty \sum_{m=1}^{\infty} m^{-2} \sum_{m=1}^{\infty} m^2 \left| \sum_{s=1}^{D_m} (a_{m,s}(x) - a_{m,s}(x+z)) \right. \right. \end{aligned}$$

$$\begin{aligned}
& \times \left| II_{m,s}(x, t, z) \right|^2 \frac{dt}{t^3} \Bigg\}^{1/2} \\
& \leq C \left\{ \int_0^\infty \sum_{m=1}^\infty m^2 \sum_{s=1}^{D_m} |a_{m,s}(x)|^2 \right. \\
& \quad \times \left. \sum_{s=1}^{D_m} |I_{m,s}(x, t, z)|^2 \frac{dt}{t^3} \right\}^{1/2} \\
& + C \left\{ \int_0^\infty \sum_{m=1}^\infty m^2 \sum_{s=1}^{D_m} |a_{m,s}(x) - a_{m,s}(x+z)|^2 \right. \\
& \quad \times \left. \sum_{s=1}^{D_m} |II_{m,s}(x, t, z)|^2 \frac{dt}{t^3} \right\}^{1/2}, \tag{48}
\end{aligned}$$

where

$$\begin{aligned}
& I_{m,s}(x, t, z) \\
& = \left(\int_{\rho(x-y) \leq t} \frac{Y_{m,s}((x-y)')}{J((x-y)') \rho(x-y)^{\alpha-1}} \right. \\
& \quad \times (b(x) - b(y)) f(y) dy \\
& \quad - \int_{\rho(x+z-y) \leq t} \frac{Y_{m,s}((x+z-y)')}{J((x+z-y)') \rho(x+z-y)^{\alpha-1}} \\
& \quad \times (b(x+z) - b(y)) f(y) dy \Bigg), \tag{49}
\end{aligned}$$

$$\begin{aligned}
& II_{m,s}(x, t, z) \\
& = \int_{\rho(x+z-y) \leq t} \frac{Y_{m,s}((x+z-y)')}{J((x+z-y)') \rho(x+z-y)^{\alpha-1}} \\
& \quad \times (b(x+z) - b(y)) f(y) dy.
\end{aligned}$$

Then by (43) and the Minkowski inequality, we get

$$\begin{aligned}
& |g_{\Omega,b}f(x) - g_{\Omega,b}f(x+z)| \\
& \leq C \sum_{m=1}^\infty m^{-3n/2} \sum_{s=1}^{D_m} \left(\int_0^\infty |I_{m,s}(x, t, z)|^2 \frac{dt}{t^3} \right)^{1/2}
\end{aligned}$$

$$\begin{aligned}
& + C \sum_{m=1}^\infty m \left(\sum_{s=1}^{D_m} |a_{m,s}(x) - a_{m,s}(x+z)|^2 \right)^{1/2} \\
& \times \sum_{s=1}^{D_m} \left(\int_0^\infty |II_{m,s}(x, t, z)|^2 \frac{dt}{t^3} \right)^{1/2} \\
& := K_1 + K_2. \tag{50}
\end{aligned}$$

We first give the L^p norm of K_2 . By (16), we have

$$|a_{m,s}(x+z) - a_{m,s}(x)| \leq C m^{-2n} \rho(z). \tag{51}$$

Moreover, by (47), we get

$$\begin{aligned}
& \left\| \left(\int_0^\infty |II_{m,s}(\cdot, t, z)|^2 \frac{dt}{t^3} \right)^{1/2} \right\|_p = \|g_{m,s;b}f(\cdot + z)\|_{L^p} \\
& \leq C \|f\|_p. \tag{52}
\end{aligned}$$

Therefore, using $D_m \simeq m^{n-2}$ and the Minkowski inequality, we get

$$\|K_2\|_p \leq C \sum_{m=1}^\infty m^{-2} \rho(z) \|f\|_p \leq C \rho(z) \|f\|_p. \tag{53}$$

