Research Article

Spectral Distribution of Transport Operator Arising in Growing Cell Populations

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Received 27 May 2014; Accepted 12 August 2014; Published 25 August 2014

Academic Editor: Leszek Olszowy

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Transport equation with partly smooth boundary conditions arising in growing cell populations is studied in L_p ($1 < p < +\infty$) space. It is to prove that the transport operator A_H generates a C_0 semigroup and the ninth-order remainder term $R_9(t)$ of the Dyson-Phillips expansion of the semigroup is compact, and the spectrum of transport operator A_H consists of only finite isolated eigenvalues with finite algebraic multiplicities in a trip Γ_ω . The main methods rely on theory of linear operators, comparison operators, and resolvent operators approach.

1. Related Knowledge

In this paper, we are concerned with the following transport equation, which was proposed by Rotenberg in [1]:

$$\begin{aligned} \frac{\partial \psi}{\partial t}(\mu, v, t) = & -v \frac{\partial \psi}{\partial \mu}(\mu, v, t) - \sigma(\mu, v) \psi(\mu, v, t) \\ & + \int_0^c r(\mu, v, v') \psi(\mu, v', t) dv', \end{aligned} \quad (1)$$

with the initial condition and the general biological rule

$$\begin{aligned} \psi(\mu, v, 0) &= \psi_0(\mu, v), \\ \psi^0(0, v, t) &= H\psi^a(a, v, t), \end{aligned} \quad (2)$$

where μ is maturity degree of cells, $\mu \in [0, a]$, $a > 0$; the degree of maturation μ is then defined in the manner that $\mu = 0$ at the birth and $\mu = a$ at the death, and their maturation velocity $v, v' \in [0, c]$, $c > 0$, $\psi(\mu, v, t)$ describes the number density of cell population as a function of the degree of maturation μ ; the maturation velocity v and the time t , $\sigma(\mu, v)$ denote the total transition cross-section while the function $r(\mu, v, v')$ represents the transition rate at which cells change their velocities from v to v' , $\psi(\mu, v, 0) = \psi_0(\mu, v)$ is initial condition, H is linear operator in boundary space and is known as transition rule in biology.

It is well known that the streaming operator generates a strongly continuous semigroup $(U_H(t))_{t \geq 0}$ (see, e.g., [2–5]). If the collision operator is bounded, then the classical perturbation theory (see, [6–8]) shows that the transport operator generates also a strongly continuous semigroup $(V_H(t))_{t \geq 0}$ (see, e.g., [9]) given by the Dyson-Phillips expansion:

$$V_H(t) = \sum_{j=0}^{n-1} U_j(t) + R_n(t), \quad (3)$$

where $U_0(t) = U_H(t)$ and

$$U_i(t) = \int_0^t U_H(s) K U_{j-1}(t-s) ds, \quad j = 1, 2, \dots, \quad (4)$$

where K represents disturbance operators, and the remainder term is given by

$$\begin{aligned} R_n(t) &= \sum_{j=n}^{+\infty} U_j(t) \\ &= \int_{t_1 + \dots + t_n \leq t, t_i \geq 0} U_H(t_1) \\ &\quad \times K U_H(t_2) K \dots U_H(t_n) K \\ &\quad \times V_H(t - t_1 \dots - t_n) dt_1 \dots dt_n. \end{aligned} \quad (5)$$

As is well known, the solution $\psi(t)$ of (1)-(2) exists and is unique for all $\psi_0 \in D(A_H)$; here A_H represents transport operator A with boundary condition H .

For the past few years, there are many research works on the spectrum analysis of (1), and some of them had been discussed in [6, 10–16]. Jeribi et al. [10] discussed Rotenberg's model of cell population with general compact boundary conditions and proved that the transport semigroup was irreducible, and a spectral decomposition of the solution into an asymptotic term was derived. Latrach and Megdiche [11] discussed the large time behavior of the solution to the Cauchy problem governed by a transport equation with Maxwell compact boundary conditions arising in growing cell population in L_1 spaces and proved that the remainder term was compact and got

$$r_e(V_H(t)) = r_e(U_H(t)). \quad (6)$$

Wang and Cheng [13] had proved that the transport operator with compact integral boundary conditions generated a C_0 semigroup and the existence of the eigenvalues in L_p ($1 \leq p < +\infty$) space in L-R's model. So, it is natural to set the following question: what happens about the transport operator spectral distribution in Rotenberg's model when boundary conditions are partly compact? In this paper, we discuss the transport equation with partly smooth boundary conditions arising in growing cell populations in L_p ($1 < p < +\infty$) space. It is to prove that the transport operator generates a C_0 semigroup and the ninth-order remainder term $R_9(t)$ of the Dyson-Phillips expansion of the semigroup is compact, and the spectrum of transport operator consists of only finite isolated eigenvalues with finite algebraic multiplicities in a trip Γ_ω .

Now, we introduce different notions and notations. Let us first make precisely the functional setting of the problem. Let

$$X_p = L_p \quad (1 < p < +\infty) \quad ([0, a] \times [0, c]; d\mu dv), \quad (7)$$

whose norm is

$$\|\psi\|_{X_p} = \left(\int_0^a \int_0^c |\psi(\mu, v)| d\mu dv \right)^{1/p}. \quad (8)$$

We denote by X_p^0, X_p^a the following boundary spaces:

$$X_p^0 = L_p \quad (\Gamma_0; v dv), \quad X_p^a = L_p \quad (\Gamma_a; v dv), \quad (9)$$

where

$$\Gamma_0 = \{0\} \times [0, c], \quad \Gamma_a = \{a\} \times [0, c]. \quad (10)$$

In the sequel, X_p^0 and X_p^a will often be identified with $L_p([0, c]; v dv)$. We define the partial Sobolev space W_p as follows:

$$W_p = \left\{ \psi \in X_p, v \frac{\partial \psi}{\partial \mu} \in X_p \right\}. \quad (11)$$

We define the disturbance operators K and streaming operator T_H by

$$(K\psi)(\mu, v) = \int_0^c r(\mu, v, v') \psi(\mu, v') dv', \quad (12)$$

$$T_H \psi(\mu, v) = -v \frac{\partial \psi}{\partial \mu}(\mu, v) - \sigma(\mu, v) \psi(\mu, v), \quad (13)$$

$$D(T_H) = \{\psi \in W_p \mid \psi^0 = H\psi^a\}.$$

So we may define the transport operator A_H by

$$A_H = T_H + K, \quad D(A_H) = D(T_H). \quad (14)$$

Let σ_0 be the real defined by

$$\sigma_0 = \text{ess inf } \{\sigma(\mu, v), (\mu, v) \in [0, a] \times [0, c]\}. \quad (15)$$

Consider now the resolvent equation for operator T_H :

$$(\lambda I - T_H)\psi = \varphi, \quad (16)$$

where φ is a given function of X_p . For $\text{Re } \lambda > -\sigma_0$ the solution is formally given by

$$\begin{aligned} \psi(\mu, v) &= \psi(0, v) e^{-(1/v) \int_0^\mu (\lambda + \sigma(\mu', v)) d\mu'} \\ &\quad + \frac{1}{v} \int_0^\mu e^{-(1/v) \int_{\mu'}^\mu (\lambda + \sigma(\xi, v)) d\xi} \varphi(\mu', v) d\mu'. \end{aligned} \quad (17)$$

