Hindawi Publishing Corporation
Journal of Function Spaces
Volume 2014, Article ID 758963, 8 pages
http://dx.doi.org/10.1155/2014/758963

Research Article

Existence of Nontrivial Solutions for a Critical Perturbed Quasilinear Elliptic System

Huixing Zhang and Xiangzhi Zhang

Department of Mathematics, China University of Mining and Technology, Xuzhou, Jiangsu 221116, China

Correspondence should be addressed to Huixing Zhang; zhx20000701cumt@163.com

Received 20 July 2014; Revised 17 October 2014; Accepted 29 October 2014; Published 17 November 2014

Academic Editor: Guozhen Lu

We consider a perturbed quasilinear elliptic system involving the p-Laplacian with critical growth terms in $\mathbb{R}^N$. Under proper conditions, we establish the existence of nontrivial solutions by using the variational methods.

1. Introduction

In this paper, we are concerned with the following perturbed quasilinear elliptic system involving the p-Laplacian:

$$\begin{aligned}
&-\varepsilon^p \Delta_p u + V(x)|u|^{p-2}u \\
&\quad = f(x)|u|^{q-2}u + \frac{\alpha}{\alpha+\beta}K(x)|u|^{\alpha-2}u|v|^{\beta}, \quad x \in \mathbb{R}^N, \\
&-\varepsilon^p \Delta_p v + V(x)|v|^{p-2}v \\
&\quad = g(x)|v|^{q-2}v + \frac{\beta}{\alpha+\beta}K(x)|u|^{\alpha}|v|^{\beta-2}v, \quad x \in \mathbb{R}^N, \\
&u(x), v(x) \longrightarrow 0 \quad \text{as } |x| \longrightarrow \infty,
\end{aligned} \tag{1}$$

where $2 < p < N$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator, $\alpha > 1$, $\beta > 1$ satisfy $\alpha + \beta = p^*$, $p^* = Np/(N-p)$ is the critical Sobolev exponent for $N \geq 3$, $p < q < p^*$ and $V(x)$, $K(x)$, $f(x)$, $g(x)$ satisfy the following assumptions:

(H_1) $V \in C(\mathbb{R}^N)$, $V(0) = \inf_{x\in\mathbb{R}^N} V(x) = 0$, and there exists $b > 0$ such that the set $v^b := \{x \in \mathbb{R}^N : V(x) < b\}$ has finite Lebesgue measure;

(H_2) $K(x) \in C(\mathbb{R}^N)$, $0 < \inf K \leq \sup K < \infty$;

(H_3) $f(x)$ and $g(x)$ are bounded and positive functions.

Set $\alpha = \beta$, $f(x) = g(x)$, and $u = v$. The problem (1) reduces to the semilinear scalar quasilinear elliptic equation

$$\begin{aligned}
&-\varepsilon^p \Delta_p u + V(x)|u|^{p-2}u \\
&\quad = f(x)|u|^{q-2}u + \frac{1}{2}K(x)|u|^{p^*-2}u, \quad x \in \mathbb{R}^N.
\end{aligned} \tag{2}$$

The type of problem (2) including $p = 2$ and $p \neq 2$ has been extensively studied in many papers involving bounded domain and unbounded domain. See, for example, [1–13] and the references therein.

In recent years, much attention has been paid to the existence of solutions for problem (1) with $\varepsilon = 1$ and $p = 2$ in bounded domain. Wu [14] was concerned with the following semilinear elliptic system with subcritical nonlinearity of concave-convex type and sign-changing weights:

$$\begin{aligned}
-\Delta u &= \lambda f(x)|u|^{q-2}u + \frac{\alpha}{\alpha+\beta}h(x)|u|^{\alpha-2}u|v|^{\beta}, \quad x \in \Omega, \\
-\Delta v &= \mu g(x)|v|^{q-2}v + \frac{\beta}{\alpha+\beta}h(x)|u|^{\alpha}|v|^{\beta-2}v, \quad x \in \Omega,
\end{aligned} \tag{3}$$

where $1 < q < 2$, $\alpha > 1$, $\beta > 1$ satisfy $2 < \alpha + \beta < 2^*$ and the functions $f(x)$, $g(x)$, $h(x)$ satisfy some suitable conditions. He established the existence of at least two positive solutions for the problem (3) when the pair of the parameters (λ, μ)