Now, we give the L^p norm of K_1 . Take an arbitrary $4^{-1} > \varepsilon^2 > 0$ and $z \in \mathbb{R}^n$ such that $\varepsilon^{-2} \rho(z) \leq 1$. Decompose $I_{m,s}$ as

$$\begin{aligned}
& I_{m,s}(x, t, z) \\
& = \int_{\substack{\rho(x-y) > \varepsilon^{-2} \rho(z) \\ \rho(x-y) \leq t, \rho(x+z-y) \geq t}} \frac{Y_{m,s}((x-y)')}{J((x-y)') \rho(x-y)^{\alpha-1}} \\
& \quad \times (b(x+z) - b(y)) f(y) dy \\
& + \int_{\substack{\rho(x-y) > \varepsilon^{-2} \rho(z) \\ \rho(x-y) \geq t, \rho(x+z-y) \leq t}} \frac{Y_{m,s}((x+z-y)')}{J((x+z-y)') \rho(x+z-y)^{\alpha-1}} \\
& \quad \times (b(y) - b(x+z)) f(y) dy \\
& + \int_{\substack{\rho(x-y) > \varepsilon^{-2} \rho(z) \\ \rho(x-y) \leq t, \rho(x+z-y) \leq t}} \left(\frac{Y_{m,s}((x-y)')}{J((x-y)') \rho(x-y)^{\alpha-1}} \right. \\
& \quad - \left(\frac{Y_{m,s}((x+z-y)')}{J((x+z-y)') \rho(x+z-y)^{\alpha-1}} \right) \\
& \quad \times (J((x+z-y)') \\
& \quad \times \rho(x+z-y)^{\alpha-1})^{-1} \Bigg) \\
& \quad \times (b(x+z) - b(y)) f(y) dy
\end{aligned}$$

$$\begin{aligned}
& + \int_{\substack{\rho(x-y) > \varepsilon^{-2} \rho(z) \\ \rho(x-y) \leq t}} \frac{Y_{m,s}((x-y)')}{J((x-y)') \rho(x-y)^{\alpha-1}} \\
& \quad \times (b(x+z) - b(y)) f(y) dy
\end{aligned}$$

$$\begin{aligned}
& \times (b(x) - b(x+z)) f(y) dy \\
& + \int_{\substack{\rho(x-y) > \varepsilon^{-2}\rho(z) \\ \rho(x-y) \leq t}} \frac{Y_{m,s}((x-y)')}{J((x-y)') \rho(x-y)^{\alpha-1}} \\
& \quad \times (b(x) - b(y)) f(y) dy \\
& - \int_{\substack{\rho(x-y) > \varepsilon^{-2}\rho(z) \\ \rho(x+z-y) \leq t}} \frac{Y_{m,s}((x+z-y)')}{J((x+z-y)') \rho(x+z-y)^{\alpha-1}} \\
& \quad \times (b(x+z) - b(y)) f(y) dy \\
& := J_1(x, t) + J_2(x, t) + J_3(x, t) + J_4(x, t) \\
& \quad + J_5(x, t) - J_6(x, t).
\end{aligned} \tag{54}$$

We will estimate each term, respectively. For J_1 , when $\rho(x-y) > 4\rho(z)$, by Lemma 8, (21), $|b(x+z) - b(y)| < C$, $J(x') \geq 1$, and the Minkowski inequality, we have

$$\begin{aligned}
& \left\{ \int_0^\infty |J_1(x, t)|^2 \frac{dt}{t^3} \right\}^{1/2} \\
& \leq C \int_{\rho(x-y) > \varepsilon^{-2}\rho(z)} \frac{|Y_{m,s}((x-y)')|}{\rho(x-y)^{\alpha-1}} |f(y)| \\
& \quad \times \left\{ \int_{\rho(x-y) \leq t, \rho(x+z-y) \geq t} \frac{dt}{t^3} \right\}^{1/2} dy \\
& \leq C \int_{\rho(x-y) > \varepsilon^{-2}\rho(z)} \frac{|Y_{m,s}((x-y)')|}{\rho(x-y)^{\alpha-1}} |f(y)| \\
& \quad \times \left\{ \frac{1}{\rho(x-y)^2} - \frac{1}{\rho(x-y+z)^2} \right\}^{1/2} dy \\
& \leq Cm^{n/2-1} \int_{\rho(x-y) > \varepsilon^{-2}\rho(z)} \frac{\rho(z)^{1/2}}{\rho(x-y)^{\alpha+1/2}} |f(y)| dy.
\end{aligned} \tag{55}$$

