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Research Article

Approximating Common Fixed Points of Bregman Weakly Relatively Nonexpansive Mappings in Banach Spaces

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Using Bregman functions, we introduce a new hybrid iterative scheme for finding common fixed points of an infinite family of Bregman weakly relatively nonexpansive mappings in Banach spaces. We prove a strong convergence theorem for the sequence produced by the method. No closedness assumption is imposed on a mapping $T : C \to C$, where C is a closed and convex subset of a reflexive Banach space E. Furthermore, we apply our method to solve a system of equilibrium problems in reflexive Banach spaces. Some application of our results to the problem of finding a minimizer of a continuously Fréchet differentiable and convex function in a Banach space is presented. Our results improve and generalize many known results in the current literature.

1. Introduction

Throughout this paper, we denote the set of real numbers and the set of positive integers by $\mathbb{R}$ and $\mathbb{N}$, respectively. Let E be a Banach space with the norm $\|\cdot\|$ and the dual space E^*. For any $x \in E$, we denote the value of $x^* \in E^*$ at x by $\langle x, x^*\rangle$. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in E; we denote the strong convergence of $\{x_n\}_{n\in\mathbb{N}}$ to $x \in E$ as $n \to \infty$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. The modulus δ of convexity of E is denoted by

$$\delta(\epsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| \le 1, \|y\| \le 1, \|x-y\| \ge \epsilon\right\} \tag{1}$$

for every ϵ with $0 \le \epsilon \le 2$. A Banach space E is said to be *uniformly convex* if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. Let $S_E = \{x \in E : \|x\| = 1\}$. The norm of E is said to be *Gâteaux differentiable* if, for each $x, y \in S_E$, the limit

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{2}$$

exists. In this case, E is called *smooth*. If the limit (2) is attained uniformly for all $x, y \in S_E$, then E is called *uniformly smooth*. The Banach space E is said to be *strictly convex* if $\|(x+y)/2\| < 1$ whenever $x, y \in S_E$ and $x \ne y$. It is well known that E is uniformly convex if and only if E^* is uniformly smooth. It is also known that if E is reflexive, then E is strictly convex if and only if E^* is smooth; for more details, see [1–3].

Let C be a nonempty subset of E. Let $T : C \to E$ be a mapping. We denote the set of fixed points of T by $F(T)$; that is, $F(T) = \{x \in C : Tx = x\}$. A mapping $T : C \to E$ is said to be *nonexpansive* if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$. A mapping $T : C \to E$ is said to be *quasinonexpansive* if $F(T) \ne \emptyset$ and $\|Tx - y\| \le \|x - y\|$ for all $x \in C$ and $y \in F(T)$. The mapping T is called *closed*, if for any sequence $\{x_n\}_{n\in\mathbb{N}} \subset C$ with $\lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty} Tx_n = y_0$, then we have $Tx_0 = y_0$. Let $T : C \to C$ be a nonexpansive mapping. Recall that the Mann-type [4] iteration is given by the following formula:

$$x_{n+1} = \gamma_n T x_n + (1 - \gamma_n) x_n, \quad x_1 \in C. \tag{3}$$

Here, $\{\gamma_n\}_{n\in\mathbb{N}}$ is a sequence of real numbers in $[0,1]$ satisfying some appropriate conditions. A more general iteration scheme is the Halpern [5] iteration given by

$$\begin{aligned} &u \in C, \quad x_1 \in C \text{ chosen arbitrarily,} \\ &y_n = (1-\beta_n)\, x_n + \beta_n T x_n, \\ &x_{n+1} = \alpha_n u + (1-\alpha_n)\, y_n, \end{aligned} \tag{4}$$

where the sequences $\{\beta_n\}_{n\in\mathbb{N}}$ and $\{\alpha_n\}_{n\in\mathbb{N}}$ satisfy some appropriate conditions. Numerous results have been proved on Mann's and Halpern's iterations for nonexpansive mappings in Hilbert and Banach spaces (see, e.g., [6–11]).

Let E be a smooth, strictly convex, and reflexive Banach space and let J be the normalized duality mapping of E. Let C be a nonempty, closed, and convex subset of E. The generalized projection Π_C from E onto C [12] is defined and denoted by

$$\Pi_C(x) = \arg\min_{y\in C} \phi(y,x), \tag{5}$$

where $\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$. Let C be a nonempty, closed, and convex subset of smooth Banach space E and let T be a mapping from C into itself. A point $p \in C$ is said to be an *asymptotic fixed point* [13] of T if there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in C which converges weakly to p and $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. We denote the set of all asymptotic fixed points of T by $\widehat{F}(T)$. A point $p \in C$ is called a *strong asymptotic fixed point* of T if there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in C which converges strongly to p and $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. We denote the set of all strong asymptotic fixed points of T by $\widetilde{F}(T)$.

Following Matsushita and Takahashi [14], a mapping $T : C \to C$ is said to be *relatively nonexpansive* if the following conditions are satisfied:

(1) $F(T)$ is nonempty;

(2) $\phi(u,Tx) \le \phi(u,x)$, $\forall u \in F(T), x \in C$;

(3) $\widehat{F}(T) = F(T)$.

The mapping T is called *relatively weak quasinonexpansive* [15, 16] if the following conditions are satisfied:

(1) $F(T)$ is nonempty;

(2) $\phi(u,Tx) \le \phi(u,x)$, $\forall u \in F(T), x \in C$.

In 2005, Matsushita and Takahashi [14] proved the following strong convergence theorem for relatively nonexpansive mappings in a Banach space.

Theorem 1. *Let E be a uniformly convex and uniformly smooth Banach space, let C be a nonempty, closed, and convex subset of E, let T be a relatively nonexpansive mapping from C into itself, and let $\{\alpha_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers such that $0 \le \alpha_n < 1$ and $\limsup_{n\to\infty}\alpha_n < 1$. Suppose that $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ is given by*

$$\begin{aligned} &x_0 = x \in C, \\ &y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T x_n), \\ &H_n = \{z \in C_n : \phi(z, y_n) \le \phi(z, x_n)\}, \\ &W_n = \{z \in C : \langle x_n - z, Jx - Jx_n\rangle \ge 0\}, \\ &x_{n+1} = \Pi_{H_n \cap W_n} x. \end{aligned} \tag{6}$$

Then $\{x_n\}_{n\in\mathbb{N}}$ converges strongly to $\Pi_{F(T)}x$.

In 2010, Plubtieng and Ungchittrakool [17] proved the following strong convergence theorem for relatively nonexpansive mappings in a Banach space.

Theorem 2. *Let E be a uniformly convex and uniformly smooth Banach space and let $\widehat{C}$ and C be two nonempty, closed, and convex subsets of E such that $C \subset \widehat{C}$. Let $\{T_n\}_{n\in\mathbb{N}}$ be a sequence of relatively nonexpansive mappings from C into E such that $\cap_{n=1}^{\infty} F(T_n)$ is nonempty and let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence defined as follows:*

$$\begin{aligned} &x_0 \in \widehat{C}, \\ &C_1 = C, \\ &x_1 = \Pi_{C_1} x_0, \\ &y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T_n x_n), \\ &C_{n+1} = \{z \in C_n : \phi(z, y_n) \le \phi(z, x_n)\}, \\ &x_{n+1} = \Pi_{C_n} x_0, \quad n \in \mathbb{N}, \end{aligned} \tag{7}$$

where $\{\alpha_n\}_{n\in\mathbb{N}} \subset [0,1]$ satisfies either

(a) $0 \le \alpha_n < 1$ *for all* $n \in \mathbb{N}$ *and* $\limsup_{n\to\infty}\alpha_n < 1$ *or*

(b) $\liminf_{n\to\infty}\alpha_n(1-\alpha_n) > 0$.

Suppose that for any bounded subset B of C, there exists an increasing, continuous, and convex function $h_B : [0,+\infty) \to [0,+\infty)$ such that $h_B(0) = 0$, and $\lim_{k,l\to\infty}\sup\{h_B(\|T_k z - T_l z\|) : z \in B\} = 0$. Let T be a mapping from C into E defined by $Tx = \lim_{n\to\infty} T_n x$ for all $x \in C$ and suppose that $F(T) = \cap_{n=1}^{\infty} F(T_n) = \cap_{n=1}^{\infty}\widetilde{F}(T_n) = \widetilde{F}(T)$. Then $\{x_n\}_{n\in\mathbb{N}}$, $\{T_n x_n\}_{n\in\mathbb{N}}$, and $\{y_n\}_{n\in\mathbb{N}}$ converge strongly to $\Pi_{F(T)}x_0$.

In 2010, Cai and Hu [15] proved the following strong convergence theorem for a finite family of closed relatively weak quasinonexpansive mappings in a Banach space.

Theorem 3. *Let C be a nonempty, closed, and let convex subset of uniformly convex and uniformly smooth Banach space E and let $\{T_1, T_2, \ldots, T_N\}$ be a finite family of closed relatively weak quasinonexpansive mappings from C into itself with $F := \cap_{i=1}^{N} F(T_i) \neq \varnothing$. Assume that T_i is uniformly continuous for all*

$i \in \{1, 2, \ldots, N\}$. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence generated by the following algorithm

$$\begin{aligned}
&x_0 = x \in C \text{ chosen arbitrarily},\\
&C_1 = C,\\
&x_1 = \Pi_{C_1} x_0,\\
&z_n = J^{-1}\left[\beta_n J x_{n-1} + (1-\beta_n) J T_n x_n\right], \quad T_n = T_{n(mod)N},\\
&y_n = J^{-1}\left[\alpha_n J x_1 + (1-\alpha_n) J z_n\right],\\
&C_{n+1} = \{z \in C_n : \phi(z, y_n) \le \phi(z, x_n) + \alpha_n \phi(z, x_1)\\
&\qquad + (1-\alpha_n)\left[\beta_n \phi(z, x_{n-1}) + (1-\beta_n)\phi(z, T_n x_n)\right]\},\\
&x_{n+1} = \Pi_{C_{n+1}} x_1, \quad n \in \mathbb{N} \cup \{0\}.
\end{aligned} \tag{8}$$

Let $\{\alpha_n\}_{n\in\mathbb{N}\cup\{0\}}$ and $\{\beta_n\}_{n\in\mathbb{N}\cup\{0\}}$ be sequences in $[0, 1)$ such that $\lim_{n\to\infty}\alpha_n = 0$ and $\lim_{n\to\infty}\beta_n = 0$. Then $\{x_n\}_{n\in\mathbb{N}}$ converge strongly to $\Pi_F x_1$ as $n \to \infty$.

1.1. Some Facts about Gradients. For any convex function $g : E \to (-\infty, +\infty]$, we denote the domain of g by $\operatorname{dom} g = \{x \in E : g(x) < \infty\}$. For any $x \in \operatorname{int}\operatorname{dom} g$ and any $y \in E$, we denote by $g^o(x, y)$ the *right-hand derivative* of g at x in the direction y; that is,

$$g^o(x, y) = \lim_{t\downarrow 0}\frac{g(x+ty) - g(x)}{t}. \tag{9}$$

The function g is said to be *Gâteaux differentiable* at x if $\lim_{t\to 0}((g(x+ty) - g(x))/t)$ exists for any y. In this case, $g^o(x, y)$ coincides with $\nabla g(x)$, the value of the *gradient* ∇g of g at x. The function g is said to be *Gâteaux differentiable* if it is Gâteaux differentiable everywhere. The function g is said to be *Fréchet differentiable* at x if this limit is attained uniformly in $\|y\| = 1$. The function g is Fréchet differentiable at $x \in E$ (see, e.g., [18, p. 13] or [19, p. 508]) if for all $\epsilon > 0$, there exists $\delta > 0$ such that $\|y - x\| \le \delta$ implies that

$$\left|g(y) - g(x) - \langle y - x, \nabla g(x)\rangle\right| \le \epsilon\|y - x\|. \tag{10}$$

The function g is said to be *Fréchet differentiable* if it is Fréchet differentiable everywhere. It is well known that if a continuous convex function $g : E \to \mathbb{R}$ is Gâteaux differentiable, then ∇g is norm-to-weak* continuous (see, e.g., [18, Proposition 1.1.10]). Also, it is known that if g is Fréchet differentiable, then ∇g is norm-to-norm continuous (see, [19, p. 508]). The function g is said to be *strongly coercive* if

$$\lim_{\|x_n\|\to\infty}\frac{g(x_n)}{\|x_n\|} = \infty. \tag{11}$$

It is also said to be *bounded* if $g(U)$ is bounded for each bounded subset U of E. Finally, g is said to be *uniformly Fréchet differentiable* on a subset X of E if the limit (9) is attained uniformly for all $x \in X$ and $\|y\| = 1$.

1.2. Some Facts about Legendre Functions. Let E be a reflexive Banach space. For any proper, lower semicontinuous, and convex function $g : E \to (-\infty, +\infty]$, the *conjugate function* g^* of g is defined by

$$g^*(x^*) = \sup_{x\in E}\{\langle x, x^*\rangle - g(x)\}, \quad \forall x^* \in E^*. \tag{12}$$

It is well known that $g(x) + g^*(x^*) \ge \langle x, x^*\rangle$ for all $(x, x^*) \in E \times E^*$. It is also known that $(x, x^*) \in \partial g$ is equivalent to

$$g(x) + g^*(x^*) = \langle x, x^*\rangle. \tag{13}$$

Here, ∂g is the subdifferential of g [20, 21]. We also know that if $g : E \to (-\infty, +\infty]$ is a proper, lower semicontinuous, and convex function, then $g^* : E^* \to (-\infty, +\infty]$ is a proper, weak* lower semicontinuous, and convex function; see [2] for more details on convex analysis.