Accordingly, for $\mu = a$, we get

$$\begin{aligned} \psi(a, v) &= \psi(0, v) e^{-(1/v) \int_0^a (\lambda + \sigma(\mu', v)) d\mu'} \\ &\quad + \frac{1}{v} \int_0^a e^{-(1/v) \int_{\mu'}^a (\lambda + \sigma(\xi, v)) d\xi} \varphi(\mu', v) d\mu'. \end{aligned} \quad (18)$$

Now, let us define the transport operators $P_\lambda, Q_\lambda, D_\lambda$, and E_λ by

$$\begin{aligned} P_\lambda &: X_p^0 \longrightarrow X_p^a, \\ (P_\lambda g)(a, v) &= g(0, v) e^{-(1/v) \int_0^a (\lambda + \sigma(\mu', v)) d\mu'}, \\ Q_\lambda &: X_p^0 \longrightarrow X_p, \\ (Q_\lambda g)(\mu, v) &= g(0, v) e^{-(1/v) \int_0^\mu (\lambda + \sigma(\mu', v)) d\mu'}, \\ D_\lambda &: X_p \longrightarrow X_p^a, \\ (D_\lambda g)(a, v) &= \frac{1}{v} \int_0^a e^{-(1/v) \int_{\mu'}^a (\lambda + \sigma(\xi, v)) d\xi} g(\mu', v) d\mu', \\ E_\lambda &: X_p \longrightarrow X_p, \\ (E_\lambda g)(\mu, v) &= \frac{1}{v} \int_0^\mu e^{-(1/v) \int_{\mu'}^\mu (\lambda + \sigma(\xi, v)) d\xi} g(\mu', v) d\mu'. \end{aligned} \quad (19)$$

Clearly, for λ satisfying $\operatorname{Re} \lambda > -\sigma_0$, the operators P_λ , Q_λ , D_λ , and E_λ are bounded and positive, and it is not difficult to check that

$$\begin{aligned} \|D_\lambda\| &\leq 1, \\ \|P_\lambda\| &\leq e^{-(a/c)(\operatorname{Re} \lambda + \sigma_0)}, \\ \|Q_\lambda\| &\leq (\operatorname{Re} \lambda + \sigma_0)^{-1}, \\ \|E_\lambda\| &\leq (\operatorname{Re} \lambda + \sigma_0)^{-1}. \end{aligned} \quad (20)$$

Because of (18), we can write

$$\psi^a = P_\lambda K \psi^a + D_\lambda \varphi. \quad (21)$$

Let λ_0 be a real number defined by

$$\lambda_0 = \begin{cases} -\sigma_0, & \text{if } \|H\| \leq 1, \\ -\sigma_0 + \frac{c}{a} \log(\|H\|), & \text{if } \|H\| > 1. \end{cases} \quad (22)$$

Clearly, for $\operatorname{Re} \lambda > -\lambda_0$, we get

$$\psi^a = (I - P_\lambda K)^{-1} D_\lambda \varphi. \quad (23)$$

Because of (17) and (23) we get

$$\psi = Q_\lambda K (I - P_\lambda K)^{-1} D_\lambda \varphi + E_\lambda \varphi. \quad (24)$$

Accordingly, for $\operatorname{Re} \lambda > -\lambda_0$, the resolvent of the operator T_H is given by

$$(\lambda I - T_H)^{-1} = \sum_{n \geq 0} Q_\lambda H (P_\lambda H)^n D_\lambda + E_\lambda. \quad (25)$$

In the following, we assume that H and K satisfy the following:

- (O₁) $H = H_1 + H_2$, where H_1 is bounded and positive operator and H_2 is compact and positive operator;
- (O₂) K is regular operator in X_p . So it can be approximated in the uniform operator topology by operators; then

$$\begin{aligned} &\int_0^c r(\mu, \nu, \nu') \psi(\mu, \nu') d\nu' \\ &= \sum_{i \in I} \alpha_i(\mu) f_i(\nu) \int_0^c g_i(\nu') \varphi(\mu, \nu') d\nu', \end{aligned} \quad (26)$$

where $\alpha_i(\cdot) \in L_\infty([0, a], d\mu)$, $f_i(\cdot) \in L_p([0, c], d\nu)$, $g_i(\cdot) \in L_q([0, c], d\nu)$, $(1/p) + (1/q) = 1$, and I is finite set.

Indeed, let us first observe that if we replace in the definition of T_H the function $\sigma(\mu, \nu)$ by the real σ_0 , we obtain a new streaming operator which we denote by $\overline{T}_H$. Arguing as above we can define the operators $\overline{P}_\lambda$, $\overline{Q}_\lambda$, $\overline{D}_\lambda$, and $\overline{E}_\lambda$, which satisfy, for any $\operatorname{Re} \lambda > \lambda_0$,

$$(\lambda I - \overline{T}_H)^{-1} = \sum_{n \geq 0} \overline{Q}_\lambda H (\overline{P}_\lambda H)^n \overline{D}_\lambda + \overline{E}_\lambda. \quad (27)$$

Lemma 1. For any $t > 0$, $n \in \mathbf{N}$, then

$$0 \leq R_n(t) \leq \overline{R_n(t)}, \quad (28)$$

where $R_n(t)$ and $\overline{R_n(t)}$ denote the n th remainder term of the Dyson-Phillips expansion of the semigroup generated by A_H and $\overline{A}_H$.

Proof. For all $\operatorname{Re} \lambda > \lambda_0$, a simple calculation shows that

$$\begin{aligned} \overline{M}_\lambda &\geq M_\lambda, & \overline{N}_\lambda &\geq N_\lambda, \\ \overline{D}_\lambda &\geq D_\lambda, & \overline{E}_\lambda &\geq E_\lambda. \end{aligned} \quad (29)$$

For (25), (27), and (29) we get

$$(\lambda I - T_H)^{-1} \leq (\lambda I - \overline{T}_H)^{-1}. \quad (30)$$

Let $t > 0$ be a fixed real; by (30), it is obvious that, for all integer $n \in \mathbf{N}$ and all $\psi \in (X_p)_+$ such that $n/t > \lambda$, we have

$$0 \leq \frac{n}{t} \left(\frac{n}{t} - T_H \right)^{-1} \psi \leq \frac{n}{t} \left(\frac{n}{t} - \overline{T}_H \right)^{-1} \psi. \quad (31)$$

Consequently,

$$\begin{aligned} 0 &\leq \lim_{n \rightarrow +\infty} \frac{n}{t} \left(\frac{n}{t} - T_H \right)^{-1} \psi \\ &\leq \lim_{n \rightarrow +\infty} \frac{n}{t} \left(\frac{n}{t} - \overline{T}_H \right)^{-1} \psi. \end{aligned} \quad (32)$$