Then by the Minkowski inequality, we get

$$\begin{aligned}
& \left\| \left\{ \int_0^\infty |J_1(x, t)|^2 \frac{dt}{t^3} \right\}^{1/2} \right\| \\
& \leq Cm^{n/2-1} \left\{ \int_{\mathbb{R}^n} \left(\int_{\rho(y) > \varepsilon^{-2}\rho(z)} \frac{\rho(z)^{1/2}}{\rho(y)^{\alpha+1/2}} \right. \right. \\
& \quad \left. \left. \times |f(x-y)| dy \right)^p dx \right\}^{1/p} \\
& \leq Cm^{n/2-1} \|f\|_p \int_{\rho(y) > \varepsilon^{-2}\rho(z)} \frac{\rho(z)^{1/2}}{\rho(y)^{\alpha+1/2}} dy
\end{aligned}$$

$$\begin{aligned}
& \leq Cm^{n/2-1} \rho(z)^{1/2} \|f\|_p \int_{\varepsilon^{-2}\rho(z)}^\infty \frac{dr}{r^{1+1/2}} \int_{S^{n-1}} d\sigma(y') \\
& \leq Cm^{n/2-1} \varepsilon \|f\|_p.
\end{aligned} \tag{56}$$

Similar to the estimate of $J_1(x, t)$, we can get

$$\begin{aligned}
& \left\{ \int_0^\infty |J_2(x, t)|^2 \frac{dt}{t^3} \right\}^{1/2} \\
& \leq Cm^{n/2-1} \int_{\rho(x-y) > \varepsilon^{-2}\rho(z)} \frac{\rho(z)^{1/2}}{\rho(x+z-y)^{\alpha+1/2}} |f(y)| dy.
\end{aligned} \tag{57}$$

Then by the Minkowski inequality, we get

$$\begin{aligned}
& \left\| \left\{ \int_0^\infty |J_2(x, t)|^2 \frac{dt}{t^3} \right\}^{1/2} \right\| \\
& \leq Cm^{n/2-1} \\
& \quad \times \left\{ \int_{\mathbb{R}^n} \left(\int_{\rho(y) > (\varepsilon^{-2}-1)\rho(z)} \frac{\rho(z)^{1/2}}{\rho(y)^{\alpha+1/2}} \right. \right. \\
& \quad \left. \left. \times |f(x+z-y)| dy \right)^p dx \right\}^{1/p} \\
& \leq Cm^{n/2-1} \|f\|_p \rho(z)^{1/2} \int_{\rho(y) > (\varepsilon^{-2}-1)\rho(z)} \frac{1}{\rho(y)^{\alpha+1/2}} dy \\
& \leq Cm^{n/2-1} (\varepsilon^{-2} - 1)^{-1/2} \|f\|_p \\
& \leq Cm^{n/2-1} \varepsilon \|f\|_p.
\end{aligned} \tag{58}$$

For J_3 , since $|b(x+z) - b(y)| < C$, by Lemma 6, the Minkowski inequality, and Lemma 8, we have

$$\begin{aligned}
& \left\{ \int_0^\infty |J_3(x, t)|^2 \frac{dt}{t^3} \right\}^{1/2} \\
& \leq C \int_{\rho(x-y) > \varepsilon^{-2}\rho(z)} \left| \frac{Y_{m,s}((x-y)')}{J((x-y)') \rho(x-y)^{\alpha-1}} \right. \\
& \quad \left. - \frac{Y_{m,s}((x+z-y)')}{J((x+z-y)') \rho(x+z-y)^{\alpha-1}} \right| \\
& \quad \times |f(y)| \left\{ \int_{\substack{\rho(x-y) \leq t \\ \rho(x+z-y) \leq t}} \frac{dt}{t^3} \right\}^{1/2} dy \\
& \leq Cm^{n/2} \int_{\rho(x-y) > \varepsilon^{-2}\rho(z)} \frac{\rho(z)}{\rho(x-y)^{\alpha+1}} dy.
\end{aligned} \tag{59}$$

Then similar to estimate (56), we get

$$\left\| \left\{ \int_0^\infty |J_3(x, t)|^2 \frac{dt}{t^3} \right\}^{1/2} \right\|_p \leq C m^{n/2} \varepsilon \|f\|_p. \quad (60)$$