The function $g : E \to (-\infty, +\infty]$ is called *Legendre* if it satisfies the following conditions:

(i) ∂g is both locally bounded and single-valued on its domain;

(ii) $(\partial g)^{-1}$ is locally bounded on its domain and g is strictly convex on every convex subset of $\operatorname{dom}\partial g$.

For more details, we refer to [22].

If E is a reflexive Banach space and $g : E \to (-\infty, +\infty]$ is a Legendre function, then in view of [23];

$$\begin{gathered}
\nabla g^* = (\nabla g)^{-1}, \quad \operatorname{ran}\nabla g = \operatorname{dom} g^* = \operatorname{int}\operatorname{dom} g^*,\\
\operatorname{ran}\nabla g = \operatorname{int}\operatorname{dom} g.
\end{gathered} \tag{14}$$

Examples of Legendre functions are given in [22, 24]. One important and interesting Legendre function is $(1/s)\|\cdot\|^s$ $(1 < s < \infty)$, where the Banach space E is smooth and strictly convex and, in particular, a Hilbert space.

1.3. Some Facts about Bregman Distances. Let E be a Banach space and let E^* be the dual space of E. Let $g : E \to \mathbb{R}$ be a convex and Gâteaux differentiable function. Then the *Bregman distance* [25, 26] corresponding to g is the function $D_g : E \times E \to \mathbb{R}$ defined by

$$D_g(x, y) = g(x) - g(y) - \langle x - y, \nabla g(y)\rangle, \quad \forall x, y \in E. \tag{15}$$

It is clear that $D_g(x, y) \ge 0$ for all $x, y \in E$. In that case when E is a smooth Banach space, setting $g(x) = \|x\|^2$ for all $x \in E$, we obtain that $\nabla g(x) = 2Jx$ for all $x \in E$ and hence $D_g(x, y) = \phi(x, y)$ for all $x, y \in E$.

Let E be a Banach space and let C be a nonempty, convex, and subset of E. Let $g : E \to \mathbb{R}$ be a convex and Gâteaux differentiable function. Then, we know from [27, 28] that for $x \in E$ and $x_0 \in C$,

$$\begin{aligned}
&D_g(x_0, x) = \min_{y\in C} D_g(y, x)\\
&\text{iff } \langle y - x_0, \nabla g(x) - \nabla g(x_0)\rangle \le 0,\\
&\qquad\qquad \forall y \in C.
\end{aligned} \tag{16}$$

Furthermore, if C is a nonempty, closed, and convex subset of a reflexive Banach space E and $g : E \to \mathbb{R}$ is a strongly coercive Bregman function, then for each $x \in E$, there exists a unique $x_0 \in C$ such that

$$D_g(x_0, x) = \min_{y \in C} D_g(y, x). \tag{17}$$

The *Bregman projection* proj_C^g from E onto C is defined by $\operatorname{proj}_C^g(x) = x_0$ for all $x \in E$. It is also well known that proj_C^g has the following property [27]:

$$D_g(y, \operatorname{proj}_C^g x) + D_g(\operatorname{proj}_C^g x, x) \le D_g(y, x) \tag{18}$$

for all $y \in C$ and $x \in E$ (see [18] for more details). Let E be a reflexive Banach space, let $g : E \to \mathbb{R}$ be a strongly coercive Bregman function, and let $D_g : E \times E \to \mathbb{R}$ be the Bregman distance corresponding to g. Then, $g^* : E^* \to \mathbb{R}$ is convex and Gâteaux differentiable [29]. Let $D_{g^*} : E^* \times E^* \to \mathbb{R}$ be the function defined by

$$D_{g^*}(x^*, y^*) = g^*(x^*) - g^*(y^*) - \langle \nabla g^*(y^*), x^* - y^* \rangle \tag{19}$$

for $x^*, y^* \in E^*$, where ∇g^* is the gradient of g^*. We know from [28] that

$$D_{g^*}(\nabla g(x), \nabla g(y)) = D_g(y, x) \tag{20}$$

for all $x, y \in E$. We have from the definition of D_{g^*} that

$$\begin{aligned} D_{g^*}(x^*, y^*) &= D_{g^*}(x^*, z^*) + D_{g^*}(z^*, y^*) \\ &\quad + \langle \nabla g^*(x^*) - \nabla g^*(z^*), z^* - y^* \rangle, \\ &\qquad \forall x^*, y^*, z^* \in E^*. \end{aligned} \tag{21}$$

In particular,

$$\begin{aligned} D_{g^*}(x^*, y^*) &= -D_{g^*}(y^*, x^*) \\ &\quad + \langle \nabla g^*(y^*) - \nabla g^*(x^*), y^* - x^* \rangle, \\ &\qquad \forall x^*, y^* \in E^*. \end{aligned} \tag{22}$$

Indeed, there exist $x, y, z \in E$ such that $\nabla g(x) = x^*$, $\nabla g(y) = y^*$ and $\nabla g(z) = z^*$. Therefore,

$$\begin{aligned} D_{g^*}(x^*, y^*) &= D_{g^*}(\nabla g(x), \nabla g(y)) = D_g(y, x) \\ &= D_g(y, z) + D_g(z, x) \\ &\quad + \langle y - z, \nabla g(z) - \nabla g(x) \rangle \\ &= D_{g^*}(\nabla g(z), \nabla g(y)) + \phi_*(\nabla g(x), \nabla g(z)) \\ &\quad + \langle \nabla g^*(y^*) - \nabla g^*(z^*), z^* - x^* \rangle \\ &= D_{g^*}(z^*, y^*) + D_{g^*}(x^*, z^*) \\ &\quad + \langle \nabla g^*(x^*) - \nabla g^*(z^*), z^* - y^* \rangle. \end{aligned} \tag{23}$$

1.4. Some Facts about Uniformly Convex Functions. Let E be a Banach space and let $B_r := \{z \in E : \|z\| \le r\}$ for all $r > 0$. Then a function $g : E \to \mathbb{R}$ is said to be *uniformly convex on bounded subsets of* E ([29, pp. 203, 221]) if $\rho_r(t) > 0$ for all $r, t > 0$, where $\rho_r : [0, +\infty) \to [0, \infty]$ is defined by

$$\begin{aligned} \rho_r(t) = \inf_{x, y \in B_r, \|x-y\| = t, \alpha \in (0,1)} \Big(&(\alpha g(x) + (1-\alpha) g(y) \\ &- g(\alpha x + (1-\alpha) y)) \\ &\times (\alpha(1-\alpha))^{-1} \Big) \end{aligned} \tag{24}$$

for all $t \ge 0$. The function ρ_r is called the gauge of uniform convexity of g. The function g is also said to be *uniformly smooth on bounded subsets of* E ([29, pp. 207, 221]) if $\lim_{t \downarrow 0}(\sigma_r(t)/t) = 0$ for all $r > 0$, where $\sigma_r : [0, +\infty) \to [0, \infty]$ is defined by

$$\begin{aligned} \sigma_r(t) &= \sup_{x \in B_r, y \in S_E, \alpha \in (0,1)} \Big(\alpha g(x + (1-\alpha) ty) \\ &\qquad + (1-\alpha) g(x - \alpha ty) - g(x)) \\ &\qquad \times (\alpha(1-\alpha))^{-1} \Big) \end{aligned} \tag{25}$$

for all $t \ge 0$.

1.5. Some Facts about Resolvents. Let E be a Banach space with the norm $\|\cdot\|$ and the dual space E^*. Let $A : E \to 2^{E^*}$ be a set-valued mapping. We define the domain and range of A by $\operatorname{dom} A = \{x \in E : Ax \neq \emptyset\}$ and $\operatorname{ran} A = \cup_{x \in E} Ax$, respectively. The graph of A is denoted by $G(A) = \{(x, x^*) \in E \times E^* : x^* \in Ax\}$. The mapping $A \subset E \times E^*$ is said to be *monotone* [30, 31] if $\langle x - y, x^* - y^* \rangle \ge 0$ whenever $(x, x^*), (y, y^*) \in A$. It is also said to be *maximal monotone* [20] if its graph is not contained in the graph of any other monotone operator on E. If $A \subset E \times E^*$ is maximal monotone, then we can show that the set $A^{-1}0 = \{z \in E : 0 \in Az\}$ is closed and convex. Let E be a reflexive Banach space with the dual space E^* and let $g : E \to (-\infty, +\infty]$ be a proper, lower semicontinuous, and convex function. Let A be a maximal monotone operator from E to E^*. For any $r > 0$, let the mapping $\operatorname{Res}_{rA}^g : E \to \operatorname{dom} A$ be defined by

$$\operatorname{Res}_{rA}^g = (\nabla g + rA)^{-1} \nabla g. \tag{26}$$

The mapping $\operatorname{Res}_{rA}^g$ is called the *g-resolvent* of A (see [32]). It is well known that $A^{-1}(0) = F(\operatorname{Res}_{rA}^g)$ for each $r > 0$ (for more details, see, e.g., [1, 33]).

1.6. Some Facts about Bregman Quasinonexpansive Mappings. Let C be a nonempty, closed, and convex subset of a reflexive Banach space E. Let $g : E \to (-\infty, +\infty]$ be a proper, lower semicontinuous, and convex function. Recall that a mapping $T : C \to C$ is said to be *Bregman quasinonexpansive*, if $F(T) \neq \emptyset$ and

$$D_g(p, Tx) \le D_g(p, x), \quad \forall x \in C,\ p \in F(T). \tag{27}$$

Nontrivial examples of such mappings are given in [34].

A mapping $T : C \to C$ is said to be *Bregman relatively nonexpansive* if the following conditions are satisfied:

(1) $F(T)$ is nonempty;

(2) $D_g(p, Tv) \leq D_g(p, v), \forall p \in F(T), v \in C$;

(3) $\widehat{F}(T) = F(T)$.

A mapping $T : C \to C$ is said to be *Bregman weakly relatively nonexpansive* if the following conditions are satisfied:

(1) $F(T)$ is nonempty;

(2) $D_g(p, Tv) \leq D_g(p, v), \forall p \in F(T), v \in C$;

(3) $\widetilde{F}(T) = F(T)$.

It is clear that any Bregman relatively nonexpansive mapping is a Bregman quasinonexpansive mapping. It is also obvious that every Bregman relatively nonexpansive mapping is a Bregman weakly relatively nonexpansive mapping, but the converse is not true in general. Indeed, for any mapping $T : C \to C$, we have $F(T) \subset \widetilde{F}(T) \subset \widehat{F}(T)$. If T is Bregman relatively nonexpansive, then $F(T) = \widetilde{F}(T) = \widehat{F}(T)$. It is easy to verify that any closed mapping $T : C \to C$ is a Bregman weakly relatively nonexpansive mapping. To this end, let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of C such that $x_n \to x \in C$ and $\|x_n - Tx_n\| \to 0$ as $n \to \infty$. This implies that $Tx_n \to x \in C$ as $n \to \infty$. From the closedness of T, we conclude that $x \in F(T)$. Below we show that there exists a Bregman weakly relatively nonexpansive mapping which is neither a Bregman relatively nonexpansive mapping nor a closed mapping.

