Research Article

Lipschitz Estimates for One-Sided Cohen's Commutators on Weighted One-Sided Triebel-Lizorkin Spaces

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We introduce one-sided Cohen's commutators of singular integral operators and fractional integral operators, respectively. Using the extrapolation of one-sided weights, we establish the boundedness of these operators from weighted Lebesgue spaces to weighted one-sided Triebel-Lizorkin spaces.

1. Introduction

The one-sided commutators considered in this paper are related to the commutators studied by Calderón in [1]. Cohen [2] defined the Cohen type commutators of Calderón-Zygmund singular integrals (for convenience, we only consider the 1-dimensional case) by

$$T^A f(x) = \text{p.v.} \int_{\mathbb{R}} \frac{A(x) - A(y) - A'(y)(x-y)}{|x-y|^2} \Omega(x-y) f(y) dy, \quad (1)$$

where Ω satisfies certain homogeneity, smoothness, and symmetry conditions. Chen and Lu [3] proved the boundedness of the commutators T^A from Lebesgue spaces to Triebel-Lizorkin spaces for $A^{(k)} \in \text{Lip}_\alpha(\mathbb{R})$ ($k = 0, 1$). A function $A \in \text{Lip}_\alpha(\mathbb{R})$, $0 < \alpha < 1$, if it satisfies

$$\|A\|_{\text{Lip}_\alpha} = \sup_{x, h \in \mathbb{R}, h \neq 0} \frac{|A(x+h) - A(x)|}{|h|^\alpha} < \infty. \quad (2)$$

Similar to an unbounded function $\log|x| \in \text{BMO}$, the functions in Lip_α are not necessarily bounded either (e.g.,

$|x|^\alpha \in \text{Lip}_\alpha$). Therefore, it is also nontrivial to investigate the commutators generated by operators and Lipschitz functions.

In the one-sided case, we will study the weighted boundedness of the commutators T_A from weighted Lebesgue spaces to weighted Triebel-Lizorkin spaces. The one-sided operators were motivated as not only the generalization of the theory of both-sided ones but also the requirement in ergodic theory. Lots of results show that, for a class of smaller operators (one-sided operators) and a class of wider weights (one-sided weights), many results in harmonic analysis still hold; see [4–14]. However, for one-sided weights, classical reverse Hölder's inequality does not hold.

A function K is called a one-sided Calderón-Zygmund kernel (OCZK) if K satisfies

$$\left| \int_{a < |x| < b} K(x) dx \right| \leq C, \quad 0 < a < b, \quad (3)$$

$$|K(x)| \leq \frac{C}{|x|}, \quad x \neq 0, \quad (4)$$

$$|K(x-y) - K(x)| \leq \frac{C|y|}{|x|^2}, \quad |x| > 2|y|, \quad (5)$$

with support in $\mathbb{R}^- = (-\infty, 0)$ or $\mathbb{R}^+ = (0, +\infty)$. An example of such a kernel is

$$K(x) = \frac{\sin(\log|x|)}{(x \log|x|)} \chi_{(-\infty, 0)}(x), \quad (6)$$

where χ_E denotes the characteristic function of a set E . In [15], Aimar et al. introduced the one-sided Calderón-Zygmund singular integrals which are defined by

$$\begin{aligned} T^+ f(x) &= \lim_{\varepsilon \rightarrow 0^+} \int_{x+\varepsilon}^{\infty} K(x-y) f(y) dy, \\ T^- f(x) &= \lim_{\varepsilon \rightarrow 0^+} \int_{-\infty}^{x-\varepsilon} K(x-y) f(y) dy, \end{aligned} \quad (7)$$

where the kernels K are OCZKs.

The study of weights for one-sided operators is motivated by their natural appearance in harmonic analysis, such as the one-sided Hardy-Littlewood maximal operator:

$$\begin{aligned} M^+ f(x) &= \sup_{h>0} \frac{1}{h} \int_x^{x+h} |f(y)| dy, \\ M^- f(x) &= \sup_{h>0} \frac{1}{h} \int_{x-h}^x |f(y)| dy. \end{aligned} \quad (8)$$

Recently, Sawyer [13] introduced the one-sided A_p classes A_p^+ , A_p^- , which are defined by

$$\begin{aligned} A_p^+ : A_p^+(\omega) &:= \sup_{a<b<c} \frac{1}{(c-a)^p} \int_a^b \omega(x) dx \left(\int_b^c \omega(x)^{1-p'} dx \right)^{p-1} < \infty, \\ A_p^- : A_p^-(\omega) &:= \sup_{a<b<c} \frac{1}{(c-a)^p} \int_b^c \omega(x) dx \left(\int_a^b \omega(x)^{1-p'} dx \right)^{p-1} < \infty, \end{aligned} \quad (9)$$

when $1 < p < \infty$; also, for $p = 1$,

$$A_1^+ : M^- \omega(x) \leq C \omega(x), \quad A_1^- : M^+ \omega(x) \leq C \omega(x), \quad (10)$$

for some constant C .