Next, in view of the exponential formula for strongly continuous semigroups, we have

$$0 \leq U_H(t) \leq \overline{U_H(t)}, \quad (33)$$

where $U_H(t)_{t \geq 0}$ and $\overline{U_H(t)}_{t \geq 0}$ denote the strongly continuous semigroup generated by T_H and $\overline{T}_H$. The positivity of K and (30) imply that

$$\begin{aligned} 0 &\leq [(\lambda I - T_H)^{-1} K]^n (\lambda I - T_H) \\ &\leq [(\lambda I - \overline{T}_H)^{-1} K]^n (\lambda I - \overline{T}_H). \end{aligned} \quad (34)$$

Now, we define transport operator $\overline{A}_H$ by

$$\overline{A}_H = \overline{T}_H + K, \quad D(\overline{A}_H) = D(\overline{T}_H). \quad (35)$$

Because

$$\begin{aligned} &(\lambda I - \overline{T}_H - K)^{-1} \\ &= \sum_{n \geq 1} [(\lambda I - \overline{T}_H)^{-1} K]^n (\lambda I - \overline{T}_H), \end{aligned} \quad (36)$$

therefore

$$0 \leq (\lambda I - A_H)^{-1} \leq (\lambda I - \overline{A}_H)^{-1}. \quad (37)$$

The exponential formula for strongly continuous semigroups leads to

$$0 \leq V_H(t) \leq \overline{V_H(t)}, \quad (38)$$

where $V_H(t)_{t \geq 0}$ and $\overline{V_H(t)}_{t \geq 0}$ denote the strongly continuous semigroup generated by A_H and $\overline{A_H}$. Because of (12),

$$\begin{aligned} \overline{R_n(t)} &= \int_{t_1 + \dots + t_n \leq t, t_i \geq 0} \overline{U_H(t_1)} \\ &\quad \times \overline{KU_H(t_2)K \dots U_H(t_n)K} \\ &\quad \times \overline{V_H(t - t_1 \dots - t_n)} dt_1 \dots dt_n. \end{aligned} \quad (39)$$

As a consequence of (12)–(33)–(38)–(39), we have (28) immediately. $\square$

Lemma 2 (see [9]). *Let B be the generator of a strongly continuous semigroup $U(t)$ on a Banach space X , and denote the K bounded linear operators in X . Assume that there exist $m \in \mathbb{N}$ and $\eta > \omega(U)$ satisfying the following:*

- (1) $(\lambda I - B)^{-1} [K(\lambda I - B)^{-1}]^m$ is compact for all λ such that $\operatorname{Re} \lambda \geq \eta$;
- (2) for every λ satisfying $\{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda \geq \eta\}$, then

$$\lim_{|\operatorname{Im} \lambda| \rightarrow +\infty} |\operatorname{Im} \lambda| \left\| (\lambda I - B)^{-1} [K(\lambda I - B)^{-1}]^m \right\| = 0. \quad (40)$$

Then $R_{2m+1}(t)$ is compact on X for each $t > 0$.

Lemma 3 (see [12]). *Let (Ω, Σ, μ) be a positive measure space and let S, T be bounded linear operators on (Ω, μ) . If T is compact and $0 \leq S \leq T$, then S is compact.*

2. Main Result

In this section we are ready to prove the main result of this paper. Let

$$\Gamma_\omega = \{\lambda \in \mathbb{C}, \operatorname{Re} \lambda \geq \lambda_0 + \omega\} \quad (\omega > 0). \quad (41)$$

Theorem 4. *If assumptions O_1 and O_2 are satisfied, for $r \in [0, 1)$, then*

$$\lim_{|\operatorname{Im} \lambda| \rightarrow +\infty} |\operatorname{Im} \lambda|^r \left\| K(\lambda I - T_H)^{-1} K \right\| = 0, \quad (42)$$

uniformly on Γ_ω .

Proof. Now we are going to divide the proof into several steps.

Step One. For any $\lambda \in \Gamma_\omega$, prove that

$$\lim_{|\operatorname{Im} \lambda| \rightarrow +\infty} |\operatorname{Im} \lambda|^r \left\| K \overline{E_\lambda} K \right\| = 0, \quad (43)$$

uniformly on Γ_ω . For all $\varphi \in X_p$,

$$\overline{E_\lambda} \varphi(\mu, \nu) = \frac{1}{\nu} \int_0^\mu e^{-(\lambda + \sigma_0)/\nu (\mu - \mu')} \varphi(\mu', \nu) d\mu'. \quad (44)$$

Using the change of variable $t = (\mu - \mu')/\nu$, for all $\varphi \in X_p$, we have

$$\overline{E_\lambda} \varphi(\mu, \nu) = \int_0^{+\infty} e^{-(\lambda + \sigma_0)t} \varphi(\mu - t\nu, \nu) \chi_{(0, \mu/\nu)} dt. \quad (45)$$

Now, we define a bounded linear operator $\overline{E_{\lambda, \varepsilon}} \varphi(\mu, \nu)$ by

$$\overline{E_{\lambda, \varepsilon}} \varphi(\mu, \nu) = \int_\varepsilon^{1/\varepsilon} e^{-(\lambda + \sigma_0)t} \varphi(\mu - t\nu, \nu) \chi_{(0, \mu/\nu)} dt. \quad (46)$$

For all $\varphi \in X_p$, clearly, $(\overline{E_{\lambda, \varepsilon}})_{\varepsilon > 0}$ converges in the operator topology to $\overline{E_\lambda}$ uniformly on Γ_ω as $\varepsilon \rightarrow 0$. So, it suffices to show that, for $\varepsilon > 0$,

$$\lim_{|\operatorname{Im} \lambda| \rightarrow +\infty} |\operatorname{Im} \lambda|^r \left\| K \overline{E_{\lambda, \varepsilon}} K \right\| = 0, \quad (47)$$

uniformly on Γ_ω . Putting

$$s = \mu - t\nu', \quad w(\cdot) = f(\cdot) g(\cdot), \quad (48)$$

we get

$$\begin{aligned} K \overline{E_{\lambda, \varepsilon}} K \varphi(\mu, \nu) &= \alpha(\mu) f(\nu) \int_0^a ds \int_\varepsilon^{1/\varepsilon} \frac{dt}{t} \\ &\quad \times e^{-(\lambda + \sigma_0)t} w\left(\frac{\mu - s}{t}\right) \alpha(s) \\ &\quad \times \int_0^c dv'' g(v'') \varphi(s, v'') \chi_{(\mu - tc, \mu)}(s). \end{aligned} \quad (49)$$

Let $K \overline{E_{\lambda, \varepsilon}} K = E_1 \cdot E_\varepsilon \cdot E_2$, where

$$E_1 : L_p[(0, a)(0, a)] \longrightarrow X_p,$$

$$E_1 \varphi(\mu, \nu) = \alpha(\mu) f(\nu) \int_0^a \varphi(s, \nu) ds,$$

$$E_2 : X_p \longrightarrow L_p(0, a),$$

$$E_2 \varphi(\mu) = \alpha(\mu) f(\nu) \int_0^c g(\nu) \varphi(\mu, \nu) d\nu, \quad (50)$$

$$E_\varepsilon : X_p \longrightarrow X_p,$$

$$\begin{aligned} (E_\varepsilon \varphi)(\mu, s) &= \int_\varepsilon^{1/\varepsilon} \frac{dt}{t} e^{-(\lambda + \sigma_0)t} \varphi(s) \\ &\quad \times w\left(\frac{\mu - s}{t}\right) \chi_{(\mu - tc, \mu)}(s). \end{aligned}$$