belongs to a certain subset of $\mathbb{R}^2$. Hsu and Lin [15] considered a similar problem and proved that the problem (3) has at least two positive solutions in $W_0^{1,2}(\Omega)\times W_0^{1,2}(\Omega)$ involving critical exponents. Subsequently, Hsu [16] extended the results of [15] to the quasilinear case $p>1$. The paper [16] was devoted to the following quasilinear elliptic system:

$$\begin{aligned} -\Delta_p u &= \lambda |u|^{q-2}u + \frac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^{\beta}, \quad x\in\Omega,\\ -\Delta_p v &= \mu |v|^{q-2}v + \frac{2\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v, \quad x\in\Omega, \end{aligned} \tag{4}$$

where $\lambda,\mu>0$, $\alpha>1$, $\beta>1$ satisfy $\alpha+\beta=p^*$ and $p^*=Np/(N-p)$ denotes the critical Sobolev exponent. He proved that the problem (4) has at least two positive solutions in $W_0^{1,p}(\Omega)\times W_0^{1,p}(\Omega)$.

However, as far as we know, there are almost no results on problem (1) involving critical exponents in whole space. In our work, we consider the problem (1) and use variational methods to get positive solutions. Our main arguments use similar ideas found in [8, 15]. The main difficulty is that some estimates and results hold for the Laplacian operator but not for p-Laplacian operator. At the same time, the corresponding functional to problem (1) lacks compactness because of unbounded domain $\mathbb{R}^N$ and critical exponent. We can prove the functional associated to (1) possesses $(\mathrm{PS})_c$ condition at some energy level c. The main result in the paper can be seen as a complement of results obtained in [8, 16].

In particular, we will mention our own work [17] and furthermore discuss the differences between these two papers. In [17], we are concerned with the following system:

$$\begin{aligned} &-\Delta_p u + \lambda V(x)|u|^{p-2}u\\ &\quad = \lambda K(x)|u|^{p^*-2}u + \lambda H_u(u,v), \quad x\in\mathbb{R}^N,\\ &-\Delta_p v + \lambda V(x)|v|^{p-2}v\\ &\quad = \lambda K(x)|v|^{p^*-2}v + \lambda H_v(u,v), \quad x\in\mathbb{R}^N,\\ &u(x),v(x)>0,\\ &u(x),v(x)\longrightarrow 0, \quad \text{as } |x|\longrightarrow\infty. \end{aligned} \tag{5}$$

The coupled system (5) shows that the coupled terms are $H_u(u,v)$ and $H_v(u,v)$. The energy functional associated with (5) is defined by

$$\begin{aligned} I_\lambda(u,v) = {}& \frac{1}{p}\int_{\mathbb{R}^N}\left(|\nabla u|^p + \lambda V(x)|u|^p + |\nabla v|^p + \lambda V(x)|v|^p\right)\\ &- \frac{\lambda}{p^*}\int_{\mathbb{R}^N}K(x)\left(|u|^{p^*}+|v|^{p^*}\right) - \lambda\int_{\mathbb{R}^N}H(u,v). \end{aligned} \tag{6}$$

We can prove that $I_\lambda(u,v)$ satisfies the $(\mathrm{PS})_c$ condition at some energy level c and possesses the mountain-pass structure. By using the mountain-pass theorem, we obtain the existence of nontrivial weak solutions for the system (5). In [17], we mostly focus on discussing the properties of the functions $H_u(u,v)$, $H_v(u,v)$ and the associated primitive function $H(u,v)$ which bring some difficulties in proving that $I_\lambda(u,v)$ satisfies the compactness condition. The difficulty is not mainly due to the critical nonlinearities $K(x)|u|^{p^*-2}u$ and $K(x)|v|^{p^*-2}v$.