Very recently, Fu and Lu [16] introduced a class of one-sided Triebel-Lizorkin spaces and their weighted version.

Definition 1. For $0 < \alpha < 1$, $1 < p < \infty$, and an appropriate weight ω , the weighted one-sided Triebel-Lizorkin spaces $\dot{F}_{p,+}^{\alpha,\infty}(\omega)$ and $\dot{F}_{p,-}^{\alpha,\infty}(\omega)$ are defined by

$$\begin{aligned} \|f\|_{\dot{F}_{p,+}^{\alpha,\infty}(\omega)} &\approx \left\| \sup_{h>0} \frac{1}{h^{1+\alpha}} \int_x^{x+h} |f - f_{[x,x+h]}| \right\|_{L^p(\omega)} < \infty, \\ \|f\|_{\dot{F}_{p,-}^{\alpha,\infty}(\omega)} &\approx \left\| \sup_{h>0} \frac{1}{h^{1+\alpha}} \int_{x-h}^x |f - f_{[x-h,x]}| \right\|_{L^p(\omega)} < \infty. \end{aligned} \quad (11)$$

In [16], the authors proved the boundedness for the one-sided commutators (with symbols $A \in \text{Lip}_\alpha(\mathbb{R})$) of Calderón-Zygmund singular integral, $T_A^+ f$, and fractional integral, $I_{\beta,A}^+ f$, respectively. $T_A^+ f$ and $I_{\beta,A}^+ f$ are defined as follows:

$$\begin{aligned} T_A^+ f(x) &= \text{p.v.} \int_x^\infty K(x-y) (A(x) - A(y)) f(y) dy, \\ I_{\beta,A}^+ f(x) &= \int_x^\infty \frac{1}{(y-x)^{1-\beta}} (A(x) - A(y)) f(y) dy. \end{aligned} \quad (12)$$

Let $A(x)$ be locally integrable functions on $\mathbb{R}$. Denote by $R_m(A; x, y)$ the m th order remainder of the Taylor series of A at x about y , precisely:

$$R_m(A; x, y) = A(x) - \sum_{|k| \leq m-1} \frac{1}{k!} A^{(k)}(y) (x-y)^k, \quad m \geq 1. \quad (13)$$

Cohen's commutators of one-sided singular integrals are defined by

$$\begin{aligned} T_{A,m}^+ f(x) &= \text{p.v.} \int_x^\infty \frac{K(x-y)}{(y-x)^{m-1}} R_m(A; x, y) f(y) dy, \\ T_{A,m}^- f(x) &= \text{p.v.} \int_{-\infty}^x \frac{K(x-y)}{(x-y)^{m-1}} R_m(A; x, y) f(y) dy. \end{aligned} \quad (14)$$

Obviously, when $m = 1$, $T_{A,m}^+ = T_A^+$. Therefore, the results of this paper are the extension of [16].

Theorem 2. Assume that $1 < p < q < \infty$ and $1/p - 1/q = \alpha$. Let $A^{(k)}(x) \in \text{Lip}_\alpha(\mathbb{R})$, $k = 0, 1, \dots, m-1$. Then, one gets the following.

(i) If $\omega \in A_{(p,q)}^+$, there exists the constant $C > 0$ such that

$$\|T_{A,m}^+ f\|_{L^q(\omega^q)} \leq C \|f\|_{L^p(\omega^p)}. \quad (15)$$

(ii) If $\omega \in A_{(p,q)}^-$, there exists the constant $C > 0$ such that

$$\|T_{A,m}^- f\|_{L^q(\omega^q)} \leq C \|f\|_{L^p(\omega^p)}. \quad (16)$$

Theorem 3. Assume that $1 < p < \infty$ and $0 < \alpha < 1$. Let $A^{(k)}(x) \in \text{Lip}_\alpha(\mathbb{R})$, $k = 0, 1, \dots, m-1$. Then one gets the following.

(i) If $\omega \in A_p^+$, there exists the constant $C > 0$ such that

$$\|T_{A,m}^+ f\|_{\dot{F}_{p,+}^{\alpha,\infty}(\omega)} \leq C \|f\|_{L^p(\omega)}. \quad (17)$$

(ii) If $\omega \in A_p^-$, there exists the constant $C > 0$ such that

$$\|T_{A,m}^- f\|_{\dot{F}_{p,-}^{\alpha,\infty}(\omega)} \leq C \|f\|_{L^p(\omega)}. \quad (18)$$

The other main objects in this paper are one-sided Cohen's commutators of fractional integral operators, which are defined by

$$\begin{aligned} I_{\beta,A}^{+,m} f(x) &= \int_x^\infty \frac{1}{(y-x)^{m-\beta}} R_m(A; x, y) f(y) dy, \\ I_{\beta,A}^{-,m} f(x) &= \int_{-\infty}^x \frac{1}{(x-y)^{m-\beta}} R_m(A; x, y) f(y) dy. \end{aligned} \quad (19)$$

Obviously, when $m = 1$, $I_{\beta,A}^{+,m} = I_{\beta,A}^+$.