Since E_1 and E_2 are bounded operators, it suffices to prove that

$$\lim_{|\operatorname{Im} \lambda| \rightarrow +\infty} |\operatorname{Im} \lambda|^{pr} \|E_\varepsilon\|^p = 0, \quad (51)$$

uniformly on Γ_ω . For all $\varphi \in L_p(0, a)$,

$$\begin{aligned} \|E_\varepsilon \varphi\|^p &= \int_0^a d\mu \left| \int_0^a ds \int_\varepsilon^{1/\varepsilon} \frac{dt}{t} e^{-(\lambda+\sigma_0)t} \right. \\ &\quad \times \left. w\left(\frac{\mu-s}{t}\right) \varphi(s) \chi_{(\mu-tc, \mu)}(s) \right|^p. \end{aligned} \quad (52)$$

Applying the Hölder inequality we get

$$\begin{aligned} &\left| \int_0^a ds \int_\varepsilon^{1/\varepsilon} \frac{dt}{t} e^{-(\lambda+\sigma_0)t} \right. \\ &\quad \times \left. w\left(\frac{\mu-s}{t}\right) \varphi(s) \chi_{(\mu-tc, \mu)}(s) \right|^p \\ &\leq \left(\int_0^a ds \left| \int_\varepsilon^{1/\varepsilon} \frac{dt}{t} \exp(-(\lambda+\sigma_0)t) \right. \right. \\ &\quad \times \left. \left. w\left(\frac{\mu-s}{t}\right) \chi_{(\mu-tc, \mu)}(s) \right|^q \right)^{p/q} \\ &\quad \times \int_0^a |\varphi(s)|^p ds \\ &= a^{p/q} \left| \int_\varepsilon^{1/\varepsilon} \frac{dt}{t} e^{-(\lambda+\sigma_0)t} \right. \\ &\quad \times \left. w\left(\frac{\mu-s}{t}\right) \chi_{(\mu-tc, \mu)}(s) \right|^p \cdot \|\varphi\|^p. \end{aligned} \quad (53)$$

Then we have

$$\begin{aligned} \|E_\varepsilon\|^p &\leq a^{p/q} \int_0^a \left| \int_\varepsilon^{1/\varepsilon} \frac{dt}{t} e^{-(\lambda+\sigma_0)t} \right. \\ &\quad \times \left. w\left(\frac{\mu-s}{t}\right) \chi_{(\mu-tc, \mu)}(s) \right|^p d\mu. \end{aligned} \quad (54)$$

Putting $\mu = \mu - s$, $w(\cdot) \subset [\delta, c]$, and $\delta > 0$, we get

$$\begin{aligned} &\int_0^a \left| \int_\varepsilon^{1/\varepsilon} \frac{dt}{t} e^{-(\lambda+\sigma_0)t} \right. \\ &\quad \times \left. w\left(\frac{\mu-s}{t}\right) \chi_{(\mu-tc, \mu)}(s) \right|^p d\mu \\ &\leq \int_{\varepsilon\delta}^{c/\varepsilon} \left| \int_\varepsilon^{1/\varepsilon} \frac{dt}{t} e^{-(\lambda+\sigma_0)t} \right. \\ &\quad \times \left. w\left(\frac{\mu}{t}\right) \chi_{(\mu-tc, \mu)}(0) \right|^p d\mu, \end{aligned} \quad (55)$$

so we have

$$\begin{aligned} \|E_\varepsilon\|^p &\leq a^{p/q} \int_{\varepsilon\delta}^{c/\varepsilon} \left| \int_\varepsilon^{1/\varepsilon} \frac{dt}{t} \right. \\ &\quad \times \left. e^{-(\lambda+\sigma_0)t} w\left(\frac{\mu}{t}\right) \chi_{(\mu-tc, \mu)}(0) \right|^p d\mu. \end{aligned} \quad (56)$$

For any $t \leq 1/\varepsilon$,

$$\chi_{(\mu-tc, \mu)}(0) = \chi_{(\mu/c, 1/\varepsilon)}(t), \quad (57)$$

and we get

$$\begin{aligned} \|E_\varepsilon\|^p &\leq a^{p/q} \int_{\varepsilon\delta}^{c/\varepsilon} \left| \int_\varepsilon^{1/\varepsilon} \frac{dt}{t} \right. \\ &\quad \times \left. e^{-(\lambda+\sigma_0)t} w\left(\frac{\mu}{t}\right) \chi_{(\mu/c, 1/\varepsilon)}(t) \right|^p d\mu. \end{aligned} \quad (58)$$

So, it remains to show that

$$\begin{aligned} \lim_{|\operatorname{Im} \lambda| \rightarrow +\infty} |\operatorname{Im} \lambda|^{pr} \int_{\varepsilon\delta}^{c/\varepsilon} \left| \int_\varepsilon^{1/\varepsilon} \frac{dt}{t} \right. \\ \times \left. e^{-(\lambda+\sigma_0)t} w\left(\frac{\mu}{t}\right) \chi_{(\mu/c, 1/\varepsilon)}(t) \right|^p d\mu = 0, \end{aligned} \quad (59)$$

uniformly on Γ_ω . We define now the functions

$$\begin{aligned} a(\cdot) : \left(\varepsilon, \frac{1}{\varepsilon}\right) &\longrightarrow \left(\varepsilon, \frac{1}{\varepsilon}\right), \quad \varepsilon \in (0, 1); \\ t &\longrightarrow \frac{1}{t}, \quad t \in \left(\varepsilon, \frac{1}{\varepsilon}\right), \\ \ell(\cdot) &= \sum_{j=0}^m d_j \chi_{(a_j, b_j)}, \end{aligned} \quad (60)$$

such that $a(t) \leq \ell(\cdot)$, $d_j \geq 0$, $(a_j, b_j) \subset [\delta, c]$, and $0 \leq j \leq m$. So, it suffices to establish that

$$\begin{aligned} \lim_{|\operatorname{Im} \lambda| \rightarrow +\infty} |\operatorname{Im} \lambda|^{pr} \int_{\varepsilon\delta}^{c/\varepsilon} \left| \int_\varepsilon^{1/\varepsilon} \frac{dt}{t} \right. \\ \times \left. e^{-(\lambda+\sigma_0)t} \ell(t) w\left(\frac{\mu}{t}\right) \right. \\ \times \left. \chi_{(\mu/c, 1/\varepsilon)}(t) \right|^p d\mu = 0, \end{aligned} \quad (61)$$

uniformly on Γ_ω . Because the functions $\ell(\cdot)$ and $w(\cdot)$ are simple functions, we fix $\mu \in (\varepsilon\delta, c/\varepsilon)$, and for all $t \in (\varepsilon, 1/\varepsilon)$, we have