About J_4 , we have

$$\begin{aligned} & \left(\int_0^\infty |J_4(x, t)|^2 \frac{dt}{t^3} \right)^{1/2} \\ & \leq |(b(x) - b(x+z))| \\ & \quad \times \left\{ \int_0^\infty \left| \int_{\substack{\rho(x-y) > \varepsilon^{-2}\rho(z) \\ \rho(x-y) \leq t}} \frac{Y_{m,s}((x-y)')}{J((x-y)') \rho(x-y)^{\alpha-1}} \right. \right. \\ & \quad \left. \left. \times f(y) dy \right|^2 \frac{dt}{t^3} \right\}^{1/2} \end{aligned} \quad (61)$$

$$= |(b(x) - b(x+z))| g_{m,s}^{\varepsilon^{-2}\rho(z)} f(x).$$

Similar to the estimate of $S_{2^{1/\varepsilon}|v|}^\rho f(x)$ in [35], we get for any fixed $1 < q < p < \infty$,

$$g_{m,s}^{\varepsilon^{-2}\rho(z)} f(x) \leq C \mathcal{M}(g_{m,s} f)(x) + C (\mathcal{M}|f|^q(x))^{1/q}, \quad (62)$$

where C is independent of m, s, ε and $\rho(z)$. So, for any $1 < p < \infty$, taking $1 < q < p$ in (62), then by (43) and the L^p boundedness of $(\mathcal{M}|f|^q)^{1/q}$, we get

$$\left\| g_{m,s}^{\varepsilon^{-2}\rho(z)} f \right\|_p \leq C \|f\|_p. \quad (63)$$

Since $b \in C_0^\infty$, we have $|b(x) - b(x+z)| \leq C\rho(z)$. Then by (63), we get

$$\left\| \left(\int_0^\infty |J_4(x, t)|^2 \frac{dt}{t^3} \right)^{1/2} \right\|_p \leq C m^{n/2} \rho(z) \|f\|_p. \quad (64)$$

For J_5 , since $J(x') \geq 1$, and $|b(x) - b(y)| \leq C\rho(x-y)$ when $\rho(x-y) \leq 1$, then by (21) and the Minkowski inequality, we get

$$\begin{aligned} & \left\{ \int_0^\infty |J_5(x, t)|^2 \frac{dt}{t^3} \right\}^{1/2} \\ & \leq C m^{n/2-1} \int_{\rho(x-y) \leq \varepsilon^{-2}\rho(z)} \frac{|f(y)|}{\rho(x-y)^{\alpha-1}} dy. \end{aligned} \quad (65)$$

Then we can get

$$\left\| \left\{ \int_0^\infty |J_5(x, t)|^2 \frac{dt}{t^3} \right\}^{1/2} \right\|_p \leq C m^{n/2-1} \varepsilon^{-2} \rho(z). \quad (66)$$

Similarly, we can get

$$\left\| \left\{ \int_0^\infty |J_6(x, t)|^2 \frac{dt}{t^3} \right\}^{1/2} \right\|_p \leq C D m^{n/2-1} (\varepsilon^{-2} \rho(z) + \rho(z)). \quad (67)$$

Combining the estimate of J_1 with J_6 , by the Minkowski inequality and $D_m \simeq C m^{(n-2)}$, we get

$$\begin{aligned} \|K_1\|_p & \leq C \sum_{m=1}^\infty m^{-3n/2} \sum_{s=1}^{D_m} \left\| \left(\int_0^\infty |I_{m,s}(x, t)|^2 \frac{dt}{t^3} \right)^{1/2} \right\|_p \\ & \leq C \sum_{m=1}^\infty m^{-3n/2} \sum_{s=1}^{D_m} m^{n/2-1} (\varepsilon^{-2} \rho(z) + \rho(z) + \varepsilon) \|f\|_p \\ & \leq C \sum_{m=1}^\infty m^{-3n/2} m^{n-2+n/2-1} (\varepsilon^{-2} \rho(z) + \rho(z) + \varepsilon) \|f\|_p \\ & \leq C \sum_{m=1}^\infty m^{-3} (\varepsilon^{-2} \rho(z) + \rho(z) + \varepsilon) \|f\|_p \\ & \leq C (\varepsilon^{-2} \rho(z) + \rho(z) + \varepsilon) \|f\|_p, \end{aligned} \quad (68)$$

which combined the L^p norm of K_2 (see (53)); then by taking $\rho(z)$ sufficiently small depending on ε , we can get

$$\|g_{\Omega,b} f(\cdot) - g_{\Omega,b} f(\cdot + z)\|_p \leq C\varepsilon. \quad (69)$$

Thus we establish the proof of Theorem 2.