Theorem 4. Assume that $1 < p < q < \infty$ and $1/p - 1/q = \alpha + \beta$. Let $A^{(k)}(x) \in \text{Lip}_\alpha(\mathbb{R})$, $k = 0, 1, \dots, m-1$. Then one gets the following.

(i) If $\omega \in A_{(p,q)}^+$, there exists the constant $C > 0$ such that

$$\|I_{\beta,A}^{+,m} f\|_{L^q(\omega^q)} \leq C \|f\|_{L^p(\omega^p)}. \quad (20)$$

(ii) If $\omega \in A_{(p,q)}^-$, there exists the constant $C > 0$ such that

$$\|I_{\beta,A}^{-,m} f\|_{L^q(\omega^q)} \leq C \|f\|_{L^p(\omega^p)}. \quad (21)$$

Theorem 5. Assume that $1 < p, q < \infty$, $0 < \beta < 1$, and $1/p - 1/q = \beta$. Let $A^{(k)}(x) \in \text{Lip}_\alpha(\mathbb{R})$, $k = 0, 1, \dots, m-1$. Then one gets the following.

(i) If $\omega \in A_{(p,q)}^+$, there exists the constant $C > 0$ such that

$$\|I_{\beta,A}^{+,m} f\|_{\dot{F}_{q,+}^{\alpha,\infty}(\omega^q)} \leq C \|f\|_{L^p(\omega^p)}. \quad (22)$$

(ii) If $\omega \in A_{(p,q)}^-$, there exists the constant $C > 0$ such that

$$\|I_{\beta,A}^{-,m} f\|_{\dot{F}_{q,-}^{\alpha,\infty}(\omega^q)} \leq C \|f\|_{L^p(\omega^p)}. \quad (23)$$

Throughout this paper the letter C will denote a positive constant that may vary from line to line.

2. Estimates for the One-Sided Cohen Type Commutators of Singular Integrals

This section begins with some necessary lemmas.

Lemma 6 (see [17]). If A is a function with derivatives of order $m-1$ in $\text{Lip}_\alpha(\mathbb{R})$ ($0 < \alpha \leq 1$), then, for the m th remainder of A , there is a constant $C > 0$ such that

$$\begin{aligned} |R_m(A; x, y)| &\leq C \|A^{(m-1)}\|_{\text{Lip}_\alpha} |x-y|^{m-1+\alpha}, \\ |R_m(A; x, y) - R_m(A; x, z)| &\leq C \|A^{(m-1)}\|_{\text{Lip}_\alpha} \sum_{l=0}^{m-1} |x-z|^l |z-y|^{m-1-l+\alpha}, \\ |R_m(A; x, y) - R_m(A; z, y)| &\leq C \|A^{(m-1)}\|_{\text{Lip}_\alpha} \\ &\quad \times \left(\sum_{l=1}^{m-1} |x-z|^l |z-y|^{m-1-l+\alpha} + |x-z|^{m-1+\alpha} \right). \end{aligned} \quad (24)$$

The primary tool in the proof of Theorem 3 is an extrapolation theorem that appeared in [18].

Lemma 7. Let T be a sublinear operator defined in $C_c^\infty(\mathbb{R})$ satisfying

$$\|\omega T f\|_\infty \leq C \|\omega f\|_\infty, \quad (25)$$

for all $\omega^{-1} \in A_1^-$. Then, for $1 < p < \infty$,

$$\|T f\|_{L^p(w)} \leq C \|f\|_{L^p(w)} \quad (26)$$

holds whenever $w \in A_p^+$.

Lemma 8 (see [9]). Suppose that $\omega \in A_1^-$; then there exists $\varepsilon_1 > 0$ such that, for all $1 < r \leq 1 + \varepsilon_1$, $\omega^r \in A_1^-$.

Proof of Theorem 2. For convenience, we only prove case (i). By Lemma 6 and assumption (4), it is easy to prove that

$$\begin{aligned} \|T_{A,m}^+ f\|_{L^q(w^q)} &\leq \left\| \int_x^\infty \frac{|K(x-y)|}{(y-x)^{m-1}} |R_m(A; x, y)| |f(y)| dy \right\|_{L^q(w^q)} \\ &\leq C \left\| \int_x^\infty \frac{|f(y)|}{(y-x)^{1-\alpha}} dy \right\|_{L^q(w^q)} \\ &\leq C \|I_\alpha^+ (|f|)\|_{L^q(w^q)} \leq C \|f\|_{L^p(w^p)}, \end{aligned} \quad (27)$$

where $1/p - 1/q = \alpha$. In the last inequality, we use the boundedness of I_α^+ that appeared in [19]. $\square$