But, in the current paper, the coupled terms of the system (1) are the critical nonlinearities $K(x)|u|^{\alpha-2}u|v|^{\beta}$ and $K(x)|u|^{\alpha}|v|^{\beta-2}v$ ($\alpha+\beta=p^*$). By using the variational methods, we can establish the existence of nontrivial weak solutions. Although we use similar ideas found in [17], the difficulty is mostly due to the effect of the coupled critical nonlinearities. By means of best Sobolev embedding constant $S_{\alpha,\beta}$ and Holder inequality, we find some energy level c and prove that the corresponding energy functional associated with the system (1) satisfies the $(\mathrm{PS})_c$ condition for all $c<\alpha_0\lambda^{1-N/p}$. Comparing with the procedure in [17], the one in this paper is complex.

Let $\lambda=\varepsilon^{-p}$. Problem (1) is equivalent to the following problem:

$$\begin{aligned} &-\Delta_p u + \lambda V(x)|u|^{p-2}u\\ &\quad = \lambda f(x)|u|^{q-2}u + \frac{\lambda\alpha}{\alpha+\beta}K(x)|u|^{\alpha-2}u|v|^{\beta}, \quad x\in\mathbb{R}^N,\\ &-\Delta_p v + \lambda V(x)|v|^{p-2}v\\ &\quad = \lambda g(x)|v|^{q-2}v + \frac{\lambda\beta}{\alpha+\beta}K(x)|u|^{\alpha}|v|^{\beta-2}v, \quad x\in\mathbb{R}^N,\\ &u(x),v(x)\longrightarrow 0 \quad \text{as } |x|\longrightarrow\infty. \end{aligned} \tag{7}$$

We will prove that problem (7) has at least one nontrivial solution under the suitable conditions on $V(x)$, $K(x)$, $f(x)$, $g(x)$.

Set the space

$$E_\lambda = \left\{u\in W^{1,p}\left(\mathbb{R}^N\right): \int_{\mathbb{R}^N}\lambda V(x)|u|^p<\infty,\ \lambda>0\right\} \tag{8}$$

and the associated norm

$$\|u\|_\lambda^p = \int_{\mathbb{R}^N}\left(|\nabla u|^p + \lambda V(x)|u|^p\right) \tag{9}$$

for any $u\in E_\lambda$. Let $E=E_\lambda\times E_\lambda$. Thus $\|(u,v)\|_E^p = \|u\|_\lambda^p + \|u\|_\lambda^p$. The problem (7) is posed in the framework of the Sobolev space E with the norm

$$\begin{aligned} &\|(u,v)\|_E\\ &\quad = \left(\int_{\mathbb{R}^N}\left(|\nabla u|^p + \lambda V(x)|u|^p + |\nabla v|^p + \lambda V(x)|v|^p\right)\right)^{1/p}. \end{aligned} \tag{10}$$

We will show the existence of nontrivial solutions of (7) by searching for critical points of the associated functional:

$$\begin{aligned} I_p(u,v) = {} & \frac{1}{p}\int_{\mathbb{R}^N}\left(|\nabla u|^p + \lambda V(x)|u|^p + |\nabla v|^p + \lambda V(x)|v|^p\right) \\ & - \frac{\lambda}{q}\int_{\mathbb{R}^N}\left(f(x)|u|^q + g(x)|v|^q\right) \\ & - \frac{\lambda}{\alpha+\beta}\int_{\mathbb{R}^N} K(x)|u|^\alpha |v|^\beta . \end{aligned} \tag{11}$$

In fact, the weak solutions of (7) are the critical points of the functional I_p. As to the weak solution (u,v) of (7), we mean that $(u,v)\in E$ which satisfies

$$\begin{aligned} & \int_{\mathbb{R}^N}\left(|\nabla u|^{p-2}\nabla u\nabla\varphi + \lambda V(x)|u|^{p-2}u\varphi \right. \\ & \quad \left. + |\nabla v|^{p-2}\nabla v\nabla\psi + \lambda V(x)|v|^{p-2}v\psi\right) \\ & = \lambda\int_{\mathbb{R}^N}\left(f(x)|u|^{q-2}u\varphi + g(x)|v|^{q-2}v\psi\right) \\ & \quad + \frac{\lambda\alpha}{\alpha+\beta}\int_{\mathbb{R}^N} K(x)|u|^{\alpha-2}u|v|^\beta\varphi \\ & \quad + \frac{\lambda\beta}{\alpha+\beta}\int_{\mathbb{R}^N} K(x)|u|^\alpha|v|^{\beta-2}v\psi, \quad \forall(\varphi,\psi)\in E. \end{aligned} \tag{12}$$

The main result of this paper reads as follows.