$$\begin{aligned} &\ell(t) w\left(\frac{\mu}{t}\right) \chi_{(\mu/c, 1/\varepsilon)}(t) \\ &= \left(\sum_{j=0}^m d_j \chi_{(a_j, b_j)}(t) \right) \left(\sum_{i=0}^n c_i \chi_{(\mu/a_i, \mu/b_i)}(t) \right) \chi_{(\mu/c, 1/\varepsilon)}(t) \\ &= \sum_{k=0}^{N(\mu, n, m)} e_k \chi_{(a_k(\mu), b_k(\mu))}(t), \end{aligned} \quad (62)$$

where $N(\mu, n, m) \leq mn$, $\mu \in (\varepsilon\delta, c/\varepsilon)$, e_k is nonnegative constant and $(a_k(\mu), b_k(\mu)) \subset (\varepsilon, 1/\varepsilon)$. We get

$$\begin{aligned} & \int_{\varepsilon}^{1/\varepsilon} \frac{dt}{t} e^{-(\lambda+\sigma_0)t} \ell(t) w\left(\frac{\mu}{t}\right) \chi_{(\mu/c, 1/\varepsilon)}(t) \\ &= \sum_{k=0}^{N(\mu, n, m)} e_k \int_{a_k(\mu)}^{b_k(\mu)} e^{-(\lambda+\sigma_0)t} dt \\ &= \sum_{k=0}^{N(\mu, n, m)} \frac{e_k}{\lambda + \sigma_0} \Xi(\lambda, \mu), \end{aligned} \quad (63)$$

where

$$\Xi(\lambda, \mu) = e^{-(\lambda+\sigma_0)a_k(\mu)} - e^{-(\lambda+\sigma_0)b_k(\mu)}. \quad (64)$$

This implies that

$$\begin{aligned} & \left| \int_{\varepsilon}^{1/\varepsilon} \frac{dt}{t} e^{-(\lambda+\sigma_0)t} \ell(t) w\left(\frac{\mu}{t}\right) \chi_{(\mu/c, 1/\varepsilon)}(t) \right| \\ & \leq \frac{2}{|\operatorname{Im} \lambda|} \sum_{k=0}^{N(\mu, n, m)} |e_k| \\ & \leq \frac{2(mn+1) \sup \|w(\cdot)\|}{\varepsilon |\operatorname{Im} \lambda|}. \end{aligned} \quad (65)$$

Consequently, we have

$$\begin{aligned} & \int_{\varepsilon\delta}^{c/\varepsilon} \left| \int_{\varepsilon}^{1/\varepsilon} \frac{dt}{t} e^{-(\lambda+\sigma_0)t} w\left(\frac{\mu}{t}\right) \chi_{(\mu/c, 1/\varepsilon)}(t) \right|^p d\mu \\ & \times |\operatorname{Im} \lambda|^{pr} \leq \frac{(\Upsilon(t, \mu, m, n))^p}{\varepsilon^p |\operatorname{Im} \lambda|^{p(1-r)}}, \end{aligned} \quad (66)$$

where

$$\Upsilon(t, \mu, m, n) = 2(mn+1) \sup \|w(\cdot)\|. \quad (67)$$

Finally, we get

$$\begin{aligned} & \lim_{|\operatorname{Im} \lambda| \rightarrow +\infty} |\operatorname{Im} \lambda|^{pr} \int_{\varepsilon\delta}^{c/\varepsilon} \left| \int_{\varepsilon}^{1/\varepsilon} \frac{dt}{t} e^{-(\lambda+\sigma_0)t} w\left(\frac{\mu}{t}\right) \chi_{(\mu/c, 1/\varepsilon)}(t) \right|^p d\mu \\ & \leq \lim_{|\operatorname{Im} \lambda| \rightarrow +\infty} \frac{(\Upsilon(t, \mu, m, n))^p}{\varepsilon^p |\operatorname{Im} \lambda|^{p(1-r)}} = 0. \end{aligned} \quad (68)$$

So we can get (43) immediately.

Step Two. For any $k \in \mathbb{N}$, for $r \in [0, 1)$, prove that

$$\lim_{|\operatorname{Im} \lambda| \rightarrow +\infty} |\operatorname{Im} \lambda|^r \left\| H_2(\overline{P_\lambda} H_1)^k \overline{D_\lambda} K \right\| = 0, \quad (69)$$

uniformly on Γ_ω . Since H_2 is positive and compact operator, so it suffices to establish the result for a positive operator of rank one; that is,

$$H_2 : \varphi(a, v) \longrightarrow H_2 \varphi(0, v), \quad (70)$$

$$H_2 \varphi(0, v) = \eta(v) \int_0^c \beta(v') \varphi(0, v') v' dv',$$

where $\eta(\cdot) \in L_p[0, c]$, $\beta(\cdot) \in L_q[0, c]$, and $(1/p) + (1/q) = 1$. So for any $\varphi \in X_p$, $f(\cdot) \subset [\delta, c]$, we get

$$\begin{aligned} & \left(H_2(\overline{P_\lambda} H_1)^k \overline{D_\lambda} K \varphi \right)(0, v) \\ &= \eta(v) \int_\delta^c f(v') dv' e^{-(\lambda+\sigma_0)(a(k+1)-\mu')/v'} \\ & \times \int_0^a d\mu \alpha(\mu) \int_0^c dv'' g(v'') \varphi(\mu, v''). \end{aligned} \quad (71)$$

Let

$$\left(H_2(\overline{P_\lambda} H_1)^k \overline{D_\lambda} K \varphi \right)(0, v) = F_1 \cdot F_{\lambda, a} \cdot F_2, \quad (72)$$

where

$$\begin{aligned} & F_1 : \gamma \in \mathbb{C} \longrightarrow \gamma \eta(\cdot) \in X_p^0, \\ & F_2 : \varphi \longrightarrow \alpha(\mu) \int_0^c g(v) \varphi(\mu, v) dv \in L_p[(0, a), d\mu], \\ & F_{\lambda, a} : L_p[(0, a), d\mu] \longrightarrow \mathbb{C}, \\ & \Phi(\cdot) = \beta(\cdot) f(\cdot), \\ & (F_{\lambda, a} \varphi)(\mu, v) = \int_0^a d\mu \varphi(\mu) \int_\delta^c dv \Phi(v) \\ & \times e^{-(\lambda+\sigma_0)(a(k+1)-\mu)/v}. \end{aligned} \quad (73)$$

Since F_1 and F_2 are bounded operators,

$$\left\| H_2(\overline{P_\lambda} H_1)^k \overline{D_\lambda} K \right\| \leq \|F_1\| \cdot \|F_{\lambda, a}\| \cdot \|F_2\|, \quad (74)$$

so it suffices to prove that

$$\lim_{|\operatorname{Im} \lambda| \rightarrow +\infty} |\operatorname{Im} \lambda|^{pr} \|F_{\lambda, a}\|^p = 0, \quad (75)$$

uniformly on Γ_ω . For any $\theta \in [a, a + \zeta]$, $\zeta > 0$, let

$$\Theta_{\theta, \mu'} = ka + \theta - \mu'; \quad (76)$$

we defined a bounded linear operator $F_{\lambda, \theta}$ by

$$\begin{aligned} & F_{\lambda, \theta} : L_p[(0, a), d\mu] \longrightarrow \mathbb{C}, \\ & F_{\lambda, \theta} \varphi = \int_0^a d\mu' \varphi(\mu') \int_\delta^c \Phi(v) \\ & \times \frac{e^{-(\lambda+\sigma_0)\Theta_{\theta, \mu'}}}{v} dv. \end{aligned} \quad (77)$$