Proof of Theorem 3. Without loss of generality, we only prove case (i). Let $x \in \mathbb{R}$, $h > 0$, and $J = [x, x+8h]$. Write $f = f_1 + f_2$, where $f_1 = f\chi_J$. Then

$$\begin{aligned}
& \frac{1}{h^{1+\alpha}} \int_x^{x+2h} \left| T_{A,m}^+ f(y) - (T_{A,m}^+ f)_{[x, x+2h]} \right| dy \\
& \leq \frac{2}{h^{1+\alpha}} \int_x^{x+2h} \left| T_{A,m}^+ f(y) - T_{A,m}^+ f_2(x+2h) \right| dy \\
& \leq \frac{2}{h^{1+\alpha}} \int_x^{x+2h} \left| T_{A,m}^+ f_1(y) \right| dy \\
& \quad + \frac{2}{h^{1+\alpha}} \int_x^{x+2h} \left| T_{A,m}^+ f_2(y) - T_{A,m}^+ f_2(x+2h) \right| dy \\
& \leq \frac{2}{h^{1+\alpha}} \int_x^{x+2h} \left| T_{A,m}^+ f_1(y) \right| dy + \frac{1}{h^{1+\alpha}} \\
& \quad \times \int_x^{x+2h} \int_{x+8h}^\infty \left| \frac{K(y-z)}{(z-y)^{m-1}} R_m(A; y, z) \right. \\
& \quad \left. - \frac{K((x+2h)-z)}{(z-(x+2h))^{m-1}} \right. \\
& \quad \left. \times R_m(A; x+2h, z) \right| |f(z)| dz dy \\
& := I(x) + II(x).
\end{aligned} \tag{28}$$

To estimate $II(x)$, we have

$$\begin{aligned}
& \left| \frac{K(y-z)}{(z-y)^{m-1}} R_m(A; y, z) \right. \\
& \quad \left. - \frac{K((x+2h)-z)}{(z-(x+2h))^{m-1}} R_m(A; x+2h, z) \right| \\
& \leq \left| \frac{K(y-z)}{(z-y)^{m-1}} [R_m(A; y, z) - R_m(A; x+2h, z)] \right| \\
& \quad + \left| R_m(A; x+2h, z) \right. \\
& \quad \times [K(y-z) - K(x+2h-z)] \frac{1}{(z-y)^{m-1}} \left| \right. \\
& \quad \left. + \left| R_m(A; x+2h, z) \right. \right. \\
& \quad \times \left[\frac{1}{(z-y)^{m-1}} - \frac{1}{(z-(x+2h))^{m-1}} \right] K(x+2h-z) \left| \right. \\
& := II_1(x) + II_2(x) + II_3(x).
\end{aligned} \tag{29}$$

Noting the fact that $|x+2h-z| \sim |y-z|$ and using Lemma 6 and (4) of kernel $K(x)$, we consider $II_1(x)$, $II_2(x)$, and $II_3(x)$, respectively:

$$\begin{aligned}
II_1(x) & \leq \frac{C}{(z-y)^m} \\
& \quad \times \left(\sum_{l=1}^{m-1} (x+2h-y)^l [z-(x+2h)]^{m-1-l+\alpha} \right. \\
& \quad \left. + (x+2h-y)^{m-1+\alpha} \right) \\
& \leq C \frac{x+2h-y}{(z-y)^{2-\alpha}}, \\
II_2(x) & \leq C \frac{(x+2h-y) [z-(x+2h)]^{m-3+\alpha}}{(z-y)^{m-1}} \\
& \leq C \frac{x+2h-y}{(z-y)^{2-\alpha}}, \\
II_3(x) & \leq C \frac{(x+2h-y) [z-(x+2h)]^{m-2+\alpha}}{(z-y)^m} \\
& \leq C \frac{x+2h-y}{(z-y)^{2-\alpha}},
\end{aligned} \tag{30}$$

where we use the differential mean value theorem for II_3 .

Combining the above estimate, we have

$$II(x) \leq C \sup_{h>0} \frac{1}{h^{1+\alpha}} \int_x^{x+2h} \int_{x+8h}^\infty \frac{x+2h-y}{(z-y)^{2-\alpha}} |f(z)| dz dy. \tag{31}$$

Consider the following two sublinear operators defined on $C_c^\infty(\mathbb{R})$:

$$\begin{aligned}
M_1^+ f(x) & = \sup_{h>0} \frac{1}{h^{1+\alpha}} \int_x^{x+2h} |T_A^+ f_1(y)| dy, \\
M_2^+ f(x) & = \sup_{h>0} \frac{1}{h^{1+\alpha}} \int_x^{x+2h} \int_{x+8h}^\infty \frac{x+2h-y}{(z-y)^{2-\alpha}} |f(z)| dz dy.
\end{aligned} \tag{32}$$

The above inequalities imply that

$$\begin{aligned}
& \frac{1}{h^{1+\alpha}} \int_x^{x+2h} \left| T_{A,m}^+ f(y) - (T_{A,m}^+ f)_{[x, x+2h]} \right| dy \\
& \leq I(x) + II(x) \\
& \leq C (M_1^+ f(x) + M_2^+ f(x)).
\end{aligned} \tag{33}$$

Thus, we will discuss the boundedness of these two operators.