Example 4. Let $E = l^2$, where

$$l^2 = \left\{\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n, \ldots) : \sum_{n=1}^{\infty} \|\sigma_n\|^2 < \infty\right\},$$
$$\|\sigma\| = \left(\sum_{n=1}^{\infty} \|\sigma_n\|^2\right)^{1/2}, \quad \forall \sigma \in l^2,$$
$$\langle \sigma, \eta \rangle = \sum_{n=1}^{\infty} \sigma_n \eta_n, \quad \forall \delta = (\sigma_1, \sigma_2, \ldots, \sigma_n, \ldots), \quad \eta = (\eta_1, \eta_2, \ldots, \eta_n, \ldots) \in l^2. \tag{28}$$

Let $\{x_n\}_{n\in\mathbb{N}\cup\{0\}} \subset E$ be a sequence defined by

$$\begin{aligned} x_0 &= (1, 0, 0, 0, \ldots) \\ x_1 &= (1, 1, 0, 0, 0, \ldots) \\ x_2 &= (1, 0, 1, 0, 0, 0, \ldots) \\ x_3 &= (1, 0, 0, 1, 0, 0, 0, \ldots) \\ &\ldots \\ x_n &= (\sigma_{n,1}, \sigma_{n,2}, \ldots, \sigma_{n,k}, \ldots) \\ &\ldots, \end{aligned} \tag{29}$$

where

$$\sigma_{n,k} = \begin{cases} 1 & \text{if } k = 1, n+1, \\ 0 & \text{if } k \neq 1, k \neq n+1, \end{cases} \tag{30}$$

for all $n \in \mathbb{N}$. It is clear that the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges weakly to x_0. Indeed, for any $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots) \in l^2 = (l^2)^*$, we have

$$\Lambda(x_n - x_0) = \langle x_n - x_0, \Lambda \rangle = \sum_{k=2}^{\infty} \lambda_k \sigma_{n,k} \longrightarrow 0 \tag{31}$$

as $n \to \infty$. It is also obvious that $\|x_n - x_m\| = \sqrt{2}$ for any $n \neq m$ with n, m sufficiently large. Thus, $\{x_n\}_{n\in\mathbb{N}}$ is not a Cauchy sequence. Let k be an even number in $\mathbb{N}$ and let $g : E \to \mathbb{R}$ be defined by

$$g(x) = \frac{1}{k}\|x\|^k, \quad x \in E. \tag{32}$$

It is easy to show that $\nabla g(x) = J_k(x)$ for all $x \in E$, where

$$J_k(x) = \left\{x^* \in E^* : \langle x, x^* \rangle = \|x\| \|x^*\|, \|x^*\| = \|x\|^{k-1}\right\}. \tag{33}$$

It is also obvious that

$$J_k(\lambda x) = \lambda^{k-1} J_k(x), \quad \forall x \in E, \lambda \in \mathbb{R}. \tag{34}$$

Now, we define a mapping $T : E \to E$ by

$$T(x) = \begin{cases} \dfrac{n}{n+1} x, & \text{if } x = x_n; \\ -x, & \text{if } x \neq x_n. \end{cases} \tag{35}$$

Then $F(T) = \{0\}$ and T is a Bregman weakly relatively nonexpansive mapping which is not a Bregman relatively nonexpansive mapping; see [35] for more details. Now, we prove that T is not a closed mapping. Indeed, let $y_n = (1 + 1/n)x_2$ for all n in $\mathbb{N}$. Then $y_n \to x_2$ as $n \to \infty$, $Ty_n = -(1 + 1/n)x_2 \to -x_2$ (since $y_n \neq x_m$ for all $n, m \in \mathbb{N}$), but $Tx_2 = (2/3)x_2 \neq -x_2$.

An example of a Bregman quasinonexpansive mapping which is neither a Bregman relatively nonexpansive mapping

nor a Bregman weakly relatively nonexpansive mapping can be found in [35].

In this paper, we investigate the problem of finding zeros of mappings $A : E \to 2^{E^*}$; that is, find $x \in \operatorname{dom} A$ such that

$$0^* \in Ax. \tag{36}$$

Recently, Sabach [36] proved the following two strong convergence theorems for the products of finitely many resolvents of maximal monotone operators in a reflexive Banach space.

Theorem 5. *Let E be a reflexive Banach space and let $A_i : E \to 2^{E^*}$, $i = 1, 2, \ldots, N$, be N maximal monotone operators such that $Z := \cap_{i=1}^{N} A_i^{-1}(0^*) \neq \varnothing$. Let $g : E \to \mathbb{R}$ be a Legendre function that is bounded, uniformly Fréchet differentiable, and totally convex on bounded subsets of E. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence defined by the following iterative algorithm:*

$$\begin{aligned} &x_0 \in E \text{ chosen arbitrarily},\\ &y_n = Res^g_{\lambda_n^N A_N}, \ldots, Res^g_{\lambda_n^1 A_1}(x_n + e_n),\\ &C_n = \left\{z \in E : D_g(z, y_n) \le D_g(z, x_n + e_n)\right\},\\ &Q_n = \left\{z \in E : \langle z - x_n, \nabla g(x_0) - \nabla g(x_n)\rangle \le 0\right\},\\ &x_{n+1} = proj^g_{C_n \cap Q_n} x_0 \text{ and } n \in \mathbb{N} \cup \{0\}. \end{aligned} \tag{37}$$

If, for each $i = 1, 2, \ldots, N$, $\liminf_{n\to\infty} \lambda_n^i > 0$, and the sequences of errors $\{e_n^i\}_{n\in\mathbb{N}} \subset E$ satisfy $\liminf_{n\to\infty} e_n^i = 0$, then each such sequence $\{x_n\}_{n\in\mathbb{N}}$ converges strongly to $proj^g_Z(x_0)$ as $n \to \infty$.

Theorem 6. *Let E be a reflexive Banach space and let $A_i : E \to 2^{E^*}$, $i = 1, 2, \ldots, N$, be N maximal monotone operators such that $Z := \cap_{i=1}^{N} A_i^{-1}(0^*) \neq \varnothing$. Let $g : E \to \mathbb{R}$ be a Legendre function that is bounded, uniformly Fréchet differentiable, and totally convex on bounded subsets of E. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence defined by the following iterative algorithm:*

$$\begin{aligned} &x_0 \in E \text{ chosen arbitrarily},\\ &H_0 = E,\\ &y_n = Res^g_{\lambda_n^N A_N}, \ldots, Res^g_{\lambda_n^1 A_1}(x_n + e_n),\\ &H_{n+1} = \left\{z \in H_n : D_g(z, y_n) \le D_g(z, x_n + e_n)\right\},\\ &x_{n+1} = proj^g_{H_{n+1}} x_0 \text{ and } n \in \mathbb{N} \cup \{0\}. \end{aligned} \tag{38}$$

If, for each $i = 1, 2, \ldots, N$, $\liminf_{n\to\infty} \lambda_n^i > 0$, and the sequences of errors $\{e_n^i\}_{n\in\mathbb{N}} \subset E$ satisfy $\liminf_{n\to\infty} e_n^i = 0$, then each such sequence $\{x_n\}_{n\in\mathbb{N}}$ converges strongly to $proj^g_Z(x_0)$ as $n \to \infty$.

The approximation of fixed points of Bregman nonexpansive type mappings via Bregman distances has been studied in the last ten years and much intensively in the last five years. For some recent articles on the existence and the construction of fixed points for Bregman nonexpansive type mappings, we refer the readers to [36–40].

But it is worth mentioning that, in all the above results for Bregman nonexpansive type mappings, the assumption $\widehat{F}(T) = F(T)$ is imposed on the map T or the closedness of T is required. So, the following question arises naturally in a Banach space setting.

Question 1. Is it possible to obtain strong convergence of modified Mann's type schemes to a common fixed point of an infinite family of Bregman quasinonexpansive mappings $\{T_j\}_{j\in\mathbb{N}}$ without imposing the closedness assumption, the uniformly continuity assumption, or the assumption $\widehat{F}(T_j) = F(T_j)$ on the mapping T_j?

In this paper, using Bregman functions, we introduce a new hybrid iterative scheme for finding common fixed points of an infinite family of Bregman weakly relatively nonexpansive mappings in Banach spaces. We prove a strong convergence theorem for the sequence produced by the method. No closedness assumption is imposed on a mapping $T : C \to C$, where C is a closed and convex subset of a reflexive Banach space E. Consequently, the above question is answered in the affirmative in reflexive Banach space setting. Furthermore, we apply our method to solve a system of equilibrium problems in reflexive Banach spaces. Some application of our results to the problem of finding a minimizer of a continuously Fréchet differentiable and convex function in a Banach space is presented. Our results improve and generalize many known results in the current literature; see, for example, [8, 11–14, 41–48].

2. Preliminaries

In this section, we begin by recalling some preliminaries and lemmas which will be used in the sequel.

Definition 7 (see [19]). Let E be a Banach space. The function $g : E \to \mathbb{R}$ is said to be a Bregman function if the following conditions are satisfied:

(1) g is continuous, strictly convex, and Gâteaux differentiable;

(2) the set $\{y \in E : D_g(x, y) \le r\}$ is bounded for all $x \in E$ and $r > 0$.

The following lemma follows from Butnariu and Iusem [18] and Zălinescu [29].

Lemma 8. *Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a strongly coercive Bregman function. Then*

(1) *$\nabla g : E \to E^*$ is one-to-one, onto and norm-to-weak* continuous;*

(2) *$\langle x - y, \nabla g(x) - \nabla g(y)\rangle = 0$ if and only if $x = y$;*

(3) *$\{x \in E : D_g(x, y) \le r\}$ is bounded for all $y \in E$ and $r > 0$;*

(4) *dom* $g^* = E^*$, g^* *is Gâteaux differentiable and* $\nabla g^* = (\nabla g)^{-1}$.

We know the following two results; see [29, Proposition 3.6.4].

Theorem 9. *Let E be a reflexive Banach space and let* $g : E \to \mathbb{R}$ *be a convex function which is bounded on bounded subsets of E. Then the following assertions are equivalent:*

(1) *g is strongly coercive and uniformly convex on bounded subsets of E;*

(2) dom $g^* = E^*$, g^* *is bounded on bounded subsets and uniformly smooth on bounded subsets of* E^*;

(3) dom $g^* = E^*$, g^* *is Fréchet differentiable and* ∇g^* *is uniformly norm-to-norm continuous on bounded subsets of* E^*.

Theorem 10. *Let E be a reflexive Banach space and let* $g : E \to \mathbb{R}$ *be a continuous convex function which is strongly coercive. Then the following assertions are equivalent:*

(1) *g is bounded on bounded subsets and uniformly smooth on bounded subsets of E;*

(2) g^* *is Fréchet differentiable and* ∇g^* *is uniformly norm-to-norm continuous on bounded subsets of* E^*;

(3) dom $g^* = E^*$, g^* *is strongly coercive and uniformly convex on bounded subsets of* E^*.

Let E be a Banach space and let $g : E \to \mathbb{R}$ be a convex and Gâteaux differentiable function. Then the Bregman distance [48] (see also [25, 26]) satisfies the *three point identity* that is

$$D_g(x,z) = D_g(x,y) + D_g(y,z) + \langle x - y, \nabla g(y) - \nabla g(z)\rangle, \quad \forall x,y,z \in E. \tag{39}$$

In particular, it can be easily seen that

$$D_g(x,y) = -D_g(y,x) + \langle y - x, \nabla g(y) - \nabla g(x)\rangle, \quad \forall x,y \in E. \tag{40}$$

Indeed, by letting $z = x$ in (39) and taking into account that $D_g(x,x) = 0$, we get the desired result.

Lemma 11 (see [39]). *Let E be a Banach space and let* $g : E \to \mathbb{R}$ *be a Gâteaux differentiable function which is uniformly convex on bounded subsets of E. Let* $\{x_n\}_{n\in\mathbb{N}}$ *and* $\{y_n\}_{n\in\mathbb{N}}$ *be bounded sequences in E. Then the following assertions are equivalent:*

(1) $\lim_{n\to\infty} D_g(x_n, y_n) = 0$;

(2) $\lim_{n\to\infty}\|x_n - y_n\| = 0$.

The following result was first proved in [49] (see also [19, 38]).

Lemma 12. *Let E be a reflexive Banach space let* $g : E \to \mathbb{R}$ *be a strongly coercive Bregman function and V the function defined by*

$$V(x,x^*) = g(x) - \langle x, x^*\rangle + g^*(x^*), \quad \forall x \in E,\ \forall x^* \in E^*. \tag{41}$$

Then the following assertions hold:

(1) $D_g(x, \nabla g^*(x^*)) = V(x,x^*)$ *for all* $x \in E$ *and* $x^* \in E^*$;

(2) $V(x,x^*) + \langle \nabla g^*(x^*) - x, y^*\rangle \le V(x, x^* + y^*)$ *for all* $x \in E$ *and* $x^*, y^* \in E^*$.

The following lemma which is a generalization of Lemma 3.2 in [50] plays a key role in our results.

Lemma 13 (see [17]). *Let C be a subset of a real Banach space E and let* $\{T_n\}_{n\in\mathbb{N}}$ *be a family of mappings from C into E. Suppose that for any bounded subset B of C, there exists a continuous increasing function* $h_B : [0,\infty) \to [0,\infty)$ *such that* $h_B(0) = 0$ *and* $\lim_{k,l\to\infty}\theta_l^k = 0$, *where* $\theta_l^k := \sup\{h_B(\|T_k z - T_l z\|) : z \in B\} < \infty$, *for all* $k,l \in \mathbb{N}$. *Then, for each* $x \in C$, $\{T_n x\}_{n\in\mathbb{N}}$ *converges strongly to some point of E. Moreover, let the mapping T be defined by*

$$Tx = \lim_{n\to\infty} T_n x, \quad \forall x \in C. \tag{42}$$

Then, $\limsup_{n\to\infty}\{h_B(\|T_n z - Tz\|) : z \in B\} = 0$.