Theorem 1. *Let (H_1)–(H_3) be satisfied. Then, for any $\sigma > 0$, there is $\Lambda_\sigma > 0$ such that if $\lambda > \Lambda_\sigma$, the problem (7) has at least one solution (u_λ, v_λ) which satisfies*

$$\begin{aligned} & \frac{q-p}{pq}\int_{\mathbb{R}^N}\left(|\nabla u_\lambda|^p + |\nabla v_\lambda|^p + \lambda V(x)\left(|u_\lambda|^p + |v_\lambda|^p\right)\right) \\ & \quad \le \sigma\lambda^{1-N/p}. \end{aligned} \tag{13}$$

This paper is organized as follows. In Section 2, we give some notations and preliminaries. Section 3 is devoted to the proof of the main result.

2. Notations and Preliminaries

In this section, we will show the range of c where the $(\mathrm{PS})_c$ condition holds for the functional I_p and prove that I_p possesses the mountain-pass structure. First, we make use of the following notations.

Let $C_0^\infty(\mathbb{R}^N)$ denote the collection of smooth functions with compact support.

Let $D^{1,p}(\mathbb{R}^N)$ be the completion of $C_0^\infty(\mathbb{R}^N)$ under the norm

$$\|u\|^p = \int_{\mathbb{R}^N}|\nabla u|^p . \tag{14}$$

$L^s(\mathbb{R}^N)$, $1\le s<\infty$, denote Lebesgue spaces and the norm of L^s is denoted by $\|\cdot\|_s$ for $1\le s<\infty$. The dual space of a Banach space E will be denoted by E^*. $B_r := \{x\in\mathbb{R}^N : |x|\le r\}$ is the ball in $\mathbb{R}^N$. $o(1)$ denotes $o(1)\to 0$ as $n\to\infty$. c, c_i represent various positive constants, the exact values of which are not important.

$S_{\alpha,\beta}$ is the best Sobolev embedding constant defined by

$$S_{\alpha,\beta} = \inf_{u,v\in W^{1,p}(\mathbb{R}^N)}\frac{\int_{\mathbb{R}^N}\left(|\nabla u|^p + |\nabla v|^p\right)}{\left(\int_{\mathbb{R}^N}|u|^\alpha|v|^\beta\right)^{p/(\alpha+\beta)}}. \tag{15}$$

By use of a similar proof of Theorem 5 in [18], we can obtain that

$$S_{\alpha,\beta} = \left(\left(\frac{\alpha}{\beta}\right)^{\beta/(\alpha+\beta)} + \left(\frac{\beta}{\alpha}\right)^{\alpha/(\alpha+\beta)}\right)S, \tag{16}$$

where S is the best Sobolev embedding constant defined by

$$S = \inf_{u\in W^{1,p}(\mathbb{R}^N)}\frac{\int_{\mathbb{R}^N}|\nabla u|^p}{\left(\int_{\mathbb{R}^N}|u|^{p^*}\right)^{p/p^*}} \tag{17}$$

which is achieved by the function

$$U_\varepsilon(x) = C_N\left(\frac{\varepsilon^{1/p}}{\varepsilon + |x|^{p/(p-1)}}\right)^{(N-p)/p}, \quad \varepsilon > 0. \tag{18}$$

Definition 2. Let $I\in C^1(E,\mathbb{R})$.

(1) A sequence $\{(u_n,v_n)\}\subset E$ which satisfies $I(u_n,v_n) = c + o(1)$ and $I'(u_n,v_n) = o(1)$ strongly in E^* as $n\to\infty$ is called a $(\mathrm{PS})_c$ sequence in E for I.

(2) We say that I satisfies $(\mathrm{PS})_c$ condition if and only if any $(\mathrm{PS})_c$ sequence $\{(u_n,v_n)\}$ in E for I has a convergent subsequence.

The main result of Section 2 is the following compactness result.