For $I(x)$, by Hölder's inequality and Theorem 2, we get

$$\begin{aligned}
& \frac{1}{h^{1+\alpha}} \int_x^{x+2h} |T_{A,m}^+ f_1(y)| dy \\
& \leq \frac{C}{h^\alpha} \left(\frac{1}{h} \int_x^{x+2h} |T_{A,m}^+ f_1(y)|^r dy \right)^{1/r} \\
& \leq \frac{C}{h^{\alpha+1/r}} \left(\int_x^{x+8h} |f(y)|^s dy \right)^{1/s} \\
& \leq C \left(\frac{1}{8h} \int_x^{x+8h} |f(y)|^s \omega^s \omega^{-s} dy \right)^{1/s} \\
& \leq C \|f\omega\|_\infty \left(\frac{1}{8h} \int_x^{x+8h} \omega^{-s} dy \right)^{1/s} \\
& \leq C \|f\omega\|_\infty \omega^{-1}(x),
\end{aligned} \tag{34}$$

where $1/s - 1/r = \alpha$, $\omega^{-s} \in A_1^-$ for $\omega^{-1} \in A_1^-$.

Next, we have

$$\|\omega M_1^+ f\|_\infty \leq C \|f\omega\|_\infty. \tag{35}$$

By Lemma 7, we obtain

$$\|M_1^+ f\|_{L^p(\omega)} \leq C \|f\|_{L^p(\omega)} \tag{36}$$

for all $\omega \in A_p^+$.

For M_2^+ , set $I_j = [x, x + 2^j h]$, $j \in \mathbb{Z}^+$. Then

$$\begin{aligned}
& \frac{1}{h^{1+\alpha}} \int_x^{x+2h} \int_{x+8h}^\infty \frac{x+2h-y}{(z-y)^{2-\alpha}} |f(z)| dz dy \\
& \leq \frac{C}{h^\alpha} \int_x^{x+2h} \sum_{j=3}^\infty \int_{x+2^j h}^{x+2^{j+1} h} \frac{1}{(z-y)^{2-\alpha}} |f(z)| dz dy \\
& \leq \frac{C}{h^{\alpha-1}} \sum_{j=3}^\infty \int_{x+2^j h}^{x+2^{j+1} h} \frac{1}{(z-x)^{2-\alpha}} |f(z)| dz \\
& \leq \frac{C}{h} \sum_{j=3}^\infty \frac{1}{2^{j(2-\alpha)}} \int_{x+2^j h}^{x+2^{j+1} h} |f(z)| dz \\
& \leq \sum_{j=3}^\infty \frac{2^{j+1}}{2^{j(2-\alpha)}} \frac{1}{2^{j+1} h} \int_x^{x+2^{j+1} h} |f(z)| dz \\
& \leq C \sum_{j=3}^\infty \frac{2^{j+1}}{2^{j(2-\alpha)}} \left(\frac{1}{2^{j+1} h} \int_x^{x+2^{j+1} h} |f(z)|^t dz \right)^{1/t} \\
& = C \sum_{j=3}^\infty \frac{2^{j+1}}{2^{j(2-\alpha)}} \left(\frac{1}{2^{j+1} h} \int_x^{x+2^{j+1} h} |f(z)|^t \omega^t \omega^{-t} dz \right)^{1/t} \\
& \leq C \|f\omega\|_\infty \sum_{j=3}^\infty \frac{2^{j+1}}{2^{j(2-\alpha)}} \left(\frac{1}{2^{j+1} h} \int_x^{x+2^{j+1} h} \omega^{-t} dz \right)^{1/t} \\
& \leq C \|f\omega\|_\infty \omega^{-1}(x),
\end{aligned} \tag{37}$$

where $0 < \alpha < 1$, $\omega^{-t} \in A_1^-$ for $\omega^{-1} \in A_1^-$. Then

$$\|\omega M_2^+ f\|_\infty \leq C \|f\omega\|_\infty. \tag{38}$$

By Lemma 7, we have

$$\|M_2^+ f\|_{L^p(\omega)} \leq C \|f\|_{L^p(\omega)} \tag{39}$$

for all $\omega \in A_p^+$.

Combining estimates (36) and (39), the proof is completed. $\square$

3. Estimates for the One-Sided Commutators of Cohen Type of Fractional Integrals

In order to prove Theorems 4 and 5, we will introduce the one-sided extrapolation lemma.

Lemma 9 (see [18]). *Let $1 < p_0 < \infty$ and let T be sublinear operator defined in $C_c^\infty(\mathbb{R})$ satisfying*

$$\|\omega T f\|_\infty \leq C \|f\omega\|_{p_0}, \tag{40}$$

for every $x \in \mathbb{R}$ and $\omega \in A_{(p_0, \infty)}^+$; then, for every $1 < p < p_0$, $1/p - 1/q = 1/p_0$, and $\omega \in A_{(p,q)}^+$, the inequality

$$\|\omega T f\|_q \leq C \|f\omega\|_p \tag{41}$$

holds.