Lemma 14. *Let E be a Banach space and let* $g : E \to \mathbb{R}$ *be a convex function which is uniformly convex on bounded subsets of E. Let* $s > 0$ *be a constant,* $B_s := \{z \in E : \|z\| \le s\}$, $B_s^* := \{z^* \in E^* : \|z^*\| \le s\}$, *let* ρ_s *be the gauge of uniform convexity of g, and let* ρ_s^* *be the gauge of uniform convexity of* g^*, *respectively. Then*

(i) *for any* $x, y \in B_s$ *and* $\alpha \in (0,1)$

$$g(\alpha x + (1-\alpha)y) \le \alpha g(x) + (1-\alpha) g(y) - \alpha(1-\alpha)\rho_s(\|x-y\|); \tag{43}$$

(ii) *for any* $x, y \in B_s$

$$\rho_s(\|x-y\|) \le D_g(x,y); \tag{44}$$

(iii) *if, in addition, g is bounded on bounded subsets and uniformly convex on bounded subsets of E, then, for any* $x \in E$, $y^*, z^* \in B_s$ *and* $\alpha \in (0,1)$

$$V(x, \alpha y^* + (1-\alpha)z^*) \le \alpha V(x,y^*) + (1-\alpha)V(x,z^*) - \alpha(1-\alpha)\rho_s^*(\|y^* - z^*\|); \tag{45}$$

(iv) *if, in addition, g is bounded on bounded subsets, uniformly convex, and uniformly smooth on bounded subsets of E, then, for any* $x \in E$, $y^*, z^* \in B_s$

$$\rho_s^*(\|x^* - y^*\|) \le D_{g^*}(x^*, y^*). \tag{46}$$

Proof. In view of (24), we get (i). Let us prove (ii). If $x, y \in B_s$ and $\alpha \in (0,1)$, then we obtain

$$\frac{g(\alpha x + (1-\alpha) y) - g(y)}{\alpha} \le g(x) - g(y) - (1-\alpha)\rho_s(\|x-y\|). \tag{47}$$

Letting $\alpha \to 0$ in the above inequality, we arrive at

$$\langle x - y, \nabla g(y)\rangle \le g(x) - g(y) - \rho_s(\|x-y\|). \tag{48}$$

This implies that

$$\rho_s(\|x-y\|) \le D_g(x,y). \tag{49}$$

(iii) Let $x \in E$, $y^*, z^* \in B_s$, and $\alpha \in (0,1)$. Then

$$\begin{aligned} V(x, \alpha y^* + (1-\alpha) z^*) &= g(x) - \langle x, \alpha y^* + (1-\alpha) z^*\rangle \\ &\quad + g^*(\alpha y^* + (1-\alpha) z^*) \\ &\le g(x) - \alpha\langle x, y^*\rangle - (1-\alpha)\langle x, z^*\rangle \\ &\quad + \alpha g^*(y^*) + (1-\alpha) g^*(z^*) \\ &\quad - \alpha(1-\alpha)\rho_s^*(\|y^* - z^*\|) \\ &= \alpha g(x) + (1-\alpha) g(x) - \alpha\langle x, y^*\rangle \\ &\quad - (1-\alpha)\langle x, z^*\rangle \\ &\quad + \alpha g^*(y^*) + (1-\alpha) g^*(z^*) \\ &\quad - \alpha(1-\alpha)\rho_s^*(\|y^* - z^*\|) \\ &= \alpha[g(x) - \langle x, y^*\rangle + g^*(y^*)] \\ &\quad + (1-\alpha)[g(x) - \langle x, z^*\rangle + g^*(z^*)] \\ &\quad - \alpha(1-\alpha)\rho_s^*(\|y^* - z^*\|) \\ &= \alpha V(x, y^*) + (1-\alpha) V(x, z^*) \\ &\quad - \alpha(1-\alpha)\rho_s^*(\|y^* - z^*\|). \end{aligned} \tag{50}$$

(iv) Since g is uniformly smooth on bounded subsets of E, g^* is uniformly convex on bounded subsets of E. Then, in view of (i), there exists a continuous, strictly increasing, and convex function $\rho_s^* : [0,\infty) \to [0,\infty)$ such that

$$\begin{aligned} g^*(\alpha x^* + (1-\alpha) y^*) &\le \alpha g^*(x^*) + (1-\alpha) g^*(y^*) \\ &\quad - \alpha(1-\alpha)\rho_s^*(\|x^* - y^*\|) \end{aligned} \tag{51}$$

for all $x^*, y^* \in B_s^*$ and all $\alpha \in (0,1)$. If $x^*, y^* \in B_s^*$, then we obtain

$$\frac{g^*(\alpha x^* + (1-\alpha) y^*) - g^*(y^*)}{\alpha} \le g^*(x^*) - g^*(y^*) - (1-\alpha)\rho_s^*(\|x^* - y^*\|). \tag{52}$$

Letting $\alpha \to 0$ in the above inequality, we conclude that

$$\langle \nabla g^*(y^*), x^* - y^*\rangle \le g^*(x^*) - g^*(y^*) - \rho_s^*(\|x^* - y^*\|). \tag{53}$$

This implies that

$$\rho_s^*(\|x^* - y^*\|) \le D_{g^*}(x^*, y^*), \tag{54}$$

which completes the proof. □

Lemma 15 (see [35]). *Let E be a Banach space, let $r > 0$ be a constant, and let $g : E \to \mathbb{R}$ be a continuous and convex function which is uniformly convex on bounded subsets of E. Then*

$$g\left(\sum_{k=0}^{\infty} \alpha_k x_k\right) \le \sum_{k=0}^{\infty} \alpha_k g(x_k) - \alpha_i \alpha_j \rho_r(\|x_i - x_j\|) \tag{55}$$

for all $i, j \in \mathbb{N} \cup \{0\}$, $x_k \in B_r$, $\alpha_k \in (0,1)$, and $k \in \mathbb{N} \cup \{0\}$ with $\sum_{k=0}^{\infty} \alpha_k = 1$, where ρ_r is the gauge of uniform convexity of g.

Lemma 16 (see [51]). *Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}_{i\in\mathbb{N}}$ of $\{n\}_{n\in\mathbb{N}}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists a subsequence $\{m_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$:*

$$a_{m_k} \le a_{m_k+1}, \qquad a_k \le a_{m_k+1}. \tag{56}$$

In fact, $m_k = \max\{j \le k : a_j < a_{j+1}\}$.

Lemma 17 (see [52–54]). *Let $\{s_n\}_{n\in\mathbb{N}}$ be a sequence of nonnegative real numbers satisfying the inequality*

$$s_{n+1} \le (1-\gamma_n) s_n + \gamma_n \delta_n, \quad \forall n \ge 1, \tag{57}$$

where $\{\gamma_n\}_{n\in\mathbb{N}}$ and $\{\delta_n\}_{n\in\mathbb{N}}$ satisfy the following conditions:

(i) *$\{\gamma_n\}_{n\in\mathbb{N}} \subset [0,1]$ and $\sum_{n=1}^{\infty} \gamma_n = \infty$, or equivalently, $\prod_{n=1}^{\infty}(1-\gamma_n) = 0$;*

(ii) *$\limsup_{n\to\infty} \delta_n \le 0$;*

(ii)' *$\sum_{n=1}^{\infty} \gamma_n \delta_n < \infty$.*

Then, $\lim_{n\to\infty} s_n = 0$.

3. Strong Convergence Theorems

In this section, we prove a strong convergence theorem concerning the approximation of fixed point of Bregman weak relatively nonexpansive mappings in a reflexive Banach space. We start with the following simple lemma which has been proved in [33].

Lemma 18. *Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a convex, continuous, strongly coercive, and Gâteaux differentiable function which is bounded on bounded subsets and uniformly convex on bounded subsets of E. Let C be a nonempty, closed, and convex subset of E. Let $T : C \to C$ be a Bregman quasinonexpansive mapping. Then $F(T)$ is closed and convex.*

Lemma 19. *Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a convex, continuous, strongly coercive, and Gâteaux differentiable function which is bounded on bounded subsets and uniformly convex on bounded subsets of E. Let C be a nonempty, closed, and convex subset of E and let $\{T_n\}_{n\in\mathbb{N}}$ be an infinite family of Bregman quasinonexpansive mappings from C into itself such that $F := \cap_{n=1}^{\infty} F(T_n) \neq \varnothing$. Let the mapping $T : C \to C$ be defined by*

$$Tx = \lim_{n\to\infty} T_n x. \tag{58}$$

Then, T is a Bregman quasinonexpansive mapping.

Proof. Let $x \in C$ and $p \in F(T)$ be fixed. Then we have that $\{T_n x\}_{n\in\mathbb{N}}$ is a bounded sequence in E. The function g is bounded on bounded subsets of E and, thus, ∇g is also bounded on bounded subsets of E^* (see, e.g., [18, Proposition 1.1.11] for more details). This implies that the sequence $\{\nabla g(T_n x)\}_{n\in\mathbb{N}}$ is bounded in E^*. Since ∇g is uniformly norm-to-norm continuous on any bounded subset of E, we obtain

$$\begin{aligned}
&D_g(p, Tx)\\
&\quad = g(p) - g(Tx) - \langle x - Tx, \nabla g(Tx)\rangle\\
&\quad = g(p) - g\left(\lim_{n\to\infty} T_n x\right)\\
&\qquad - \left\langle x - \lim_{n\to\infty} T_n x, \nabla g\left(\lim_{n\to\infty} T_n x\right)\right\rangle\\
&\quad = \lim_{n\to\infty}\left[g(p) - g(T_n x) - \langle x - T_n x, \nabla g(T_n x)\rangle\right]\\
&\quad = \lim_{n\to\infty} D_g(p, T_n x)\\
&\quad \le D_g(p, x).
\end{aligned} \tag{59}$$

Thus, T is a Bregman quasinonexpansive mapping, which completes the proof. □

Theorem 20. *Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a strongly coercive Bregman function which is bounded on bounded subsets and uniformly convex and uniformly smooth on bounded subsets of E. Let C be a nonempty, closed and convex subset of E and let $\{T_n\}_{n\in\mathbb{N}}$ be an infinite family of Bregman weak relatively nonexpansive mappings from C into itself such that $F := \cap_{n=1}^{\infty} F(T_n) \neq \varnothing$. Suppose in addition that $T_0 = I$, where I is the identity mapping on E. Let $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ be a sequence generated by*

$$\begin{aligned}
x_0 &= x \in C \text{ chosen arbitrarily},\\
C_0 &= C,\\
y_n &= \nabla g^*\left[\alpha_n \nabla g(x_n) + (1-\alpha_n)\nabla g(T_n x_n)\right],\\
C_{n+1} &= \left\{z \in C_n : D_g(z, y_n) \le D_g(z, x_n)\right\},\\
x_{n+1} &= proj^g_{C_{n+1}} x \text{ and } n \in \mathbb{N}\cup\{0\},
\end{aligned} \tag{60}$$

where ∇g is the gradient of g. Let $\{\alpha_n\}_{n\in\mathbb{N}\cup\{0\}}$ be a sequence in $[0,1)$ such that $\liminf_{n\to\infty}\alpha_n(1-\alpha_n) > 0$.

Suppose that for any bounded subset B of C, there exists an increasing, continuous, and convex function $h_B : [0,+\infty) \to [0,+\infty)$ such that $h_B(0) = 0$, and $\lim_{k,l\to\infty}\sup\{h_B(\|T_k z - T_l z\|) : z \in B\} = 0$. Let T be a mapping from C into E defined by $Tx = \lim_{n\to\infty} T_n x$ for all $x \in C$ and suppose that $F(T) = \cap_{n=1}^{\infty} F(T_n) = \cap_{n=1}^{\infty}\widetilde{F}(T_n) = \widetilde{F}(T)$. Then $\{x_n\}_{n\in\mathbb{N}}$, $\{T_n x_n\}_{n\in\mathbb{N}}$, and $\{y_n\}_{n\in\mathbb{N}}$ converge strongly to $proj^g_F x_0$.

Proof. We divide the proof into several steps.

Step 1. We prove that C_n is closed and convex for each $n \in \mathbb{N}\cup\{0\}$.

It is clear that $C_0 = C$ is closed and convex. Let C_m be closed and convex for some $m \in \mathbb{N}$. For $z \in C_m$, we see that

$$D_g(z, y_m) \le D_g(z, x_m) \tag{61}$$

is equivalent to

$$\begin{aligned}
&\langle z, \nabla g(x_m) - \nabla g(y_m)\rangle\\
&\quad \le g(y_m) - g(x_m) + \langle x_m, \nabla g(x_m)\rangle - \langle y_m, \nabla g(y_m)\rangle.
\end{aligned} \tag{62}$$

It could easily be seen that C_{m+1} is closed and convex. Therefore, C_n is closed and convex for each $n \in \mathbb{N}\cup\{0\}$.

Step 2. We claim that $F \subset C_n$ for all $n \in \mathbb{N}\cup\{0\}$.

It is obvious that $F \subset C_0 = C$. Assume now that $F \subset C_m$ for some $m \in \mathbb{N}$. Employing Lemma 12, for any $w \in F \subset C_m$, we obtain

$$\begin{aligned}
&D_g(w, y_m)\\
&\quad = D_g\left(w, \nabla g^*\left[\alpha_m \nabla g(x_m) + (1-\alpha_m)\nabla g(T_m x_m)\right]\right)\\
&\quad = V\left(w, \alpha_m \nabla g(x_m) + (1-\alpha_m)\nabla g(T_m x_m)\right)\\
&\quad = g(w) - \langle w, \alpha_m \nabla g(x_m) + (1-\alpha_m \nabla g(T_m x_m))\rangle\\
&\qquad + g^*\left(\alpha_m \nabla g(x_m) + (1-\alpha_m)\nabla g(T_m x_m)\right)\\
&\quad \le \alpha_m g(w) + (1-\alpha_m) g(w)\\
&\qquad + \alpha_m g^*(\nabla g(x_m)) + (1-\alpha_m) g^*(\nabla g(T_m x_m))\\
&\quad = \alpha_m V(w, \nabla g(x_m)) + (1-\alpha_m) V(w, \nabla g(T_m x_m))\\
&\quad = \alpha_m D_g(w, x_m) + (1-\alpha_m) D_g(w, T_m x_m)\\
&\quad \le \alpha_m D_g(w, x_m) + (1-\alpha_m) D_g(w, x_m)\\
&\quad = D_g(w, x_m).
\end{aligned} \tag{63}$$

This proves that $w \in C_{m+1}$ and hence $F \subset C_n$ for all $n \in \mathbb{N}\cup\{0\}$.