4. Proof of Theorem 3

In this proof for $j = 1, 2, \dots, 13$, B_j is a positive constant depending only on $\Omega, p, n, \delta, \Theta$ and B_i ($1 \leq i < j$). Suppose that $g_{\Omega,b}$ is a bounded operator from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$; we are going to prove that $b \in \text{BMO}$. Without loss of generality, we can assume $\|g_{\Omega,b}\|_{L^p \rightarrow L^p} = 1$. To prove that $b \in \text{BMO}(\mathbb{R}^n, \rho)$, we need to show there exists a constant $B > 0$ such that for any $x_0 \in \mathbb{R}^n$ and $r > 0$,

$$\begin{aligned} N &= \frac{1}{|\mathcal{E}(x_0, r)|} \int_{\mathcal{E}(x_0, r)} |b(y) - b_{\mathcal{E}(x_0, r)}| dy \\ &\leq B(p, \Omega, \Theta, \delta, n), \end{aligned} \quad (70)$$

where $b_{\mathcal{E}(x_0, r)} = (1/|\mathcal{E}(x_0, r)|) \int_{\mathcal{E}(x_0, r)} b(y) dy$. Since $g_{\Omega,b-b_{\mathcal{E}(x_0, r)}} = g_{\Omega,b}$, we may assume $b_{\mathcal{E}(x_0, r)} = 0$. So it suffices to prove that for any fixed $x_0 \in \mathbb{R}^n$, $r \in \mathbb{R}_+$,

$$N = \frac{1}{|\mathcal{E}(x_0, r)|} \int_{\mathcal{E}(x_0, r)} |b(y)| dy \leq B(p, \Omega, \Theta, \delta, n), \quad (71)$$

where $(1/|\mathcal{E}(x_0, r)|) \int_{\mathcal{E}(x_0, r)} b(y) dy = 0$. Let

$$f(y) = [\text{sgn}(b(y)) - c_0] \chi_{\mathcal{E}(x_0, r)}(y), \quad (72)$$

where $c_0 = |\mathcal{E}(x_0, r)|^{-1} \int_{\mathcal{E}(x_0, r)} \text{sgn}(b(y)) dy$. We get $|c_0| < 1$. Then f satisfies the following properties:

$$\|f\|_\infty \leq 2, \quad (73)$$

$$\text{supp } f \subset \mathcal{E}(x_0, r), \quad (74)$$

$$\int_{\mathbb{R}^n} f(y) dy = 0, \quad (75)$$

$$f(y)b(y) > 0, \quad (76)$$

$$\frac{1}{|\mathcal{E}(x_0, r)|} \int_{\mathbb{R}^n} f(y)b(y) dy = N. \quad (77)$$

Then for $x \in G = \{x \in \mathbb{R}^n : \rho(x - x_0) > B_1 r = (3\delta^{-1} + 1)r \text{ and } (x - x_0)' \in \Theta\}$,

$$\begin{aligned} & |g_{\Omega; b} f(x)| \\ & \geq |g_\Omega(bf)(x)| - |b(x)| |g_\Omega f(x)| \\ & = \left(\int_0^\infty \left| \int_{\rho(x-y) \leq t} \Omega(x, (x-y)') \right. \right. \\ & \quad \times \left. \left. \rho(x-y)^{1-\alpha} b(y) f(y) dy \right|^2 \frac{dt}{t^3} \right)^{1/2} \\ & \quad - |b(x)| \left(\int_0^\infty \left| \int_{\rho(x-y) \leq t} \Omega(x, (x-y)') \right. \right. \\ & \quad \times \left. \left. \rho(x-y)^{1-\alpha} f(y) dy \right|^2 \frac{dt}{t^3} \right)^{1/2} \\ & := I_1 - I_2. \end{aligned} \quad (78)$$