Lemma 10 (see [20]). *Suppose that $\omega \in A_{(p,q)}^+$; then $\omega^q \in A_q^+$ and $\omega^p \in A_p^+$ for all $1 < p < q \leq \infty$.*

Proof of Theorem 4. For convenience, we only give the proof of case (i). Using Lemma 6 and the boundedness of one-sided fractional integral operators $I_{\alpha+\beta}^+$ that appeared in [19], we get

$$\begin{aligned}
& \|I_{\beta,A}^{+,m} f\|_{L^q(w^q)} \\
& \leq \left\| \int_x^\infty \frac{1}{(y-x)^{m-\beta}} |R_m(A; x, y)| |f(y)| dy \right\|_{L^q(w^q)} \\
& \leq C \left\| \int_x^\infty \frac{1}{(y-x)^{1-\alpha-\beta}} |f(y)| dy \right\|_{L^q(w^q)} \\
& \leq C \|f\|_{L^p(w^p)},
\end{aligned} \tag{42}$$

where $1/p - 1/q = \alpha + \beta$. $\square$

Proof of Theorem 5. Let $x \in \mathbb{R}$, $h > 0$. Write $f = f_1 + f_2$, where $f_1 = f\chi_J$, $J = [x, x + 4h]$. Then

$$\begin{aligned} & \frac{1}{h^{1+\alpha}} \int_x^{x+2h} \left| I_{\beta,A}^{+,m} f(y) - \left(I_{\beta,A}^{+,m} f \right)_{[x,x+2h]} \right| dy \\ & \leq \frac{2}{h^{1+\alpha}} \int_x^{x+2h} \left| I_{\beta,A}^{+,m} f(y) - I_{\beta,A}^{+,m} f_2(x+2h) \right| dy \\ & \leq \frac{2}{h^{1+\alpha}} \int_x^{x+2h} \left| I_{\beta,A}^{+,m} f_1(y) \right| dy \\ & \quad + \frac{2}{h^{1+\alpha}} \int_x^{x+2h} \left| I_{\beta,A}^{+,m} f_2(y) - I_{\beta,A}^{+,m} f_2(x+2h) \right| dy. \\ & := III(x) + IV(x). \end{aligned} \quad (43)$$

By Lemma 6, we have

$$\begin{aligned} & \left| I_{\beta,A}^{+,m} f_2(y) - I_{\beta,A}^{+,m} f_2(x+2h) \right| \\ & \leq \int_{x+4h}^{\infty} \left| \frac{1}{(z-y)^{m-\beta}} R_m(A; y, z) \right. \\ & \quad \left. - \frac{1}{(z-(x+2h))^{m-\beta}} R_m(A; x+2h, z) \right| |f(z)| dz \\ & \leq C \int_{x+4h}^{\infty} \frac{1}{(z-y)^{m-\beta}} \\ & \quad \times |R_m(A; y, z) - R_m(A; x+2h, z)| |f(z)| dz \\ & \quad + C \int_{x+4h}^{\infty} |R_m(A; x+2h, z)| \\ & \quad \times \left| \frac{1}{(z-y)^{m-\beta}} - \frac{1}{(z-(x+2h))^{m-\beta}} \right| |f(z)| dz \\ & \leq C \int_{x+4h}^{\infty} \frac{x+2h-y}{(z-y)^{2-\alpha-\beta}} |f(z)| dz. \end{aligned} \quad (44)$$

Consider the following two sublinear operators defined on $C_c^{\infty}(\mathbb{R})$:

$$\begin{aligned} M_3^+ f(x) &= \sup_{h>0} \frac{1}{h^{1+\alpha}} \int_x^{x+2h} \left| I_{\beta,A}^{+,m} f_1(y) \right| dy, \\ M_4^+ f(x) &= \sup_{h>0} \frac{1}{h^{1+\alpha}} \\ & \quad \times \int_x^{x+2h} \int_{x+4h}^{\infty} \frac{x+2h-y}{(z-y)^{2-\alpha-\beta}} |f(z)| dz dy. \end{aligned} \quad (45)$$

We conclude from (43) and (45) that

$$\begin{aligned} & \frac{1}{h^{1+\alpha}} \int_x^{x+2h} \left| I_{\beta,A}^{+,m} f(y) - \left(I_{\beta,A}^{+,m} f \right)_{[x,x+2h]} \right| dy \\ & \leq III(x) + IV(x) \leq C(M_3^+ f(x) + M_4^+ f(x)). \end{aligned} \quad (46)$$