Let $v \rightarrow \zeta = 1/v$ and $\varphi \in L_p(-a, a)$; then

$$F_{\lambda, \theta} \varphi = \int_0^a d\mu' \varphi(\mu') \int_{1/c}^{1/\delta} \overline{\Phi(\zeta)} \times e^{-(\lambda + \sigma_0)\zeta \Theta_{\theta, \mu'}} d\zeta, \quad (78)$$

where $\overline{\Phi(\zeta)} \in L_p(1/c, 1/\delta)$ and

$$\overline{\Phi(\zeta)} = \frac{\Phi(1/\zeta)}{\zeta^2}, \quad \zeta \in (\delta, c). \quad (79)$$

So, we can assume that

$$\overline{\Phi(\cdot)} = \sum_{i=0}^n q_k \chi_{(\alpha_k, \beta_k)}(\cdot), \quad (80)$$

where $q_k \in (\alpha_k, \beta_k)$. Clearly, using the Lebesgue dominated convergence theorem, we get the nets of operators $(F_{\lambda, \theta})_{\theta > \alpha} \rightarrow F_{\lambda, a}$ ($\theta \rightarrow a^+$). And for all $\lambda \in \Gamma_\omega$, it suffices to prove that

$$\lim_{|\operatorname{Im} \lambda| \rightarrow +\infty} |\operatorname{Im} \lambda|^{pr} \|F_{\lambda, \theta}\|^p = 0, \quad (81)$$

uniformly on Γ_ω . Because

$$\overline{\Phi(\zeta)} = \sum_{i=0}^n q_k \chi_{(\alpha_k, \beta_k)}(\zeta), \quad (82)$$

we get

$$\begin{aligned} & \int_{1/c}^{1/\delta} \overline{\Phi(\zeta)} e^{-(\lambda + \sigma_0)\zeta \Theta_{\theta, \mu'}} d\zeta \\ &= \sum_{i=0}^n \frac{q_k}{-(\lambda + \sigma_0) \Theta_{\theta, \mu'}} \\ & \quad \times (e^{-(\lambda + \sigma_0)\alpha_k \Theta_{\theta, \mu'}} - e^{-(\lambda + \sigma_0)\beta_k \Theta_{\theta, \mu'}}). \end{aligned} \quad (83)$$

And we get

$$\left| \int_{1/c}^{1/\delta} \overline{\Phi(\zeta)} e^{-(\lambda + \sigma_0)\zeta \Theta_{\theta, \mu'}} d\zeta \right| \leq \frac{2(n+1)}{(\theta - a) |\operatorname{Im} \lambda|}. \quad (84)$$

Applying the Hölder inequality we get

$$\|F_{\lambda, \theta} \varphi\|^p \leq a^{p/q} \left| \int_{1/c}^{1/\delta} \overline{\Phi(\zeta)} e^{-(\lambda + \sigma_0)\zeta \Theta_{\theta, \mu'}} d\zeta \right|^p \|\varphi\|^p. \quad (85)$$

And we have

$$\begin{aligned} & \lim_{|\operatorname{Im} \lambda| \rightarrow +\infty} |\operatorname{Im} \lambda|^{pr} \|F_{\lambda, \theta}\|^p \\ & \leq \lim_{|\operatorname{Im} \lambda| \rightarrow +\infty} \frac{a^{p/q} (2n+2)^p}{(\theta - a)^p |\operatorname{Im} \lambda|^{p(1-r)}} = 0. \end{aligned} \quad (86)$$

So we can get (69) immediately.

Step Three. For any $N \in \mathbb{N}$, for $r \in [0, 1)$, prove that

$$\lim_{|\operatorname{Im} \lambda| \rightarrow +\infty} |\operatorname{Im} \lambda|^r \|K \overline{Q_\lambda} H_1 (\overline{P_\lambda} H_1)^N K\| = 0, \quad (87)$$

uniformly on Γ_ω . For all $\varphi \in X_p$, easy calculations show that

$$\begin{aligned} & K \overline{Q_\lambda} H_1 (\overline{P_\lambda} H_1)^N \overline{D_\lambda} K \varphi(\mu, v) \\ &= \alpha(\mu) f(v) \int_\delta^c \frac{w(v')}{v'} dv' e^{-(\lambda + \sigma_0)(n\mu + \mu')/v'} \\ & \quad \times \int_0^a d\mu' \alpha(\mu') e^{-(\lambda + \sigma_0)(a - \mu')/v'} \\ & \quad \times \int_\delta^c dv'' g(v'') \varphi(\mu', v''). \end{aligned} \quad (88)$$

Let

$$K \overline{Q_\lambda} H_1 (\overline{P_\lambda} H_1)^N \overline{D_\lambda} K = G_1 \cdot G_{\lambda, a} \cdot G_2, \quad (89)$$

where

$$G_1 : \varphi \rightarrow \alpha(\mu) f(v) \varphi(\mu) \in X_p,$$

$$G_2 : \varphi \rightarrow \alpha(\mu) \int_0^c g(v) \varphi(\mu, v) dv \in L_p[(0, a), d\mu],$$

$$G_{\lambda, a} : L_p[(0, a), d\mu] \rightarrow L_p[(0, a), d\mu],$$

$$\begin{aligned} (G_{\lambda, a} \varphi)(\mu, v) &= \int_0^a d\mu' \varphi(\mu') \int_\delta^c \frac{w(v)}{v} \\ & \quad \times e^{-(\lambda + \sigma_0)((n+1)a + \mu - \mu')/v} dv. \end{aligned} \quad (90)$$

Because G_1 and G_2 are bounded operators, it suffices to prove that

$$\lim_{|\operatorname{Im} \lambda| \rightarrow +\infty} |\operatorname{Im} \lambda|^{pr} \|G_{\lambda, a}\| = 0, \quad (91)$$

uniformly on Γ_ω . In the same principle as (75), we can get (87) immediately.