Step 3. We prove that $\{x_n\}_{n\in\mathbb{N}}$, $\{y_n\}_{n\in\mathbb{N}}$, $\{z_n\}_{n\in\mathbb{N}}$, and $\{T_n x_n\}_{n\in\mathbb{N}}$ are bounded sequences in C.

In view of (18), we conclude that

$$\begin{aligned} D_g(x_n, x) &= D_g\left(\operatorname{proj}^g_{C_n} x, x\right) \\ &\le D_g(w, x) - D_g(w, x_n) \\ &\le D_g(w, x), \quad \forall w \in F \subset C_n,\ n \in \mathbb{N} \cup \{0\}. \end{aligned} \tag{64}$$

This implies that the sequence $\{D_g(x_n, x)\}_{n\in\mathbb{N}}$ is bounded and hence there exists $M_1 > 0$ such that

$$D_g(x_n, x) \le M_1, \quad \forall n \in \mathbb{N}. \tag{65}$$

In view of Lemma 8(3), we conclude that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is bounded. Since $\{T_n\}_{n\in\mathbb{N}}$ is an infinite family of Bregman weak relatively nonexpansive mappings from C into itself, we have for any $q \in F$ that

$$D_g(q, T_n x_n) \le D_g(q, x_n), \quad \forall n \in \mathbb{N}. \tag{66}$$

This, together with Definition 7 and the boundedness of $\{x_n\}_{n\in\mathbb{N}}$, implies that the sequence $\{T_n x_n\}_{n\in\mathbb{N}}$ is bounded.

Step 4. We show that $x_n \to u$ for some $u \in F$, where $u = \operatorname{proj}^g_F x$.

From Step 3 it follows that $\{x_n\}_{n\in\mathbb{N}}$ is bounded. By the construction of C_n, we conclude that $C_m \subset C_n$ and $x_m = \operatorname{proj}^g_{C_m} x \in C_m \subset C_n$ for any positive integer $m \ge n$. This, together with (18), implies that

$$\begin{aligned} D_g(x_m, x_n) &= D_g\left(x_m, \operatorname{proj}^g_{C_n} x\right) \\ &\le D_g(x_m, x) - D_g\left(\operatorname{proj}^g_{C_n} x, x\right) \\ &= D_g(x_m, x) - D_g(x_n, x). \end{aligned} \tag{67}$$

In view of (16), we conclude that

$$\begin{aligned} D_g(x_n, x) &= D_g\left(\operatorname{proj}^g_{C_n} x, x\right) \\ &\le D_g(w, x) - D_g(w, x_n) \\ &\le D_g(w, x), \quad \forall w \in F \subset C_n,\ n \in \mathbb{N} \cup \{0\}. \end{aligned} \tag{68}$$

It follows from (68) that the sequence $\{D_g(x_n, x)\}_{n\in\mathbb{N}}$ is bounded and hence there exists $M_2 > 0$ such that

$$D_g(x_n, x) \le M_2, \quad \forall n \in \mathbb{N}. \tag{69}$$

In view of (67), we conclude that

$$\begin{aligned} D_g(x_n, x) &\le D_g(x_n, x) + D_g(x_m, x_n) \\ &\le D_g(x_m, x), \quad \forall m \ge n. \end{aligned} \tag{70}$$

This proves that $\{D_g(x_n, x)\}_{n\in\mathbb{N}}$ is an increasing sequence in $\mathbb{R}$ and hence by (69) the limit $\lim_{n\to\infty} D_g(x_n, x)$ exists. Letting $m, n \to \infty$ in (67), we deduce that $D_g(x_m, x_n) \to 0$. In view of Lemma 11, we obtain that $\|x_m - x_n\| \to 0$ as $m, n \to \infty$. This means that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Since E is a Banach space and C is closed and convex, we conclude that there exists $v \in C$ such that

$$\lim_{n\to\infty} \|x_n - v\| = 0. \tag{71}$$

Now, we show that $v \in F$. In view of (67), we obtain

$$\lim_{n\to\infty} D_g(x_{n+1}, x_n) = 0. \tag{72}$$

Since $x_{n+1} \in C_{n+1}$, we conclude that

$$D_g(x_{n+1}, y_n) \le D_g(x_{n+1}, x_n). \tag{73}$$

This, together with (72), implies that

$$\lim_{n\to\infty} D_g(x_{n+1}, y_n) = 0. \tag{74}$$

It follows from Lemma 11, (72), and (74) that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0, \qquad \lim_{n\to\infty} \|x_{n+1} - y_n\| = 0. \tag{75}$$

In view of (71), we get

$$\lim_{n\to\infty} \|y_n - u\| = 0. \tag{76}$$

From (71) and (76), it follows that

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{77}$$

Since ∇g is uniformly norm-to-norm continuous on any bounded subset of E, we obtain

$$\lim_{n\to\infty} \|\nabla g(x_n) - \nabla g(y_n)\| = 0. \tag{78}$$

Applying Lemma 11, we derive that

$$\lim_{n\to\infty} D_g(y_n, x_n) = 0. \tag{79}$$

It follows from the three point identity (see (39)) that

$$\begin{aligned} &\left|D_g(w, x_n) - D_g(w, y_n)\right| \\ &\quad = \left|D_g(w, y_n) + D_g(y_n, x_n) \right. \\ &\qquad \left. + \langle w - y_n, \nabla g(y_n) - \nabla g(x_n)\rangle - D_g(w, y_n)\right| \\ &\quad = \left|D_g(y_n, x_n) - \langle w - y_n, \nabla g(y_n) - \nabla g(x_n)\rangle\right| \\ &\quad \le D_g(y_n, x_n) + \|w - y_n\| \, \|\nabla g(y_n) - \nabla g(x_n)\| \\ &\quad \longrightarrow 0 \end{aligned} \tag{80}$$

as $n \to \infty$.

The function g is bounded on bounded subsets of E and, thus, ∇g is also bounded on bounded subsets of E^* (see, e.g., [18, Proposition 1.1.11] for more details). This implies that the sequences $\{\nabla g(x_n)\}_{n\in\mathbb{N}}$, $\{\nabla g(y_n)\}_{n\in\mathbb{N}}$, and $\{\nabla g(T_n x_n) : n \in \mathbb{N} \cup \{0\}\}$ are bounded in E^*.

In view of Theorem 10(3), we know that $\operatorname{dom} g^* = E^*$ and g^* is strongly coercive and uniformly convex on bounded subsets. Let $s_1 = \sup\{\|\nabla g(x_n)\|, \|\nabla g(T_n x_n)\| : n \in \mathbb{N} \cup \{0\}\}$

and $\rho_{s_1}^* : E^* \to \mathbb{R}$ be the gauge of uniform convexity of the conjugate function g^*. We prove that for any $w \in F$,

$$D_g(w, y_n) \le D_g(w, x_n) - \alpha_n(1-\alpha_n)\rho_{s_1}^* \times (\|\nabla g(x_n) - \nabla g(T_n x_n)\|). \tag{81}$$

Let us show (81). For any given $w \in F(T)$, in view of the definition of the Bregman distance (see (13)), (9), Lemma 14, we obtain

$$\begin{aligned}
&D_g(w, y_n)\\
&\quad= D_g(w, \nabla g^*[\alpha_n \nabla g(x_n) + (1-\alpha_n)\nabla g(T_n x_n)])\\
&\quad= V(w, \alpha_n \nabla g(x_n) + (1-\alpha_n)\nabla g(T_n x_n))\\
&\quad= g(w) - \langle w, \alpha_n \nabla g(x_n) + (1-\alpha_n)\nabla g(T_n x_n)\rangle\\
&\qquad+ g^*(\alpha_n \nabla g(x_n) + (1-\alpha_n)\nabla g(T_n x_n))\\
&\quad\le \alpha_n g(w) + (1-\alpha_n) g(w) - \alpha_n \langle w, \nabla g(x_n)\rangle\\
&\qquad- (1-\alpha_n)\langle w, \nabla g(T_n x_n)\rangle\\
&\qquad+ \alpha_n g^*(\nabla g(x_n)) + (1-\alpha_n) g^*(\nabla g(T_n x_n))\\
&\qquad- \alpha_n(1-\alpha_n)\rho_{s_1}^*(\|\nabla g(x_n) - \nabla g(T_n x_n)\|)\\
&\quad= \alpha_n V(w, \nabla g(x_n)) + (1-\alpha_n) V(w, \nabla g(T_n x_n))\\
&\qquad- \alpha_n(1-\alpha_n)\rho_{s_1}^*(\|\nabla g(x_n) - \nabla g(T_n x_n)\|)\\
&\quad= \alpha_n D_g(w, x_n) + (1-\alpha_n) D_g(w, T_n x_n)\\
&\qquad- \alpha_n(1-\alpha_n)\rho_{s_1}^*(\|\nabla g(x_n) - \nabla g(T_n x_n)\|)\\
&\quad\le \alpha_n D_g(w, x_n) + (1-\alpha_n) D_g(w, x_n)\\
&\qquad- \alpha_n(1-\alpha_n)\rho_{s_1}^*(\|\nabla g(x_n) - \nabla g(T_n x_n)\|)\\
&\quad= D_g(w, x_n) - \alpha_n(1-\alpha_n)\rho_{s_1}^*(\|\nabla g(x_n) - \nabla g(T_n x_n)\|).
\end{aligned} \tag{82}$$

In view of (80), we obtain

$$D_g(w, x_n) - D_g(w, y_n) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{83}$$

In view of (81) and (83), we conclude that

$$\begin{aligned}
&\alpha_n(1-\alpha_n)\rho_{s_1}^*(\|\nabla g(x_n) - \nabla g(T_n x_n)\|)\\
&\quad\le D_g(w, x_n) - D_g(w, y_n) \longrightarrow 0
\end{aligned} \tag{84}$$

as $n \to \infty$. From the assumption $\liminf_{n\to\infty}\alpha_n(1-\alpha_n) > 0$, we get

$$\lim_{n\to\infty}\rho_{s_1}^*(\|\nabla g(x_n) - \nabla g(T_n x_n)\|) = 0. \tag{85}$$

Therefore, from the property of $\rho_{s_1}^*$ we deduce that

$$\lim_{n\to\infty}\|\nabla g(x_n) - \nabla g(T_n x_n)\| = 0. \tag{86}$$

Since ∇g^* is uniformly norm-to-norm continuous on bounded subsets of E^*, we arrive at

$$\lim_{n\to\infty}\|x_n - T_n x_n\| = 0. \tag{87}$$

From the boundedness of $\{x_n\}_{n\in\mathbb{N}}$, it follows that there exists a bounded subset B of C such that $\{x_n\}_{n\in\mathbb{N}} \subset B$. Let $Tx = \lim_{n\to\infty} T_n x$ for all $x \in C$. In view of Lemma 19, T is a Bregman quasinonexpansive mapping. On the other hand, we have

$$\begin{aligned}
&\frac{1}{2}\|x_n - Tx_n\|\\
&\quad\le \frac{1}{2}\|x_n - T_n x_n\| + \frac{1}{2}\|T_n x_n - Tx_n\|, \quad \forall n \in \mathbb{N}.
\end{aligned} \tag{88}$$

Since $h_B : [0, +\infty) \to [0, +\infty)$ is an increasing, continuous, and convex function, we have

$$\begin{aligned}
&h_B\left(\frac{1}{2}\|x_n - Tx_n\|\right)\\
&\quad\le h_B\left(\frac{1}{2}\|x_n - T_n x_n\|\right) + h_B\left(\frac{1}{2}\|T_n x_n - Tx_n\|\right)\\
&\quad\le \frac{1}{2}h_B(\|x_n - T_n x_n\|)\\
&\qquad+ \frac{1}{2}\sup\{h_B(\|T_n z - Tz\|) : z \in B\}.
\end{aligned} \tag{89}$$

Exploiting Lemma 13 and (87), we obtain

$$\lim_{n\to\infty} h_B(\|x_n - Tx_n\|) = 0. \tag{90}$$

By the properties of h_B, we conclude that

$$\lim_{n\to\infty}\|x_n - Tx_n\| = 0. \tag{91}$$

This, together with Lemma 19 and (71), implies that $v \in F(T) = \cap_{n=1}^{\infty} F(T_n) = F$.