Proposition 3. *Assume that (H_1)–(H_3) are satisfied. Then, for any $(PS)_c$ sequence $\{(u_n,v_n)\}$ for I_p, there exists a constant $\alpha_0 > 0$ (independent of λ) such that either $(u_n,v_n)\to(u,v)$ or $c - I_p(u,v)\ge\alpha_0\lambda^{1-N/p}$; that is to say, the functional I_p satisfies the $(PS)_c$ condition for all $c\le\alpha_0\lambda^{1-N/p}$.*

To prove Proposition 3, we need the following lemmas.

Lemma 4. *Assume that (H_1)–(H_3) hold. Let the sequence $\{(u_n,v_n)\}\subset E$ be a $(PS)_c$ sequence for I_p; then we get that $c\ge 0$ and $\{(u_n,v_n)\}$ is bounded in the space E.*

Proof. By direct computation, we have

$$\begin{aligned} I_p(u_n,v_n) &- \frac{1}{q} I_p'(u_n,v_n)(u_n,v_n) \\ &= \frac{1}{p}\|(u_n,v_n)\|_E^p - \frac{\lambda}{q}\int_{\mathbb{R}^N}\left(f(x)|u_n|^q + g(x)|v_n|^q\right) \\ &\quad - \frac{\lambda}{\alpha+\beta}\int_{\mathbb{R}^N} K(x)|u_n|^\alpha |v_n|^\beta \\ &\quad - \frac{1}{q}\left[\|(u_n,v_n)\|_E^p - \lambda\int_{\mathbb{R}^N}\left(f(x)|u_n|^q + g(x)|v_n|^q\right)\right. \\ &\qquad \left. -\lambda\int_{\mathbb{R}^N} K(x)|u_n|^\alpha|v_n|^\beta\right] \\ &= \left(\frac{1}{p}-\frac{1}{q}\right)\|(u_n,v_n)\|_E^p \\ &\quad + \left(\frac{1}{q}-\frac{1}{\alpha+\beta}\right)\lambda\int_{\mathbb{R}^N} K(x)|u_n|^\alpha|v_n|^\beta. \end{aligned} \tag{19}$$

Together with (H_2) and $p < q < p^*$, we get

$$\begin{aligned} I_p(u_n,v_n) &- \frac{1}{q} I_p'(u_n,v_n)(u_n,v_n) \\ &\geq \left(\frac{1}{p}-\frac{1}{q}\right)\|(u_n,v_n)\|_E^p. \end{aligned} \tag{20}$$

By the fact that $I_p(u_n,v_n) = c + o(1)$ and $I_p'(u_n,v_n) = o(1)$, we easily obtain the desired conclusion. □

Lemma 4 shows that a $(\mathrm{PS})_c$ sequence $\{(u_n,v_n)\}$ is bounded. Hence, we may assume that $(u_n,v_n) \rightharpoonup (u,v)$ in E, $u_n \to u$, $v_n \to v$ a.e. in $\mathbb{R}^N$ and $(u_n,v_n) \to (u,v)$ in $L^q_{\mathrm{loc}}(\mathbb{R}^N) \times L^q_{\mathrm{loc}}(\mathbb{R}^N)$ for any $p < q < p^*$.

Lemma 5. *We can choose a subsequence $\{(u_{n_j},v_{n_j})\}$ such that, for any $\varepsilon > 0$, there is $r_\varepsilon > 0$ with $r \geq r_\varepsilon$:*

$$\lim_{j\to\infty}\sup\int_{B_j\setminus B_r}\left(\left|u_{n_j}\right|^q + \left|v_{n_j}\right|^q\right) \leq \varepsilon, \tag{21}$$

where $p < q < p^*$.