Step Four. For any $\operatorname{Re} \lambda > \lambda_0$, because of (30) and assumption O_2 , we have

$$\begin{aligned} & \|(\lambda - T_H)^{-1}\| \leq \|(\lambda - \overline{T_H})^{-1}\|, \\ & \|K(\lambda - T_H)^{-1} K\| \leq \|K(\lambda - \overline{T_H})^{-1} K\|. \end{aligned} \quad (92)$$

According to (25)–(27)–(29)–(30)–(43), it suffices to establish that, for $r \in [0, 1)$, prove that

$$\lim_{|\operatorname{Im} \lambda| \rightarrow +\infty} |\operatorname{Im} \lambda|^r \|K \overline{Q_\lambda} H (\overline{P_\lambda} H)^N \overline{D_\lambda} K\| = 0, \quad (93)$$

uniformly on Γ_ω . Note that $\overline{P_\lambda} H_1$ and $\overline{P_\lambda} H_2$ do not commute, so we have

$$(\overline{P_\lambda} H)^n = \sum_{j=1}^{2^n} P_j, \quad (94)$$

where each P_j is the product of n factors involving both $\overline{P_\lambda}H_1$ and $\overline{P_\lambda}H_2$ except the term

$$P_{2^n} = (\overline{P_\lambda}H_1)^n. \quad (95)$$

So, for $j \in \{1, 2, \dots, 2^n - 1\}$, the operator H_2 appears at least one time in the expression of P_j . So, there exists $k \in \{1, 2, \dots, 2^n - 1\}$, such that

$$P_j = Q_j \overline{P_\lambda} H_2 (\overline{P_\lambda} H_1)^k, \quad (96)$$

where Q_j is uniformly bounded on Γ_ϵ . Now using the inequality

$$\begin{aligned} \|K \overline{Q_\lambda} H P_j \overline{D_\lambda} K\| &\leq \|K \overline{Q_\lambda} H\| \\ &\times \|Q_j \overline{P_\lambda}\| \cdot \|H_2 (\overline{P_\lambda} H_1)^n \overline{D_\lambda} K\| \end{aligned} \quad (97)$$

and (53), for $j \in \{1, 2, \dots, 2^n - 1\}$, $r \in [0, 1)$, we get

$$\lim_{|\operatorname{Im} \lambda| \rightarrow +\infty} |\operatorname{Im} \lambda|^r \|K \overline{Q_\lambda} H P_j \overline{D_\lambda} K\| = 0, \quad (98)$$

uniformly on Γ_ω . On the other hand, for $j = 2^n$, we have

$$\begin{aligned} \|K \overline{Q_\lambda} H P_{2^n} \overline{D_\lambda} K\| &\leq \|K \overline{Q_\lambda} H_1 (\overline{P_\lambda} H_1)^n \overline{D_\lambda} K\| \\ &+ \|K \overline{Q_\lambda} H_2 (\overline{P_\lambda} H_1)^n \overline{D_\lambda} K\|. \end{aligned} \quad (99)$$

Using (69) again, it remains to prove that

$$\lim_{|\operatorname{Im} \lambda| \rightarrow +\infty} |\operatorname{Im} \lambda|^r \|K \overline{Q_\lambda} H_1 (\overline{P_\lambda} H_1)^n \overline{D_\lambda} K\| = 0, \quad (100)$$

uniformly on Γ_ω . This follows from (87). This ends the proof. $\square$

Theorem 5. *If assumptions O_1 and O_2 are satisfied and $r_\sigma(P_{\lambda_0} H_2) > 1$, then the spectrum of transport operator A_H consists of only finite isolated eigenvalues which have finite algebraic multiplicities in trip Γ_ω ; furthermore a real $\lambda^* > \lambda_0$ exists such that $\lambda^* \in \sigma_p(A_H)$, where $\sigma_p(A_H)$ denotes point spectrum set of transport operator A_H .*

Proof. Consider the following.

Step One. Prove that the spectrum of the transport operator A_H consists of, at most, finite isolated eigenvalues which have finite algebraic multiplicities in trip Γ_ω . By Lemmas 1–3, it suffices to prove that $R_9(t)$ is compact operator on X_p .

On one hand, because of hypothesis O_2 , we have $K(\lambda I - \overline{T_H})^{-1}$ and $(\lambda I - \overline{T_H})^{-1} K$ is compact operator on X_p . Let

$$\overline{\Pi(\lambda)} = (\lambda I - \overline{T_H})^{-1} [K(\lambda I - \overline{T_H})^{-1}]^4. \quad (101)$$

Since $(\lambda I - \overline{T_H})^{-1}$ is bounded operator and

$$(\lambda I - \overline{T_H})^{-1} \leq \frac{1}{\operatorname{Re} \lambda - \lambda_0}, \quad (102)$$

we get that $\overline{\Pi(\lambda)}$ is compact operator on X_p .

On the other hand, because of

$$\overline{\Pi(\lambda)} \leq \|(\lambda I - \overline{T_H})^{-1}\|^3 \cdot \|K(\lambda I - \overline{T_H})^{-1} K\|^2, \quad (103)$$

thanks to (29), and for $\operatorname{Re} \lambda \geq \lambda_0 + \omega$, we have

$$(\lambda I - \overline{T_H})^{-1} \leq \frac{1}{\omega}. \quad (104)$$

So for $\operatorname{Re} \lambda \geq \lambda_0 + \omega$, together with (102), (103), and (104) we have

$$\begin{aligned} |\operatorname{Im} \lambda| \|\overline{\Pi(\lambda)}\| &\leq \frac{1}{\omega^3} \left[|\operatorname{Im} \lambda|^{1/2} \|K(\lambda I - \overline{T_H})^{-1} K\| \right]^2. \end{aligned} \quad (105)$$

By Theorem 4, we get

$$\begin{aligned} \lim_{|\operatorname{Im} \lambda| \rightarrow +\infty} |\operatorname{Im} \lambda| \|\overline{\Pi(\lambda)}\| &\leq \frac{1}{\omega^3} \left[\lim_{|\operatorname{Im} \lambda| \rightarrow +\infty} |\operatorname{Im} \lambda|^{1/2} \|K(\lambda I - \overline{T_H})^{-1} K\| \right]^2 \\ &= 0, \end{aligned} \quad (106)$$

uniformly on Γ_ω . Now, applying Lemma 3 we conclude that, for each $t > 0$, $R_9(t)$ is compact operator on X_p , together with [3]; we end Step one. Now we consider again the resolvent equation which is equivalent to solving in X_p^a the following one:

$$\psi^a = P_\lambda H_1 \psi^a + P_\lambda H_2 \psi^a + E_\lambda \varphi. \quad (107)$$

If $\lambda > \lambda_0$, then the operator $(I - P_\lambda H_1)^{-1}$ is bounded invertible and (104) becomes

$$\psi^a = \Xi_\lambda \psi^a + L_\lambda \varphi, \quad (108)$$

where

$$\begin{aligned} \Xi_\lambda &= (I - P_\lambda H_1)^{-1} P_\lambda H_2, \\ L_\lambda &= (I - P_\lambda H_1)^{-1} E_\lambda. \end{aligned} \quad (109)$$

Step Two. Prove that $(I - \Xi_\lambda)^{-1}$ exists for λ in the half plane

$$\Gamma = \{\lambda \in \mathbb{C}, \operatorname{Re} \lambda > -\sigma_0\} \quad (110)$$

with $|\operatorname{Im} \lambda|$ sufficiently large.