Finally, we show that $v = \mathrm{proj}_F^g x$. From $x_n = \mathrm{proj}_{C_n}^g x$, we conclude that

$$\langle z - x_n, \nabla g(x_n) - \nabla g(x)\rangle \ge 0, \quad \forall z \in C_n. \tag{92}$$

Since $F \subset C_n$ for each $n \in \mathbb{N}$, we obtain

$$\langle z - x_n, \nabla g(x_n) - \nabla g(x)\rangle \ge 0, \quad \forall z \in F. \tag{93}$$

Letting $n \to \infty$ in (93), we deduce that

$$\langle z - v, \nabla g(u) - \nabla g(x)\rangle \ge 0, \quad \forall z \in F. \tag{94}$$

In view of (16), we have $v = \mathrm{proj}_F^g x$, which completes the proof. □

Remark 21. Theorem 20 improves Theorems 1, 2, and 3 in the following aspects.

(1) For the structure of Banach spaces, we extend the duality mapping to more general case, that is, a convex, continuous, and strongly coercive Bregman function which is bounded on bounded subsets, and uniformly convex and uniformly smooth on bounded subsets.

(2) For the mappings, we extend the mapping from a relatively nonexpansive mapping to a countable family of Bregman weak relatively nonexpansive mappings. We remove the assumption $\widehat{F}(T) = F(T)$ on the mapping T and extend the result to a countable family of Bregman weak relatively nonexpansive mappings, where $\widehat{F}(T)$ is the set of asymptotic fixed points of the mapping T.

(3) For the algorithm, we remove the set W_n in Theorem 1.

(4) Theorem 20 extends and improves Theorem 3.1 in [17]. We note that the proof of Theorem 3.3 (lines 24-25) in [17] is not valid in our discussion.

(5) We note also that the main result of the paper cannot be deduced from the results of [35].

We end this section with the following simple example in order to support Theorem 20.

Example 22. Let E, $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ and g be as in Example 4. We define a countable family of mappings $T_j : E \to E$ by

$$T_j(x) = \begin{cases} \dfrac{n}{n+1}x, & \text{if } x = x_n; \\ \dfrac{-j}{j+1}x, & \text{if } x \neq x_n, \end{cases} \tag{95}$$

for all $j \geq 1$ and $n \geq 0$. It is clear that $F(T_j) = \{0\}$ for all $j \geq 1$. Choose $j \in \mathbb{N}$; then, for any $n \in \mathbb{N}$,

$$\begin{aligned} &D_g\left(0, T_j x_n\right) \\ &\quad = g(0) - g\left(T_j x_n\right) - \left\langle 0 - T_j x_n, \nabla g\left(T_j x_n\right)\right\rangle \\ &\quad = -\frac{n^k}{(n+1)^k} g(x_n) + \frac{n^k}{(n+1)^k}\langle x_n, \nabla g(x_n)\rangle \\ &\quad = \frac{n^k}{(n+1)^k}\left[-g(x_n) + \langle x_n, \nabla g(x_n)\rangle\right] \\ &\quad = \frac{n^k}{(n+1)^k}\left[D_g(0, x_n)\right] \\ &\quad \leq D_g(0, x_n). \end{aligned} \tag{96}$$

If $x \neq x_n$, then we have

$$\begin{aligned} &D_g\left(0, T_j x\right) \\ &\quad = g(0) - g\left(T_j x\right) - \left\langle 0 - T_j x, \nabla g\left(T_j x\right)\right\rangle \\ &\quad = -\frac{j^k}{(j+1)^k} g(x) - \frac{j^k}{(j+1)^k}\langle x, -\nabla g(x)\rangle \\ &\quad = \frac{j^k}{(j+1)^k}\left[-g(x) - \langle -x, \nabla g(x)\rangle\right] \\ &\quad \leq D_g(0, x). \end{aligned} \tag{97}$$

Therefore, T_j is a Bregman quasinonexpansive mapping. Next, we claim that T_j is a Bregman weak relatively nonexpansive mapping. Indeed, for any sequence $\{z_n\}_{n\in\mathbb{N}} \subset E$ such that $z_n \to z_0$ and $\|z_n - T_j z_n\| \to 0$ as $n \to \infty$, there exists a sufficiently large number $N_0 \in \mathbb{N}$ such that $z_n \neq x_m$, for any $n, m > N_0$. If we suppose that there exists $m \leq N$ such that $z_n = x_m$ for infinitely many $n \in \mathbb{N}$, then a subsequence $\{x_{n_i}\}_{i\in\mathbb{N}}$ would satisfy $z_{n_i} = x_m$, so $z_0 = \lim_{i\to\infty} z_{n_i} = x_m$ and $z_0 = \lim_{i\to\infty} T_j z_{n_i} = T_j x_m = (m/(m+1))x_m$ which is impossible. This implies that $T_j z_n = -(j/(j+1))z_n$ for all $n > N_0$. It follows from $\|z_n - T_j z_n\| \to 0$ that $((2j+1)/(j+1))z_n \to 0$ and hence $z_n \to z_0 = 0$. Since $z_0 \in F(T_j)$, we conclude that T_j is a Bregman weak relatively nonexpansive mapping. It is clear that $\cap_{j=1}^{\infty}\widetilde{F}(T_j) = \cap_{j=1}^{\infty}F(T_j) = \{0\}$. Thus $\{T_j\}_{j\in\mathbb{N}}$ is a countable family of Bregman weak relatively nonexpansive mappings. Next, we show that $\{T_j\}_{j\in\mathbb{N}}$ is not a countable family of Bregman relatively nonexpansive mappings. In fact, though $x_n \rightharpoonup x_0$ and

$$\left\|x_n - T_j x_n\right\| = \left\|x_n - \frac{n}{n+1}x_n\right\| = \frac{1}{n+1}\|x_n\| \longrightarrow 0 \tag{98}$$

as $n \to \infty$, $x_0 \notin F(T_j)$ for all $j \in \mathbb{N}$. Therefore, $\widehat{F}(T_j) \neq F(T_j)$ for all $j \in \mathbb{N}$. This implies that $\cap_{j=1}^{\infty}\widehat{F}(T_j) \neq \cap_{j=1}^{\infty}F(T_j)$. Let $Tx = \lim_{j\to\infty} T_j x$ for all $x \in E$. It is easy to see that

$$T(x) = \begin{cases} \dfrac{n}{n+1}x, & \text{if } x = x_n; \\ -x, & \text{if } x \neq x_n. \end{cases} \tag{99}$$

In view of Example 4, we obtain that T is a Bregman weak relatively nonexpansive mapping with $F(T) = \{0\} = \widetilde{F}(T)$. Let B be a bounded subset of E. Then there exists $r > 0$ such that $B \subset B_r = \{z \in E : \|z\| \leq r\}$. Let ρ_r be the gauge of uniform convexity of g. Then, in view of Lemma 14, we obtain

$$\rho_r\left(\|x - y\|\right) \leq D_g(x, y), \quad \forall x, y \in B. \tag{100}$$

On the other hand, for any $k, l \in \mathbb{N}$, we have

$$\begin{aligned} &\sup\left\{\|T_k z - T_l z\| : z \in B\right\} \\ &\quad = \sup\left\{\left\|\frac{-k}{k+1}z - \frac{-l}{l+1}z\right\| : z \in B\right\} \\ &\quad = \frac{|k-l|}{(k+1)(l+1)}\sup\left\{\|z\| : z \in B\right\}. \end{aligned} \tag{101}$$

This implies that

$$\lim_{k,l\to\infty}\sup\left\{\|T_k z - T_l z\| : z \in B\right\} = 0. \tag{102}$$

Furthermore, we have

$$\lim_{k,l\to\infty}\sup\left\{\rho_r\left(\|T_k z - T_l z\|\right) : z \in B\right\} = 0. \tag{103}$$

It is clear that, for any $j \in \mathbb{N}$, T_j is not continuous. Finally, it is obvious that the family $\{T_j\}_{j\in\mathbb{N}}$ satisfies all the aspects of the hypothesis of Theorem 20.

4. Equilibrium Problems

Let C be a nonempty, closed, and convex of a reflexive Banach space E. Let $f : C \times C \to \mathbb{R}$ be a bifunction. Consider the following equilibrium problem [55]. Find $p \in C$ such that

$$f(p, y) \geq 0, \quad \forall y \in C. \tag{104}$$

For solving the equilibrium problem, let us assume that $f : C \times C \to \mathbb{R}$ satisfies the following conditions:

(A1) $f(x, x) = 0$ for all $x \in C$;

(A2) f is monotone; that is, $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;

(A3) for each $y \in C$, the function $x \mapsto f(x, y)$ is upper semicontinuous;

(A4) for each $x \in C$, the function $y \mapsto f(x, y)$ is convex and lower semicontinuous.

The set of solutions of problem (104) is denoted by $\mathrm{EP}(f)$.

Let $g : E \to \mathbb{R}$ be a Legendre function. The *resolvent* of a bifunction $f : C \times C \to \mathbb{R}$ [36] is the operator $\mathrm{Res}_f^g : E \to 2^{E^*}$, defined by

$$\mathrm{Res}_f^g(x) = \{z \in C : f(z, y) + \langle y - z, \nabla g(z) - \nabla g(x) \rangle \geq 0 \ \forall y \in C\} \tag{105}$$

for all $x \in E$. We also define the mapping $A_f : E \to 2^{E^*}$ in the following way:

$$A_f(x) = \begin{cases} \{\xi \in E^* : f(x, y) \geq \langle \xi, y - x \rangle \ \forall y \in C\}, & x \in C, \\ \varnothing, & x \notin C. \end{cases} \tag{106}$$

Lemma 23 (see [36, 56]). *Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a convex, continuous, and strongly coercive function which is bounded on bounded subsets and uniformly convex on bounded subsets of E. Let C be a nonempty, closed, and convex subset of E and let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)–(A4) and $\mathrm{EP}(f) \neq \varnothing$. Then, the following statements hold:*

(1) $\mathrm{dom}(Res_f^g) = E$;

(2) *Res_f^g is single-valued;*

(3) *Res_f^g is a Bregman firmly nonexpansive mapping [57]; that is, for all $x, y \in E$,*

$$\left\langle Res_f^g(x) - Res_f^g(y), \nabla g\left(Res_f^g(x)\right) - \nabla g\left(Res_f^g(y)\right) \right\rangle \leq \left\langle Res_f^g(x) - Res_f^g(y), \nabla g(x) - \nabla g(y) \right\rangle; \tag{107}$$

(4) *the set of fixed points of Res_f^g is the solution of the corresponding equilibrium problem; that is, $F(Res_f^g) = \mathrm{EP}(f)$;*

(5) *$EP(f)$ is a closed and convex subset of C;*

(6) *$D_g(q, Res_f^g x) + D_g(Res_f^g x, x) \leq D_g(q, x), \forall q \in F(Res_f^g)$.*

Lemma 24 (see [36]). *Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a convex, continuous, and strongly coercive function which is bounded on bounded subsets and uniformly convex on bounded subsets of E. Let C be a nonempty, closed, and convex subset of E and let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)–(A4) and $EP(f) \neq \varnothing$. Then, the following statements hold:*

(1) *$EP(f) = A_f^{-1}(0^*)$;*

(2) *A_f is a maximal monotone operator;*

(3) *$Res_f^g = Res_{A_f}^g$.*

In this section, we propose a new Halpern-type iterative scheme for finding common zeros of an infinite family of maximal monotone operators and prove the following strong convergence theorem in a Banach space.

Theorem 25. *Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a strongly coercive Bregman function which is bounded on bounded subsets and uniformly convex and uniformly smooth on bounded subsets of E. For any $j \in \mathbb{N}$, let $A_j : E \to 2^{E^*}$ be a maximal monotone operator such that $Z := \cap_{j=1}^{\infty} A_j^{-1}(0^*) \neq \varnothing$. Let $\{\alpha_n\}_{n\in\mathbb{N}}$, $\{\beta_{n,j}\}_{n\in\mathbb{N}, j\in\mathbb{N}\cup\{0\}}$ be sequences in $[0, 1]$ satisfying the following control conditions:*

(a) $\lim_{n\to\infty} \alpha_n = 0$;

(b) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(c) $\beta_{n,0} + \sum_{j=1}^{\infty} \beta_{n,j} = 1, \forall n \in \mathbb{N}$;

(d) $\liminf_{n\to\infty} \beta_{n,0}\beta_{n,j} > 0, \forall j \in \mathbb{N}$.

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence generated by

$$u \in E, \qquad x_1 \in E \text{ chosen arbitrarily},$$
$$y_n = \nabla g^* \left[\beta_{n,0} \nabla g(x_n) + \sum_{j=1}^{\infty} \beta_{n,j} \nabla g\left(Res_{rA_j}^g(x_n)\right) \right],$$
$$x_{n+1} = \nabla g^* \left[\alpha_n \nabla g(u) + (1 - \alpha_n) \nabla g(y_n) \right] \text{ and } n \in \mathbb{N}, \tag{108}$$

where ∇g is the gradient of g and $r > 0$ is a constant. Then the sequence $\{x_n\}_{n\in\mathbb{N}}$ defined in (108) converges strongly to $proj_Z^g u$ as $n \to \infty$.

Proof. We divide the proof into several steps.