Proof. Note that, for each $j \in \mathbb{N}$, we have

$$\int_{B_j}\left(|u_n|^q + |v_n|^q\right) \longrightarrow \int_{B_j}\left(|u|^q + |v|^q\right). \tag{22}$$

So there exists $n_0 \in \mathbb{N}$ such that

$$\int_{B_j}\left(|u_n|^q + |v_n|^q - |u|^q - |v|^q\right) < \frac{1}{j}, \tag{23}$$

for all $n \geq n_0 + 1$. We may choose $n_j = n_0 + j$ such that

$$\int_{B_j}\left(\left|u_{n_j}\right|^q + \left|v_{n_j}\right|^q - |u|^q - |v|^q\right) < \frac{1}{j}. \tag{24}$$

It is obvious that there is r_ε satisfying

$$\int_{\mathbb{R}^N\setminus B_r}\left(|u|^q + |v|^q\right) < \varepsilon \quad \forall r \geq r_\varepsilon. \tag{25}$$

Furthermore

$$\begin{aligned} &\int_{B_j\setminus B_r}\left(\left|u_{n_j}\right|^q + \left|v_{n_j}\right|^q\right) \\ &\quad < \frac{1}{j} + \int_{\mathbb{R}^N\setminus B_r}\left(|u|^q + |v|^q\right) \\ &\qquad + \int_{B_r}\left(|u|^q - \left|u_{n_j}\right|^q + |v|^q - \left|v_{n_j}\right|^q\right), \end{aligned} \tag{26}$$

the lemma follows. □

Let $\eta : \mathbb{R}^+ \to [0,1]$ be a smooth function satisfying $\eta(t) = 1$ if $t \leq 1$ and $\eta(t) = 0$ if $t \geq 3$. Define $\tilde{u}_j(x) = \eta(3|x|/j)u(x)$ and $\tilde{v}_j(x) = \eta(3|x|/j)v(x)$. It is obvious that

$$\left(\tilde{u}_j, \tilde{v}_j\right) \longrightarrow (u,v) \text{ in } E \quad \text{as } j \longrightarrow \infty. \tag{27}$$

Lemma 6. *One has*

$$\begin{aligned} \lim_{j\to\infty}\Bigg|\int_{\mathbb{R}^N} f(x)\Big(&\left|u_{n_j}\right|^{q-2}u_{n_j} - \left|u_{n_j}-\tilde{u}_j\right|^{q-2}\left(u_{n_j}-\tilde{u}_j\right) \\ &- \left|\tilde{u}_j\right|^{q-2}\tilde{u}_j\Big)\varphi\Bigg| = 0, \end{aligned} \tag{28}$$

$$\begin{aligned} \lim_{j\to\infty}\Bigg|\int_{\mathbb{R}^N} g(x)\Big(&\left|v_{n_j}\right|^{q-2}v_{n_j} - \left|v_{n_j}-\tilde{v}_j\right|^{q-2}\left(v_{n_j}-\tilde{v}_j\right) \\ &- \left|\tilde{v}_j\right|^{q-2}\tilde{v}_j\Big)\psi\Bigg| = 0 \end{aligned} \tag{29}$$

uniformly in $(\varphi,\psi) \in E$ with $\|(\varphi,\psi)\|_E \leq 1$.

Proof. Because the proof is similar to the one of Lemma 3.4 [8], we omit it. □

Lemma 7. *One has along a subsequence*

$$\begin{aligned} &I_p\left(u_n-\tilde{u}_n, v_n-\tilde{v}_n\right) \longrightarrow c - I_p(u,v), \\ &I_p'\left(u_n-\tilde{u}_n, v_n-\tilde{v}_n\right) \longrightarrow 0 \text{ in } E^* \quad (\text{the dual space of } E). \end{aligned} \tag{30}$$

Proof. Together with the fact that $(\tilde{u}_j,\tilde{v}_j) \to (u,v)$ in E, we get

$$\begin{aligned} &I_p\left(u_n-\tilde{u}_n, v_n-\tilde{v}_n\right) \\ &\quad = I_p(u_n,v_n) - I_p(\tilde{u}_n,\tilde{v}_n) \\ &\qquad + \frac{\lambda}{p^*}\int_{\mathbb{R}^N} K(x)\Big(|u_n|^\alpha|v_n|^\beta - |u_n-\tilde{u}_n|^\alpha|v_n-\tilde{v}_n|^\beta \\ &\qquad\qquad - |\tilde{u}_n|^\alpha|\tilde{v}_n|^\beta\Big) \end{aligned}$$

$$+\frac{\lambda}{q}\int_{\mathbb{R}^N} f(x)\left(|u_n|^q-|u_n-\tilde{u}_n|^q-|\tilde{u}_n|^q\right)$$
$$+\frac{\lambda}{q}\int_{\mathbb{R}^N} g(x)\left(|v_n|^q-|v_n-\tilde{v}_n|^q-|\tilde{v}_n|^q\right)+o(1). \tag{31}$$