(i) Let λ be an element of

$$\Gamma_0 = \{\lambda \in \mathbb{C}, -\sigma_0 < \operatorname{Re} \lambda \leq \lambda_0\}, \quad (111)$$

and set

$$\begin{aligned} P_\lambda^* : X_q^a &\longrightarrow X_q^0, \\ (P_\lambda^* g)(0, v) &= g(a, v) e^{-(1/v) \int_0^a (\lambda + \sigma(\mu', v)) d\mu'}, \\ H_2^* : X_q^a &\longrightarrow X_q^0, \\ (H_2^* g)(a, v) &= \eta(v) \int_0^c \beta(v') g(0, v') dv', \end{aligned} \quad (112)$$

where P_λ^* and H_2^* are the dual operators of P_λ and H_2 . Easy calculations show that the dual $(P_\lambda H_2)^*$ of the operator $P_\lambda H_2$ is $H_2^* P_\lambda^*$. Let

$$\lambda_n = \zeta + it_n \quad (113)$$

be a sequence in Γ such that

$$\lim_{n \rightarrow +\infty} |t_n| = +\infty. \quad (114)$$

We consider $\varphi \in X_q^a$; then (112) show that

$$\begin{aligned} H_2^* P_{\lambda_n}^* \varphi &= \int_0^c e^{-(1/v') \int_0^a (\zeta + it_n + \sigma(\mu', v)) d\mu'} \\ &\quad \times \varphi(a, v') \eta(v) \beta(v') v' dv'. \end{aligned} \quad (115)$$

By the Riemann-Lebesgue theorem, we obtain

$$\lim_{n \rightarrow +\infty} |H_2^* P_{\lambda_n}^* \varphi(a, v)| = 0 \quad (116)$$

a.e. on $\{a\} \times (0, c)$. Furthermore, for every integer n , we have

$$|H_2^* P_{\lambda_n}^* \varphi(a, v)| \leq |\eta(v)| \int_0^c |\varphi(a, v')| |\beta(v')| v' dv'. \quad (117)$$

By the dominated convergence theorem, we get

$$\lim_{|\operatorname{Im} \lambda| \rightarrow +\infty} |H_2^* P_\lambda^* \varphi| = 0, \quad (118)$$

for all $\varphi \in X_p^a$ ($\lambda \in \Gamma$).

(ii) Prove that the family $H_2^* P_\lambda^*$, $\lambda \in \Gamma_0$, is collectively compact. Let Θ be the unit ball of the space X_p^a and let (x_n) be a sequence in $\cup_{\lambda \in \Gamma_0} (H_2^* P_\lambda^* \Theta)$; then there exists a sequence (φ_n) in Θ such that

$$x_n = P_{\lambda_n}^* \varphi_n. \quad (119)$$

Since H_2^* is compact and (x_n) is bounded in the space X_p^a , there exists a converging subsequence of $(H_2^* x_n)$ in $\overline{\cup_{\lambda \in \Gamma_0} (H_2^* P_\lambda^* \Theta)}$. This is the wanted.

(iii) The use of (i) and (ii) together with Anselone's Proposition [17, Proposition 3.1] gives

$$\lim_{|\operatorname{Im} \lambda| \rightarrow +\infty} \|(H_2^* P_\lambda^*)^2\| = 0 \quad (120)$$

uniformly on Γ_0 . Then

$$\lim_{|\operatorname{Im} \lambda| \rightarrow +\infty} r_\sigma(H_2^* P_\lambda^*) = 0 \quad (121)$$

uniformly on Γ_0 . Since

$$\begin{aligned} r_\sigma(H_2^* P_\lambda^*) &\leq \|(H_2^* P_\lambda^*)^2\|^{1/2}, \\ r_\sigma(H_2^* P_\lambda^*) &= r_\sigma(P_\lambda H_2), \end{aligned} \quad (122)$$

therefore, we obtain

$$\lim_{|\operatorname{Im} \lambda| \rightarrow +\infty} r_\sigma(P_\lambda H_2) = 0, \quad (123)$$

uniformly on Γ_0 . This ends Step two by (iii).

Step Three. We will prove that $\sigma_P(A_H) \neq \emptyset$.

First, because of

$$\Xi_\lambda = (I - P_\lambda H_1)^{-1} P_\lambda H_2, \quad (124)$$

there exists $\lambda_1 > \lambda_0$ for $\lambda > \lambda_1$ and we get

$$(I - P_\lambda H_1)^{-1} \leq (I - P_{\lambda_1} H_1)^{-1}. \quad (125)$$

Hence

$$(\Xi_\lambda)^3 \leq (I - P_\lambda H_1)^{-1} P_\lambda H_2 (\Xi_\lambda)^2. \quad (126)$$

Because H_2 is compact on X_p and

$$\lim_{\lambda \rightarrow +\infty} P_\lambda = 0, \quad (127)$$

we get that $(P_{\lambda_1} H_2)^2$ is compact on X_p , and we have

$$\begin{aligned} \|\Xi_\lambda\|^3 &\leq \|(I - P_{\lambda_1} H_1)^{-1}\| \cdot \|P_{\lambda_1} H_2\| \\ &\quad \times \|(\Xi_{\lambda_1})^2\| \longrightarrow 0 \quad (\lambda \longrightarrow +\infty). \end{aligned} \quad (128)$$

And we get

$$\lim_{\lambda \rightarrow +\infty} r_\sigma(\Xi_\lambda) = 0. \quad (129)$$

Next, by spectral mapping theorem, we get that Ξ_λ is continuous decreasing function on λ , since

$$\lambda \in \sigma_P(A_H) \neq \emptyset \iff 1 \in \sigma(\Xi_\lambda). \quad (130)$$

Because Ξ_λ is continuous decreasing function on λ , it suffices to prove that

$$\lambda \in \sigma_P(A_H) \neq \emptyset \iff \lim_{\lambda \rightarrow \lambda_0} r_\sigma(\Xi_\lambda) > 1. \quad (131)$$

Let

$$\overline{\Xi}_\lambda = (I - \overline{P}_\lambda H_1)^{-1} \overline{P}_\lambda H_2; \quad (132)$$

we have $\overline{\Xi}_\lambda \rightarrow \Xi_\lambda$ ($\lambda > \lambda_0$), since

$$\overline{\Xi}_\lambda = \sum_{n \geq 0} (\overline{P}_\lambda H_1)^n \overline{P}_\lambda H_2 \geq \overline{P}_\lambda H_2, \quad (133)$$

and we get

$$\begin{aligned} \Xi_\lambda &\geq P_\lambda H_2, \\ \lim_{\lambda \rightarrow \lambda_0} r_\sigma(\Xi_\lambda) &\geq \lim_{\lambda \rightarrow \lambda_0} r_\sigma(P_\lambda H_2). \end{aligned} \quad (134)$$

Due to the hypothesis $r_\sigma(P_{\lambda_0} H_2) > 1$, we get

$$\lim_{\lambda \rightarrow \lambda_0} r_\sigma(\Xi_\lambda) > 1. \quad (135)$$

Finally, according to (135) and the fact that Ξ_λ is continuous decreasing function on λ , we know that there must be a real $\lambda^* > \lambda_0$ such that

$$r_\sigma(\Xi_{\lambda^*}) = 1. \quad (136)$$

Hence, $\lambda^* \in \sigma_P(A_H) \neq \emptyset$. This ends the proof. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The authors would like to express their gratitude to the referee for his/her valuable suggestions. The research was supported by National Natural Science Foundation of China (no. 11461055), the Natural Science Foundation of Jiangxi Province of China (no. 20132BAB201002) and the Office of Education of Jiangxi Province of China (no. GJJ13706).

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