For I_1 , by (18), we know

$$\Omega(x, (x - x_0)') \geq 2\delta. \quad (79)$$

Noting that if $\rho(y - x_0) < r$ and $\rho(x - x_0) > B_1 r$, by Lemma 9, we get $\rho((x - y)' - (x - x_0)') \leq \delta$. Since

$$\begin{aligned} & \sup_{x \in \mathbb{R}^n} |\Omega(x, (x - y)') - \Omega(x, (x - x_0)')| \\ & \leq \rho((x - y)' - (x - x_0)') \leq \delta, \end{aligned} \quad (80)$$

if $\Omega(x, (x - x_0)') \geq 2\delta$, we can get $\Omega(x, (x - y)') \geq \delta$. Then by (74), (76), (77), $(3/4)\rho(x - x_0) \leq \rho(x - y) \leq (5/4)\rho(x - x_0)$ and the Hölder inequality, we get

$$\begin{aligned} I_1 & \geq \delta \left\{ \int_{\rho(x-x_0)}^\infty \left(\int_{\mathcal{E}(x_0, r)} b(y) f(y) \rho(x-y)^{1-\alpha} \right. \right. \\ & \quad \times \left. \left. \chi_{\{\rho(x-y) \leq t\}} dy \right)^2 \frac{dt}{t^3} \right\}^{1/2} \\ & \geq \delta \int_{\rho(x-x_0)}^\infty \int_{\mathcal{E}(x_0, r)} b(y) f(y) \rho(x-y)^{1-\alpha} \chi_{\{\rho(x-y) \leq t\}} dy \frac{dt}{t^3} \\ & \quad \times \left(\int_{\rho(x-x_0)}^\infty \frac{dt}{t^3} \right)^{-1/2} \\ & = C\delta \rho(x - x_0) \int_{\mathcal{E}(x_0, r)} \rho(x - y)^{1-\alpha} b(y) f(y) \\ & \quad \times \int_{\substack{\rho(x-x_0) \leq t \\ \rho(x-y) \leq t}} \frac{dt}{t^3} dy \\ & \geq B_2 \delta \rho(x - x_0)^{-\alpha} \int_{\mathcal{E}(x_0, r)} b(y) f(y) dy \\ & = B_3 N r^\alpha \rho(x - x_0)^{-\alpha}. \end{aligned} \quad (81)$$

By (73), (74), (75), $\rho(x - x_0) \simeq \rho(x - y)$, $\rho(y - x_0) < r$ and $\Omega \in L^\infty(\mathbb{R}^n) \times L^\infty(S^{n-1})$, then by Lemma 13 and Lemma 8, we have

$$\begin{aligned} I_2 & = |b(x)| \\ & \quad \times \left(\int_0^\infty \left| \int_{\mathbb{R}^n} \left(\frac{\Omega(x, x - y)}{\rho(x - y)^{\alpha-1}} \chi_{\{\rho(x-y) \leq t\}} \right. \right. \right. \\ & \quad \times \left. \left. \left. \frac{-\Omega(x, x - x_0)}{\rho(x - x_0)^{\alpha-1}} \chi_{\{\rho(x-x_0) \leq t\}} \right) \right. \right. \\ & \quad \times \left. \left. f(y) dy \right|^2 \frac{dt}{t^3} \right)^{1/2} \\ & \leq |b(x)| \int_{\mathcal{E}(x_0, r)} |f(y)| \left| \frac{\Omega(x, x - y)}{\rho(x - y)^{\alpha-1}} - \frac{\Omega(x, x - x_0)}{\rho(x - x_0)^{\alpha-1}} \right| \\ & \quad \times \left(\int_{\substack{\rho(x-y) \leq t \\ \rho(x-x_0) \leq t}} \frac{dt}{t^3} \right)^{1/2} dy \\ & \quad + |b(x)| \int_{\mathcal{E}(x_0, r)} \frac{|\Omega(x, x - y)|}{\rho(x - y)^{\alpha-1}} |f(y)| \\ & \quad \times \left(\int_{\substack{\rho(x-y) \leq t \\ \rho(x-x_0) \geq t}} \frac{dt}{t^3} \right)^{1/2} dy \end{aligned}$$

$$\begin{aligned}
& + |b(x)| \int_{\mathcal{E}(x_0, r)} \frac{|\Omega(x, x - x_0)|}{\rho(x - x_0)^{\alpha-1}} |f(y)| \\
& \quad \times \left(\int_{\substack{\rho(x-y) \geq t \\ \rho(x-x_0) \leq t}} \frac{dt}{t^3} \right)^{1/2} dy \\
& \leq C |b(x)| r \int_{\mathcal{E}(x_0, r)} \frac{|f(y)|}{\rho(x - x_0)^{\alpha+1}} dy \\
& \quad + C |b(x)| r^{1/2} \int_{\mathcal{E}(x_0, r)} \frac{|f(y)|}{\rho(x - x_0)^{\alpha+1/2}} dy \\
& \leq B_4 |b(x)| r^{\alpha+1/2} \rho(x - x_0)^{-\alpha-1/2}.
\end{aligned} \tag{82}$$