By using similar ideas of proving the Brézis-Lieb Lemma [19], we easily get

$$\begin{aligned}&\lim_{n\to\infty}\int_{\mathbb{R}^N} K(x)\left(|u_n|^\alpha|v_n|^\beta-|u_n-\tilde{u}_n|^\alpha|v_n-\tilde{v}_n|^\beta\right.\\&\qquad\left.-|\tilde{u}_n|^\alpha|\tilde{v}_n|^\beta\right)=0,\\&\lim_{n\to\infty}\int_{\mathbb{R}^N} f(x)\left(|u_n|^q-|u_n-\tilde{u}_n|^q-|\tilde{u}_n|^q\right)=0,\\&\lim_{n\to\infty}\int_{\mathbb{R}^N} g(x)\left(|v_n|^q-|v_n-\tilde{v}_n|^q-|\tilde{v}_n|^q\right)=0.\end{aligned} \tag{32}$$

In connection with the fact that $I_p(u_n,v_n)=c+o(1)$ and $I_p(\tilde{u}_n,\tilde{v}_n)=I_p(u,v)+o(1)$, we obtain

$$I_p(u_n-\tilde{u}_n,v_n-\tilde{v}_n)\longrightarrow c-I_p(u,v). \tag{33}$$

For any $(\varphi,\psi)\in E$, it follows that

$$\begin{aligned}&I_p'(u_n-\tilde{u}_n,v_n-\tilde{v}_n)(\varphi,\psi)\\&\quad=I_p'(u_n,v_n)(\varphi,\psi)-I_p'(\tilde{u}_n,\tilde{v}_n)(\varphi,\psi)\\&\qquad+\frac{\lambda\alpha}{\alpha+\beta}\int_{\mathbb{R}^N} K(x)\left(|u_n|^{\alpha-2}u_n|v_n|^\beta-|u_n-\tilde{u}_n|^{\alpha-2}\right.\\&\qquad\qquad\times(u_n-\tilde{u}_n)|v_n-\tilde{v}_n|^\beta\\&\qquad\qquad\left.-|\tilde{u}_n|^{\alpha-2}\tilde{u}_n|\tilde{v}_n|^\beta\right)\varphi\\&\qquad+\frac{\lambda\beta}{\alpha+\beta}\int_{\mathbb{R}^N} K(x)\left(|u_n|^\alpha|v_n|^{\beta-2}v_n-|u_n-\tilde{u}_n|^\alpha\right.\\&\qquad\qquad\times|v_n-\tilde{v}_n|^{\beta-2}(v_n-\tilde{v}_n)\\&\qquad\qquad\left.-|\tilde{u}_n|^\alpha|\tilde{v}_n|^{\beta-2}\tilde{v}_n\right)\psi\\&\qquad+\lambda\int_{\mathbb{R}^N} f(x)\left(|u_n|^{q-2}u_n-|u_n-\tilde{u}_n|^{q-2}(u_n-\tilde{u}_n)\right.\\&\qquad\qquad\left.-|\tilde{u}_n|^{q-2}\tilde{u}_n\right)\varphi\\&\qquad+\lambda\int_{\mathbb{R}^N} g(x)\left(|v_n|^{q-2}v_n-|v_n-\tilde{v}_n|^{q-2}(v_n-\tilde{v}_n)\right.\\&\qquad\qquad\left.-|\tilde{v}_n|^{q-2}\tilde{v}_n\right)\psi+o(1).\end{aligned} \tag{34}$$