Let $z = \mathrm{proj}_Z^g u$. For every $j \in \mathbb{N}$, we denote by T_j the resolvent $\mathrm{Res}_{rf_j}^g$. Therefore,

$$y_n = \nabla g^* \left[\beta_{n,0} \nabla g(x_n) + \sum_{j=1}^{\infty} \beta_{n,j} \nabla g(T_j x_n) \right]. \tag{109}$$

Step 1. We prove that $\{x_n\}_{n\in\mathbb{N}}$, $\{y_n\}_{n\in\mathbb{N}}$, and $\{T_j x_n : n \in \mathbb{N}, j \in \mathbb{N}\}$ are bounded sequences in E. We first show that $\{x_n\}_{n\in\mathbb{N}}$ is bounded. Let $p \in Z$ be fixed. In view of Lemma 12 and (108), we have

$$\begin{aligned}
&D_g(p, y_n) \\
&\quad = D_g\left(p, \nabla g^* \left[\beta_{n,0} \nabla g(x_n) + \sum_{j=1}^{\infty} \beta_{n,j} \nabla g(T_j x_n) \right]\right) \\
&\quad = V\left(p, \beta_{n,0} \nabla g(x_n) + \sum_{j=1}^{\infty} \beta_{n,j} \nabla g(T_j x_n)\right) \\
&\quad \le \beta_{n,0} V(p, \nabla g(x_n)) + \sum_{j=1}^{\infty} \beta_{n,j} V(p, \nabla g(T_j x_n)) \\
&\quad = \beta_{n,0} D_g(p, x_n) + \sum_{j=1}^{\infty} \beta_{n,j} D_g(p, T_j x_n) \\
&\quad \le \beta_{n,0} D_g(p, x_n) + \sum_{j=1}^{\infty} \beta_{n,j} D_g(p, x_n) \\
&\quad = D_g(p, x_n).
\end{aligned} \tag{110}$$

This implies that

$$\begin{aligned}
&D_g(p, x_{n+1}) \\
&\quad = D_g(p, \nabla g^* [\alpha_n \nabla g(u) + (1-\alpha_n) \nabla g(y_n)]) \\
&\quad = V(p, \alpha_n \nabla g(u) + (1-\alpha_n) \nabla g(y_n)) \\
&\quad \le \alpha_n V(p, \nabla g(u)) + (1-\alpha_n) V(p, \nabla g(y_n)) \\
&\quad = \alpha_n D_g(p, u) + (1-\alpha_n) D_g(p, y_n) \\
&\quad \le \alpha_n D_g(p, u) + (1-\alpha_n) D_g(p, y_n) \\
&\quad \le \alpha_n D_g(p, u) + (1-\alpha_n) D_g(p, x_n) \\
&\quad \le \max\{D_g(p, u), D_g(p, x_n)\}.
\end{aligned} \tag{111}$$

By induction, we obtain

$$D_g(p, x_{n+1}) \le \max\{D_g(p, u), D_g(p, x_1)\} \tag{112}$$

for all $n \in \mathbb{N}$. It follows from (112) that the sequence $\{D_g(x_n, x)\}_{n\in\mathbb{N}}$ is bounded and hence there exists $M_3 > 0$ such that

$$D_g(x_n, x) \le M_3, \quad \forall n \in \mathbb{N}. \tag{113}$$

In view of Lemma 11(3), we get that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is bounded. Since $\{T_j\}_{j\in\mathbb{N}}$ is an infinite family of Bregman relatively nonexpansive mappings from E into itself, we conclude that

$$D_g(p, T_j x_n) \le D_g(p, x_n), \quad \forall j, n \in \mathbb{N}. \tag{114}$$

This, together with Definition 7 and the boundedness of $\{x_n\}_{n\in\mathbb{N}}$, implies that the sequence $\{T_j x_n : j, n \in \mathbb{N}\}$ is bounded. The function g is bounded on bounded subsets of E and therefore ∇g is also bounded on bounded subsets of E^* (see, e.g., [18, Proposition 1.1.11] for more details). This, together with Step 1, implies that the sequences $\{\nabla g(x_n)\}_{n\in\mathbb{N}}$, $\{\nabla g(y_n)\}_{n\in\mathbb{N}}$, and $\{\nabla g(T_j x_n) : j, n \in \mathbb{N}\}$ are bounded in E^*. In view of Theorem 10(3), we obtain that $\mathrm{dom}\, g^* = E^*$ and g^* is strongly coercive and uniformly convex on bounded subsets of E. Let $s_2 = \sup\{\|\nabla g(x_n)\|, \|\nabla g(T_j x_n)\| : j, n \in \mathbb{N}\}$ and $\rho_{s_2}^* : E^* \to \mathbb{R}$ be the gauge of uniform convexity of the conjugate function g^*.

Step 2. We prove that for any $j, n \in \mathbb{N}$

$$\begin{aligned}
D_g(z, y_n) \le{}& D_g(z, x_n) \\
&- \beta_{n,0} \beta_{n,j} \rho_{s_2}^* \left(\left\|\nabla g(x_n) - \nabla g(T_j x_n)\right\|\right).
\end{aligned} \tag{115}$$

Let us show (115). For each $n \in \mathbb{N}$, in view of the definition of Bregman distance (see (15)), Lemmas 13, 14, 15, and (110), we obtain

$$\begin{aligned}
&D_g(z, y_n) \\
&\quad = g(z) - g(y_n) - \langle z - y_n, \nabla g(y_n)\rangle \\
&\quad = g(z) + g^*(\nabla g(y_n)) - \langle y_n, \nabla g(y_n)\rangle \\
&\qquad - \langle z, \nabla g(y_n)\rangle + \langle y_n, \nabla g(y_n)\rangle \\
&\quad = g(z) + g^*\left(\beta_{n,0} \nabla g(x_n) + \sum_{j=1}^{\infty} \beta_{n,j} \nabla g(T_j x_n)\right) \\
&\qquad - \left\langle z, \beta_{n,j} \nabla g(x_n) + \sum_{j=1}^{\infty} \beta_{n,j} \nabla g(T_j x_n)\right\rangle \\
&\quad \le \beta_{n,0} g(z) + \sum_{j=1}^{\infty} \beta_{n,j} g(z) + \beta_{n,0} g^*(\nabla g(x_n)) \\
&\qquad + \sum_{j=1}^{\infty} \beta_{n,j} g^*(\nabla g(T_j x_n)) \\
&\qquad - \beta_{n,0} \beta_{n,j} \rho_{s_2}^* \left(\left\|\nabla g(x_n) - \nabla g(T_j x_n)\right\|\right) \\
&\qquad - \beta_{n,0} \langle z, \nabla g(x_n)\rangle - \sum_{j=1}^{\infty} \beta_{n,j} \langle z, \nabla g(T_j x_n)\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \beta_{n,0}\left[g(z) + g^*(\nabla g(x_n)) - \langle z, \nabla g(x_n)\rangle\right] \\
&\quad + \sum_{j=1}^{\infty}\beta_{n,j}\left[g(z) + g^*\left(\nabla g\left(T_j x_n\right)\right) - \left\langle z, \nabla g\left(T_j x_n\right)\right\rangle\right] \\
&\quad - \beta_{n,0}\beta_{n,j}\rho_{s_2}^*\left(\left\|\nabla g(x_n) - \nabla g\left(T_j x_n\right)\right\|\right) \\
&= \beta_{n,0}\left[g(z) - g(x_n) + \langle x_n, \nabla g(x_n)\rangle - \langle z, \nabla g(x_n)\rangle\right] \\
&\quad + \sum_{j=1}^{\infty}\beta_{n,j}\left[g(z) - g\left(T_j x_n\right) + \left\langle T_j x_n, \nabla g\left(T_j x_n\right)\right\rangle\right. \\
&\qquad\qquad \left. - \left\langle z, \nabla g\left(T_j x_n\right)\right\rangle\right] \\
&\quad - \beta_{n,0}\beta_{n,j}\rho_{s_2}^*\left(\left\|\nabla g(x_n) - \nabla g\left(T_j x_n\right)\right\|\right) \\
&= \beta_{n,0}D_g(z, x_n) + \sum_{j=1}^{\infty}\beta_{n,j}D_g\left(z, T_j x_n\right) \\
&\quad - \beta_{n,0}\beta_{n,j}\rho_{s_2}^*\left(\left\|\nabla g(x_n) - \nabla g\left(T_j x_n\right)\right\|\right) \\
&\le \beta_{n,0}D_g(z, x_n) + \sum_{j=1}^{\infty}\beta_{n,j}D_g(z, x_n) \\
&\quad - \beta_{n,0}\beta_{n,j}\rho_{s_2}^*\left(\left\|\nabla g(x_n) - \nabla g\left(T_j x_n\right)\right\|\right) \\
&= D_g(z, x_n) - \beta_{n,0}\beta_{n,j}\rho_{s_2}^*\left(\left\|\nabla g(x_n) - \nabla g\left(T_j x_n\right)\right\|\right).
\end{aligned}
\tag{116}
$$

It then follows from Lemma 12 and (115) that

$$
\begin{aligned}
&D_g(z, x_{n+1}) \\
&\quad = D_g\left(z, \nabla g^*\left[\alpha_n \nabla g(u) + (1-\alpha_n)\nabla g(y_n)\right]\right) \\
&\quad = D_g\left(z, \nabla g^*\left[\alpha_n \nabla g(u) + (1-\alpha_n)\nabla g(y_n)\right]\right) \\
&\quad = V\left(z, \alpha_n \nabla g(u) + (1-\alpha_n)\nabla g(y_n)\right) \\
&\quad \le \alpha_n V(z, \nabla g(u)) + (1-\alpha_n)V(z, \nabla g(y_n)) \\
&\quad = \alpha_n D_g(z, u) + (1-\alpha_n)D_g(z, y_n) \\
&\quad \le \alpha_n D_g(z, u) + (1-\alpha_n)D_g(z, y_n) \\
&\quad \le \alpha_n D_g(z, u) \\
&\qquad + (1-\alpha_n)\left[D_g(z, x_n) - \beta_{n,0}\beta_{n,j}\rho_{s_2}^*\right. \\
&\qquad\qquad \left. \times \left\|\nabla g(x_n) - \nabla g\left(T_j x_n\right)\right\|\right].
\end{aligned}
\tag{117}
$$

Let $M_4 := \sup\{|D_g(z,u) - D_g(z,x_n)| + \beta_{n,0}\beta_{n,j}\rho_{s_2}^*(\|\nabla g(x_n) - \nabla g(T_j x_n)\|) : j, n \in \mathbb{N}\}$. Then we get from (117) that

$$
\begin{aligned}
&\beta_{n,0}\beta_{n,j}\rho_{s_2}^*\left(\left\|\nabla g(x_n) - \nabla g\left(T_j x_n\right)\right\|\right) \\
&\quad \le D_g(z, x_n) - D_g(z, x_{n+1}) + \alpha_n M_4, \quad \forall j \in \mathbb{N}.
\end{aligned}
\tag{118}
$$

In view of Lemma 12 and (115), we obtain

$$
\begin{aligned}
&D_g(z, x_{n+1}) \\
&\quad = D_g\left(z, \nabla g^*\left[\alpha_n \nabla g(u) + (1-\alpha_n)\nabla g(y_n)\right]\right) \\
&\quad = V\left(z, \alpha_n \nabla g(u) + (1-\alpha_n)\nabla g(y_n)\right) \\
&\quad \le V\left(z, \alpha_n \nabla g(u) + (1-\alpha_n)\nabla g(y_n)\right. \\
&\qquad \left. - \alpha_n\left(\nabla g(u) - \nabla g(z)\right)\right) \\
&\qquad - \left\langle \nabla g^*\left[\alpha_n \nabla g(u) + (1-\alpha_n)\nabla g(y_n)\right]\right. \\
&\qquad \left. - z, -\alpha_n\left(\nabla g(u) - \nabla g(z)\right)\right\rangle \\
&\quad = V\left(z, \alpha_n \nabla g(z) + (1-\alpha_n)\nabla g(y_n)\right) \\
&\qquad + \alpha_n\left\langle x_{n+1} - z, \nabla g(u) - \nabla g(z)\right\rangle \\
&\quad = D_g\left(z, \nabla g^*\left[\alpha_n \nabla g(z) + (1-\alpha_n)\nabla g(y_n)\right]\right) \\
&\qquad + \alpha_n\left\langle x_{n+1} - z, \nabla g(u) - \nabla g(z)\right\rangle \\
&\quad \le \alpha_n D_g(z, z) + (1-\alpha_n)D_g(z, y_n) \\
&\qquad + \alpha_n\left\langle x_{n+1} - z, \nabla g(u) - \nabla g(z)\right\rangle \\
&\quad = (1-\alpha_n)D_g(z, x_n) \\
&\qquad + \alpha_n\left\langle x_{n+1} - z, \nabla g(u) - \nabla g(z)\right\rangle.
\end{aligned}
\tag{119}
$$

We will show that $D_g(z, x_n) \to 0$ as $n \to \infty$ by considering two possible cases on the sequence $\{D_g(z, x_n)\}_{n\in\mathbb{N}}$.