Then using the same argument of the proof of Theorem 1.2 in [29], we get $N \leq B(\Omega, n, \delta, \Theta, p)$.

5. Proof of Theorem 4

Since $g_{\Omega; b}$ is a compact operator on $L^p(\mathbb{R}^n)$, then $g_{\Omega; b}$ is bounded on $L^p(\mathbb{R}^n)$. Thus from Theorem 3, $b \in \text{BMO}(\mathbb{R}^n, \rho)$. Thus we may assume $\|b\|_{*, \rho} = 1$. By Lemma 12, to prove that $b \in \text{CMO}(\mathbb{R}^n, \rho)$, it suffices to show that b satisfies conditions (i), (ii), and (iii) of Lemma 12.

First, suppose that b does not satisfy (i) of Lemma 12. Then there exists $\zeta > 0$ and a sequence of ellipsoids $\{\mathcal{E}_j(y_j, r_j)\}_{j=1}^\infty$ with $\lim_{j \rightarrow \infty} r_j = 0$, such that for every j

$$M(b, \mathcal{E}_j) = |\mathcal{E}_j|^{-1} \int_{\mathcal{E}_j} |b(y) - b_{\mathcal{E}_j}| dy > \zeta. \tag{83}$$

Let $f_j(y) = |\mathcal{E}_j|^{-1/p} \{[\text{sgn}(b(y) - b_{\mathcal{E}_j}) - c_0] \chi_{\mathcal{E}_j}(y)\}$, where $c_0 = |\mathcal{E}_j|^{-1} \int_{\mathcal{E}_j} \text{sgn}(b(y) - b_{\mathcal{E}_j}) dy$. So $|c_0| < 1$. Then f_j has the following properties:

$$\text{supp } f_j \subset \mathcal{E}_j; \tag{84}$$

$$f_j(y) (b(y) - b_{\mathcal{E}_j}) > 0; \tag{85}$$

$$\int_{\mathbb{R}^n} f_j(y) dy = 0; \tag{86}$$

$$|f_j(y)| \leq 2 |\mathcal{E}_j|^{-1/p}, \tag{87}$$

for $y \in \mathcal{E}_j$. It is easy to see that $\|f_j\|_p \leq C$, where C is independent of j .

Let $B_1 = (3/\delta) + 1$. If $\rho(y - y_j) < r_j$, then

$$x \in (B_1 \mathcal{E}_j)^c \cap \{x : (x - y_j)' \in \Theta\}. \tag{88}$$

Similar to the proof of I_1 in Theorem 3, by (84), (85) and noting that $\rho(x - y_j) \approx \rho(x - y)$, we get