If $\omega \in A_{(1/\beta, \infty)}^+$, then $\omega^{-1/(1-\beta)} \in A_1^-$; see [16]. By Lemma 8, there exists $t > 1$, such that $\omega^{-t/(1-\beta)} \in A_1^-$. Let $r = t/(1-\beta)$ and $1/s - 1/r = \beta$. Using the boundedness of I_{β}^+ from L^s to L^r , together with Lemma 6, we have

$$\begin{aligned} & \frac{1}{h^{1+\alpha}} \int_x^{x+2h} \left| I_{\beta,A}^{+,m} f_1(y) \right| dy \\ & \leq \frac{C}{h^{\alpha}} \left(\frac{1}{h} \int_x^{x+2h} \left| I_{\beta,A}^{+,m} f_1(y) \right|^r dy \right)^{1/r} \\ & \leq \frac{C}{h^{\alpha+1/r}} \left(\int_x^{x+2h} \left(\int_y^{\infty} \frac{|R_m(A; y, z)|}{(z-y)^{m-\beta}} |f_1(z)| dz \right)^r dy \right)^{1/r} \\ & \leq \frac{C}{h^{\alpha+1/r}} \left(\int_x^{x+2h} \left(\int_y^{\infty} \frac{h^{\alpha}}{(z-y)^{1-\beta}} |f_1(z)| dz \right)^r dy \right)^{1/r} \\ & \leq Ch^{\beta} \left(\frac{1}{h} \int_x^{x+4h} |f(y)|^s dy \right)^{1/s} \\ & \leq Ch^{\beta} \left(\frac{1}{h} \int_x^{x+4h} |f(y)|^s \omega^s \omega^{-s} dy \right)^{1/s} \\ & \leq Ch^{\beta} \left(\frac{1}{h} \int_x^{x+4h} |f(y)|^{1/\beta} \omega^{1/\beta} dy \right)^{\beta} \\ & \quad \times \left(\frac{1}{h} \int_x^{x+4h} \omega^{-r} dy \right)^{1/r} \leq C \|f\omega\|_{L^{1/\beta}} \omega^{-1}(x). \end{aligned} \quad (47)$$

In the last inequality, we use the fact $\omega^{-r} = \omega^{-t/(1-\beta)} \in A_1^-$ for all $\omega \in A_{(1/\beta, \infty)}^+$.

By Lemma 9, there exists $p_0 = 1/\beta$ such that

$$\|\omega M_3^+ f\|_{\infty} \leq C \|f\omega\|_{L^{1/\beta}}; \quad (48)$$

thus

$$\|M_3^+ f\|_{L^q(\omega^q)} \leq C \|f\|_{L^p(\omega^p)} \quad (49)$$

for all $\omega \in A_{(p,q)}^+$.

For M_4^+ , let $I_j = [x, x + 2^j h]$, $j \in \mathbb{Z}^+$. Then

$$\begin{aligned} & \frac{1}{h^{1+\alpha}} \int_x^{x+2h} \int_{x+4h}^{\infty} \frac{x+2h-y}{(z-y)^{2-\alpha-\beta}} |f(z)| dz dy \\ & \leq \frac{C}{h^{\alpha}} \int_x^{x+2h} \sum_{j=2}^{\infty} \int_{x+2^j h}^{x+2^{j+1} h} \frac{1}{(z-y)^{2-\alpha-\beta}} |f(z)| dz dy \\ & \leq \frac{C}{h^{\alpha-1}} \sum_{j=2}^{\infty} \int_{x+2^j h}^{x+2^{j+1} h} \frac{1}{(z-x)^{2-\alpha-\beta}} |f(z)| dz \end{aligned}$$

$$\begin{aligned}
&\leq Ch^\beta \sum_{j=2}^{\infty} \frac{2^{j+1}}{2^{j(2-\alpha-\beta)}} \frac{1}{2^{j+1}h} \int_x^{x+2^{j+1}h} |f(z)| dz \\
&\leq Ch^\beta \sum_{j=2}^{\infty} \frac{2^{j+1}}{2^{j(2-\alpha-\beta)}} \left(\frac{1}{2^{j+1}h} \int_x^{x+2^{j+1}h} |f(z)|^s dz \right)^{1/s} \\
&= Ch^\beta \sum_{j=2}^{\infty} \frac{2^{j+1}}{2^{j(2-\alpha-\beta)}} \\
&\quad \times \left(\frac{1}{2^{j+1}h} \int_x^{x+2^{j+1}h} |f(z)|^s \omega^s \omega^{-s} dz \right)^{1/s} \\
&\leq C \|f\omega\|_{L^{1/\beta}} \sum_{j=2}^{\infty} \frac{2^{(j+1)(1-\beta)}}{2^{j(2-\alpha-\beta)}} \\
&\quad \times \left(\frac{1}{2^{j+1}h} \int_x^{x+2^{j+1}h} \omega^{-r} dz \right)^{1/r} \\
&\leq C \|f\omega\|_{L^{1/\beta}} \omega^{-1}(x),
\end{aligned} \tag{50}$$

where $\omega^{-r} = \omega^{-t/(1-\beta)} \in A_1^-(t > 1)$ for all $\omega \in A_{(1/\beta, \infty)}^+$. Then

$$\|\omega M_4^+ f\|_{\infty} \leq C \|f\omega\|_{L^{1/\beta}}. \tag{51}$$

By Lemma 9, the inequality

$$\|M_4^+ f\|_{L^q(\omega^q)} \leq C \|f\|_{L^p(\omega^p)} \tag{52}$$

holds for $\omega \in A_{(p,q)}^+$.