Case 1. If $\{D_g(z, x_n)\}_{n\in\mathbb{N}}$ is eventually decreasing, then there exists $n_0 \in \mathbb{N}$ such that $\{D_g(z, x_n)\}_{n=n_0}^{\infty}$ is decreasing and hence $\{D_g(z, x_n)\}_{n\in\mathbb{N}}$ is convergent. Thus, we have $D_g(z, x_n) - D_g(z, x_{n+1}) \to 0$ as $n \to \infty$. This, together with condition (c) and (118), implies that

$$
\lim_{n\to\infty}\rho_{s_2}^*\left(\left\|\nabla g(x_n) - \nabla g\left(T_j x_n\right)\right\|\right) = 0. \tag{120}
$$

Therefore, from the property of $\rho_{s_2}^*$, we deduce that

$$
\lim_{n\to\infty}\left\|\nabla g(x_n) - \nabla g\left(T_j x_n\right)\right\| = 0. \tag{121}
$$

Since ∇g^* is uniformly norm-to-norm continuous on bounded subsets of E^*, we arrive at

$$
\lim_{n\to\infty}\left\|x_n - T_j x_n\right\| = 0, \quad \forall j \in \mathbb{N}. \tag{122}
$$

Since, for any $j \in \mathbb{N}$, T_j is a Bregman relatively nonexpansive mapping, there exists a subsequence $\{x_{n_i}\}_{i\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ converging weakly to some $y \in Z$ such that

$$
\begin{aligned}
&\limsup_{n\to\infty}\left\langle x_{n+1} - z, \nabla g(u) - \nabla g(z)\right\rangle \\
&\quad = \lim_{i\to\infty}\left\langle x_{n_i+1} - z, \nabla g(u) - \nabla g(z)\right\rangle.
\end{aligned}
\tag{123}
$$

This, together with (13), implies that

$$\begin{aligned}\limsup_{n\to\infty}\langle x_{n+1}-z,\nabla g(u)-\nabla g(z)\rangle \\ =\langle y-z,\nabla g(u)-\nabla g(z)\rangle\le 0.\end{aligned}\tag{124}$$

In view of Lemma 11 and (122), we obtain that

$$\lim_{n\to\infty}D_g\left(T_jx_n,x_n\right)=0.\tag{125}$$

This implies that

$$\begin{aligned}D_g\left(T_jx_n,y_n\right)&\le\beta_{n,0}D_g\left(T_jx_n,x_n\right)+\sum_{j=1}^{\infty}\beta_{n,j}D_g\left(T_jx_n,T_jx_n\right)\\&=\beta_{n,0}D_g\left(T_jx_n,x_n\right)\longrightarrow 0\end{aligned}\tag{126}$$

as $n\to\infty$. Observe also that

$$\begin{aligned}D_g\left(y_n,x_{n+1}\right)&\le\alpha_nD_g\left(y_n,u\right)+\left(1-\alpha_n\right)D_g\left(y_n,y_n\right)\\&=\alpha_nD_g\left(y_n,u\right)\longrightarrow 0\end{aligned}\tag{127}$$

as $n\to\infty$. In view of Lemma 11 and (119), (126), and (127), we conclude that

$$\begin{aligned}\lim_{n\to\infty}\left\|y_n-T_jx_n\right\|&=\lim_{n\to\infty}\left\|y_n-x_{n+1}\right\|=0,\\\lim_{n\to\infty}\left\|x_{n+1}-x_n\right\|&=\lim_{n\to\infty}\left\|y_n-x_n\right\|=0.\end{aligned}\tag{128}$$

Moreover, from (124) and (128), we deduce that

$$\begin{aligned}\limsup_{n\to\infty}\langle x_n-z,\nabla g(u)-\nabla g(z)\rangle \\ =\limsup_{n\to\infty}\langle x_{n+1}-z,\nabla g(u)-\nabla g(z)\rangle\le 0.\end{aligned}\tag{129}$$

Thus we have the desired result by Lemma 17.

Case 2. If $\{D_g(z,x_n)\}_{n\in\mathbb{N}}$ is not eventually decreasing, then there exists a subsequence $\{n_i\}_{i\in\mathbb{N}}$ of $\{n\}_{n\in\mathbb{N}}$ such that

$$D_g\left(z,x_{n_i}\right)<D_g\left(z,x_{n_i+1}\right)\tag{130}$$

for all $i\in\mathbb{N}$. Applying Lemma 16, we can find a nondecreasing sequence $\{m_k\}_{k\in\mathbb{N}}\subset\mathbb{N}$ such that $m_k\to\infty$,

$$\begin{aligned}D_g\left(z,x_{m_k}\right)&<D_g\left(z,x_{m_k+1}\right),\\D_g\left(z,x_k\right)&\le D_g\left(z,x_{m_k+1}\right)\end{aligned}\tag{131}$$

for all $k\in\mathbb{N}$. This, together with (118), implies that

$$\begin{aligned}&\beta_{m_k}\left(1-\beta_{m_k}\right)\rho_{s_2}^*\left(\left\|\nabla g\left(x_{m_k}\right)-\nabla g\left(T_jx_{m_k}\right)\right\|\right)\\&\le D_g\left(z,x_{m_k}\right)-D_g\left(z,x_{m_k+1}\right)+\alpha_{m_k}M_4\\&\le\alpha_{m_k}M_4\end{aligned}\tag{132}$$

for all $k\in\mathbb{N}$. Then, by conditions (a) and (c), we get

$$\lim_{k\to\infty}\rho_{s_2}^*\left(\left\|\nabla g\left(x_{m_k}\right)-\nabla g\left(T_jx_{m_k}\right)\right\|\right)=0.\tag{133}$$

By the same argument, as in Case 1, we arrive at

$$\begin{aligned}&\limsup_{k\to\infty}\left\langle x_{m_k+1}-z,\nabla g(u)-\nabla g(z)\right\rangle\\&=\limsup_{k\to\infty}\left\langle x_{m_k}-z,\nabla g(u)-\nabla g(z)\right\rangle\le 0.\end{aligned}\tag{134}$$

It follows from (119) that

$$\begin{aligned}D_g\left(z,x_{m_k+1}\right)\le&\left(1-\alpha_{m_k}\right)D_g\left(z,x_{m_k}\right)\\&+\alpha_{m_k}\left\langle x_{m_k+1}-z,\nabla g(u)-\nabla g(z)\right\rangle.\end{aligned}\tag{135}$$

Since $D_g(z,x_{m_k})\le D_g(z,x_{m_k+1})$, it follows that

$$\begin{aligned}\alpha_{m_k}D_g\left(z,x_{m_k}\right)&\le D_g\left(z,x_{m_k}\right)-D_g\left(z,x_{m_k+1}\right)\\&\quad+\alpha_{m_k}\left\langle x_{m_k+1}-z,\nabla g(u)-\nabla g(z)\right\rangle\\&\le\alpha_{m_k}\left\langle x_{m_k+1}-z,\nabla g(u)-\nabla g(z)\right\rangle.\end{aligned}\tag{136}$$

In particular, since $\alpha_{m_k}>0$, we obtain

$$D_g\left(z,x_{m_k}\right)\le\left\langle x_{m_k+1}-z,\nabla g(u)-\nabla g(z)\right\rangle.\tag{137}$$

In view of (135), we deduce that

$$\lim_{k\to\infty}D_g\left(z,x_{m_k}\right)=0.\tag{138}$$

This, together with (136), implies that

$$\lim_{k\to\infty}D_g\left(z,x_{m_k+1}\right)=0.\tag{139}$$

On the other hand, we have $D_g(z,x_k)\le D_g(z,x_{m_k+1})$ for all $k\in\mathbb{N}$ which implies that $x_k\to z$ as $k\to\infty$. Thus, we have $x_n\to z$ as $n\to\infty$. □

In the following, we propose a new Halpern-type iterative scheme for finding common solutions of a system of equilibrium problems in a reflexive Banach space and obtain a strong convergence theorem.

Theorem 26. *Let E be a reflexive Banach space and let $g:E\to\mathbb{R}$ be a strongly coercive Bregman function which is bounded on bounded subsets and uniformly convex and uniformly smooth on bounded subsets of E. Let $\{C_j\}_{j\in\mathbb{N}}$ be an infinite family of nonempty, closed, and convex subsets of E. For any $j\in\mathbb{N}$, let $f_j:C_j\times C_j\to\mathbb{R}$ be a bifunction that satisfies conditions (A1)–(A4) such that $Z:=\cap_{j=1}^{\infty}EP(f_j)\neq\varnothing$, where $EP(f_j)$ is the set of solutions to the equilibrium problem* (104). *Let $\{\alpha_n\}_{n\in\mathbb{N}}$, $\{\beta_{n,j}\}_{n\in\mathbb{N},j\in\mathbb{N}\cup\{0\}}$ be sequences in $[0,1]$ satisfying the following control conditions:*

(a) $\lim_{n\to\infty}\alpha_n=0$;

(b) $\sum_{n=1}^{\infty}\alpha_n=\infty$;

(c) $\beta_{n,0} + \sum_{j=1}^{\infty} \beta_{n,j} = 1, \quad \forall n \in \mathbb{N}$;

(d) $\liminf_{n\to\infty} \beta_{n,0}\beta_{n,j} > 0, \forall j \in \mathbb{N}$.

Let $\{x_n\}_{n\in\mathbb{N}}$ *be a sequence generated by*

$$u \in E, \quad x_1 \in E \text{ chosenarbitrarily},$$
$$y_n = \nabla g^* \left[\beta_{n,0} \nabla g(x_n) + \sum_{j=1}^{\infty} \beta_{n,j} \nabla g\left(Res^g_{rf_j}(x_n)\right) \right],$$
$$x_{n+1} = \nabla g^* \left[\alpha_n \nabla g(u) + (1-\alpha_n) \nabla g(y_n) \right], \quad n \in \mathbb{N}, \tag{140}$$

where ∇g *is the gradient of* g *and* $r > 0$ *is a constant. Then, the sequence* $\{x_n\}_{n\in\mathbb{N}}$ *defined in* (140) *converges strongly to* $proj^g_Z u$ *as* $n \to \infty$.

Remark 27. (1) We propose a new type of Halpern iterative scheme for finding common zeros of an infinite family of maximal monotone operators in a reflexive Banach space. This scheme has an advantage that we do not use any projection which creates some difficulties in a practical calculation of the iterative sequence.

(2) In Theorem 25, we present a strong convergence theorem for an infinite family of maximal monotone operators with a new algorithm and new control conditions. We remove the sets C_n and Q_n in Theorems 5 and 6.

(3) Theorem 20 improves and extends the corresponding results of [45, 46] from one maximal monotone operator in Hilbert spaces to more general an infinite family of maximal monotone operators in Banach spaces.

(4) Theorem 20 improves and extends the corresponding result of [47] from two maximal monotone operators in Hilbert spaces to more general an infinite family of maximal monotone operators in Banach spaces.

(5) Theorems 25 and 26 improve and generalize Theorems 5 and 6, respectively.

5. Applications

In this section, we consider the problem of finding a minimizer of a continuously Fréchet differentiable and convex function in a Banach space.

Theorem 28. *Let* E *be a reflexive Banach space and let* $g : E \to \mathbb{R}$ *be a convex, continuous, and strongly coercive function which is bounded on bounded subsets and uniformly convex on bounded subsets of* E. *Let* $\{g_j\}_{j\in\mathbb{N}}$ *be an infinite family of continuously Fréchet differentiable and convex functions on* E *such that the gradient of* g_j, ∇g_j *is continuous and monotone for each* $j \in \mathbb{N}$. *Assume that* $\Omega := \cap_{j=1}^{\infty} \arg\min_{y\in E} g_j(y) = \{z \in E : g_j(z) = \cap_{j=1}^{\infty} \min_{y\in C} g_j(y)\} \neq \varnothing$. *Let* $\{\alpha_n\}_{n\in\mathbb{N}}$, $\{\beta_{n,j}\}_{n\in\mathbb{N}, j\in\mathbb{N}\cup\{0\}}$ *be sequences in* $[0,1]$ *satisfying the following control conditions:*

(a) $\lim_{n\to\infty} \alpha_n = 0$;

(b) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(c) $\beta_{n,0} + \sum_{j=1}^{\infty} \beta_{n,j} = 1, \forall n \in \mathbb{N}$;

(d) $\liminf_{n\to\infty} \beta_{n,0}\beta_{n,j} > 0, \forall j \in \mathbb{N}$.

Let $\{x_n\}_{n\in\mathbb{N}}$ *be a sequence generated by*

$$u \in E, \quad x_1 \in C \text{ chosen arbitrarily},$$
$$u_{n,j} \in C \text{ suchthat } \left\langle y - u_{n,j}, \nabla g_j\left(u_{n,j}\right)\right\rangle + \left\langle y - u_{n,j}, \nabla g\left(u_{n,j}\right) - \nabla g(x_n)\right\rangle \geq 0, \quad \forall y \in C,$$
$$y_n = \nabla g^* \left[\beta_{n,0} \nabla g(x_n) + \sum_{j=1}^{\infty} \beta_{n,j} \nabla g\left(u_{n,j}\right) \right],$$
$$x_{n+1} = \nabla g^* \left[\alpha_n \nabla g(u) + (1-\alpha_n) \nabla g(y_n) \right] \text{ and } n \in \mathbb{N}. \tag{141}$$

Then the sequence $\{x_n\}_{n\in\mathbb{N}}$ *defined in* (141) *converges strongly to* $proj^g_\Omega u$ *as* $n \to \infty$.

Conflict of Interests

The authors declare that they have no competing interests.

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