$$\begin{aligned}
& |g_\Omega((b - b_{\mathcal{E}_j}) f_j)(x)| \\
& \geq \delta \left\{ \int_0^\infty \left(\int_{\mathcal{E}_j} \frac{(b(y) - b_{\mathcal{E}_j}) f_j(y)}{\rho(x - y)^{\alpha-1}} \chi_{\{\rho(x-y) \leq t\}} dy \right)^2 \frac{dt}{t^3} \right\}^{1/2} \\
& \geq \delta \int_{\rho(x-y_j)}^\infty \int_{\mathcal{E}_j} \frac{(b(y) - b_{\mathcal{E}_j}) f_j(y)}{\rho(x - y)^{\alpha-1}} \chi_{\{\rho(x-y) \leq t\}} dy \frac{dt}{t^3} \\
& \quad \times \left(\int_{\rho(x-y_j)}^\infty \frac{dt}{t^3} \right)^{-1/2} \\
& \geq \delta |\mathcal{E}_j|^{-1/p} \rho(x - y_j) \\
& \quad \times \int_{\rho(x-y_j)}^\infty \int_{\mathcal{E}_j} \left((b(y) - b_{\mathcal{E}_j}) \right. \\
& \quad \times \left. \left\{ [\text{sgn}(b(y) - b_{\mathcal{E}_j}) - c_0] \chi_{\mathcal{E}_j}(y) \right\} \right) \\
& \quad \times (\rho(x - y)^{\alpha-1})^{-1} \\
& \quad \times \chi_{\{\rho(x-y) \leq t\}} dy \frac{dt}{t^3} \\
& \geq \delta |\mathcal{E}_j|^{-1/p} \rho(x - y_j)^{-\alpha} \\
& \quad \times \int_{\mathcal{E}_j} |b(y) - b_{\mathcal{E}_j}| - c_0 (b(y) - b_{\mathcal{E}_j}) dy \\
& = \delta |\mathcal{E}_j|^{-1/p} \rho(x - y_j)^{-\alpha} \int_{\mathcal{E}_j} |b(y) - b_{\mathcal{E}_j}| dy \\
& \geq B_2 |\mathcal{E}_j|^{-1/p'} \rho(x - y_j)^{-\alpha}.
\end{aligned} \tag{89}$$

On the other hand, for $x \in (B_1 \mathcal{E}_j)^c$, by (84), (87), $\Omega \in L^\infty(\mathbb{R}^n) \times L^\infty(S^{n-1})$, and $\rho(x - y_j) \approx \rho(x - y)$ when $y \in \mathcal{E}_j$, we have

$$\begin{aligned}
& |g_\Omega((b - b_{\mathcal{E}_j}) f_j)(x)| \\
& \leq \int_{\mathcal{E}_j} |b(y) - b_{\mathcal{E}_j}| |f_j(y)| \frac{|\Omega(x, x - y)|}{\rho(x - y)^{\alpha-1}} \\
& \quad \times \left(\int_{\rho(x-y) \leq t} \frac{dt}{t^3} \right)^{1/2} dy \\
& \leq C |\mathcal{E}_j|^{1/p'} \rho(x - y_j)^{-\alpha}.
\end{aligned} \tag{90}$$

On the other hand, for $x \in (B_1 \mathcal{E}_j)^c$, by (84), (86), (87), and $\rho(x - y_j) \approx \rho(x - y)$, we have

$$\begin{aligned}
 & \left| (b(x) - b_{\mathcal{E}_j}) g_{\Omega}(f_j)(x) \right| \\
 & \leq |b(x) - b_{\mathcal{E}_j}| \\
 & \quad \times \left(\int_0^\infty \left| \int_{\substack{\rho(x-y) \leq t \\ \rho(x-y_j) \leq t}} \left(\frac{\Omega(x, x-y)}{\rho(x-y)^{\alpha-1}} - \frac{\Omega(x, x-y_j)}{\rho(x-y_j)^{\alpha-1}} \right) \right. \right. \\
 & \quad \left. \left. \times f_j(y) dy \right|^2 \frac{dt}{t^3} \right)^{1/2} \\
 & + |b(x) - b_{\mathcal{E}_j}| \\
 & \quad \times \left(\int_0^\infty \left| \int_{\substack{\rho(x-y) \leq t \\ \rho(x-y_j) > t}} \frac{\Omega(x, x-y)}{\rho(x-y)^{\alpha-1}} f_j(y) dy \right|^2 \frac{dt}{t^3} \right)^{1/2} \\
 & + |b(x) - b_{\mathcal{E}_j}| \\
 & \quad \times \left(\int_0^\infty \left| \int_{\substack{\rho(x-y) > t \\ \rho(x-y_j) \leq t}} \frac{\Omega(x, x-y_j)}{\rho(x-y_j)^{\alpha-1}} f_j(y) dy \right|^2 \frac{dt}{t^3} \right)^{1/2} \\
 & \leq C |\mathcal{E}_j|^{1/p'} \frac{|b(x) - b_{\mathcal{E}_j}| r_j^{1/2}}{\rho(x - y_j)^{\alpha+1/2}}.
 \end{aligned} \tag{91}$$

In fact, in the above estimate we have used Lemma 13 for $x \in (B_1 \mathcal{E}_j)^c$. Then similar to the proof of Theorem 1.3 in [29], we show that b satisfies (i)–(iii) of Lemma 12.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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