This completes the proof of case (i). For case (ii), we omit the details since they are similar to those of the proof of (i) with $\omega \in A_{(p,q)}^-$ instead of $\omega \in A_{(p,q)}^+$. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

- [1] A. P. Calderón, "Commutators of singular integral operators," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 53, pp. 1092–1099, 1965.
- [2] J. Cohen, "A sharp estimate for a multilinear singular integral in $\mathbb{R}^n$," *Indiana University Mathematics Journal*, vol. 30, no. 5, pp. 693–702, 1981.
- [3] W. G. Chen and S. Z. Lu, "On multilinear singular integrals in $\mathbb{R}^n$," *Advances in Mathematics (China)*, vol. 29, no. 4, pp. 325–330, 2000.
- [4] Z. Fu, S. Lu, Y. Pan, and S. Shi, "Some one-sided estimates for oscillatory singular integrals," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 108, pp. 144–160, 2014.
- [5] Z. Fu, S. Lu, S. Sato, and S. Shi, "On weighted weak type norm inequalities for one-sided oscillatory singular integrals," *Studia Mathematica*, vol. 207, no. 2, pp. 137–151, 2011.
- [6] Z. W. Fu, G. L. Wang, and Q. Y. Wu, "Weighted Lipschitz estimates for commutators of one-sided operators on one-sided Triebel-Lizorkin spaces," *Mathematical Inequalities & Applications*, vol. 17, no. 2, pp. 611–625, 2014.
- [7] Z. Fu, S. Lu, Y. Pan, and S. Shi, "Boundedness of one-sided oscillatory integral operators on weighted Lebesgue spaces," *Abstract and Applied Analysis*, vol. 2014, Article ID 291397, 7 pages, 2014.
- [8] M. Lorente and M. S. Riveros, "Weights for commutators of the one-sided discrete square function, the Weyl fractional integral and other one-sided operators," *Proceedings of the Royal Society of Edinburgh, Section A: Mathematics*, vol. 135, no. 4, pp. 845–862, 2005.
- [9] R. A. Macías and M. S. Riveros, "One-sided extrapolation at infinity and singular integrals," *Proceedings of the Royal Society of Edinburgh: Section A Mathematics*, vol. 130, no. 5, pp. 1081–1102, 2000.
- [10] F. J. Martín-Reyes and A. de la Torre, "One-sided BMO spaces," *Journal of the London Mathematical Society*, vol. 49, no. 3, pp. 529–542, 1994.
- [11] C. Pérez, "Sharp estimates for commutators of singular integrals via iterations of the Hardy-Littlewood maximal function," *The Journal of Fourier Analysis and Applications*, vol. 3, no. 6, pp. 743–756, 1997.
- [12] S. Shi and Z. Fu, "Estimates of some operators on one-sided weighted Morrey spaces," *Abstract and Applied Analysis*, vol. 2013, Article ID 829218, 9 pages, 2013.
- [13] E. Sawyer, "Weighted inequalities for the one-sided Hardy-Littlewood maximal functions," *Transactions of the American Mathematical Society*, vol. 297, no. 1, pp. 53–61, 1986.
- [14] A. de la Torre and J. L. Torrea, "One-sided discrete square function," *Studia Mathematica*, vol. 156, no. 3, pp. 243–260, 2003.
- [15] H. Aimar, L. Forzani, and F. J. Martín-Reyes, "On weighted inequalities for singular integrals," *Proceedings of the American Mathematical Society*, vol. 125, no. 7, pp. 2057–2064, 1997.
- [16] Z. W. Fu and S. Z. Lu, "One-sided Triebel-Lizorkin space and its applications," *Scientia Sinica Mathematica*, vol. 41, pp. 43–52, 2011 (Chinese).
- [17] Y. Ding, S. Z. Lu, and K. Yabuta, "Multilinear singular and fractional integrals," *Acta Mathematica Sinica*, vol. 22, no. 2, pp. 347–356, 2006.
- [18] M. Lorente and M. S. Riveros, "Two extrapolation theorems for related weights and applications," *Mathematical Inequalities & Applications*, vol. 10, no. 3, pp. 643–660, 2007.
- [19] K. F. Andersen and E. T. Sawyer, "Weighted norm inequalities for the Riemann-Liouville and Weyl fractional integral operators," *Transactions of the American Mathematical Society*, vol. 308, no. 2, pp. 547–558, 1988.
- [20] F. J. Martín-Reyes, L. Pick, and A. de la Torre, " A_{∞}^+ condition," *Canadian Journal of Mathematics*, vol. 45, no. 6, pp. 1231–1244, 1993.