It is standard to check

$$\begin{aligned}&\lim_{n\to\infty}\int_{\mathbb{R}^N} K(x)\left(|u_n|^{\alpha-2}u_n|v_n|^\beta-|u_n-\tilde{u}_n|^{\alpha-2}(u_n-\tilde{u}_n)\right.\\&\qquad\left.\times|v_n-\tilde{v}_n|^\beta-|\tilde{u}_n|^{\alpha-2}\tilde{u}_n|\tilde{v}_n|^\beta\right)\varphi=0,\end{aligned} \tag{35}$$

$$\begin{aligned}&\lim_{n\to\infty}\int_{\mathbb{R}^N} K(x)\left(|u_n|^\alpha|v_n|^{\beta-2}v_n-|u_n-\tilde{u}_n|^\alpha|v_n-\tilde{v}_n|^{\beta-2}\right.\\&\qquad\left.\times(v_n-\tilde{v}_n)-|\tilde{u}_n|^\alpha|\tilde{v}_n|^{\beta-2}\tilde{v}_n\right)\psi=0\end{aligned} \tag{36}$$

uniformly in $\|(\varphi,\psi)\|_E\le 1$.

Combining Lemma 6 and $I_p'(u_n,v_n)\to 0$, we complete the proof of Lemma 7. □

Set $u_n^1=u_n-\tilde{u}_n$ and $v_n^1=v_n-\tilde{v}_n$; then $u_n-u=u_n^1+(\tilde{u}_n-u)$ and $v_n-v=v_n^1+(\tilde{v}_n-v)$. In order to check $(u_n,v_n)\to(u,v)$ in E, we only prove $(u_n^1,v_n^1)\to(0,0)$ in E.

Observe that

$$\begin{aligned}&I_p(u_n^1,v_n^1)-\frac{1}{p}I_p'(u_n^1,v_n^1)(u_n^1,v_n^1)\\&\quad=\left(\frac{1}{p}-\frac{1}{\alpha+\beta}\right)\lambda\int_{\mathbb{R}^N} K(x)|u_n^1|^\alpha|v_n^1|^\beta\\&\qquad+\left(\frac{1}{p}-\frac{1}{q}\right)\lambda\int_{\mathbb{R}^N}\left(f(x)|u_n^1|^q+g(x)|v_n^1|^q\right)\\&\quad\ge\frac{\lambda}{N}K_{\min}\int_{\mathbb{R}^N}|u_n^1|^\alpha|v_n^1|^\beta,\end{aligned} \tag{37}$$

where $K_{\min}=\inf_{x\in\mathbb{R}^N}K(x)>0$. By Lemma 7, we get

$$\int_{\mathbb{R}^N}|u_n^1|^\alpha|v_n^1|^\beta\le\frac{N\left(c-I_p(u,v)\right)}{\lambda K_{\min}}+o(1). \tag{38}$$

In addition, by (H_1)

$$\begin{aligned}&\int_{\mathbb{R}^N} V(x)\left(|u_n^1|^p+|v_n^1|^p\right)\\&\quad=\int_{\mathbb{R}^N} V_b(x)\left(|u_n^1|^p+|v_n^1|^p\right)+o(1),\end{aligned} \tag{39}$$

where $V_b(x):=\max\{V(x),b\}$. Furthermore, by using Holder inequality, $\alpha+\beta=p^*$, $p<q<p^*$, (H_2) and (H_3), for any $b>0$, there is a constant $C_b>0$ such that

$$\begin{aligned}&\int_{\mathbb{R}^N}\left(K(x)|u_n^1|^\alpha|v_n^1|^\beta+f(x)|u_n^1|^q+g(x)|v_n^1|^q\right)\\&\quad\le b\left(\|u_n^1\|_p^p+\|v_n^1\|_p^p\right)+C_b\int_{\mathbb{R}^N}|u_n^1|^\alpha|v_n^1|^\beta.\end{aligned} \tag{40}$$

Thus

$$\begin{aligned}
&S_{\alpha,\beta}\left(\int_{\mathbb{R}^N}\left|u_n^1\right|^\alpha\left|v_n^1\right|^\beta\right)^{p/(\alpha+\beta)}\\
&\quad\le\int_{\mathbb{R}^N}\left(\left|\nabla u_n^1\right|^p+\left|\nabla v_n^1\right|^p\right)\\
&\quad=\int_{\mathbb{R}^N}\left(\left|\nabla u_n^1\right|^p+\left|\nabla v_n^1\right|^p+\lambda V(x)\left|u_n^1\right|^p+\lambda V(x)\left|v_n^1\right|^p\right)\\
&\qquad-\int_{\mathbb{R}^N}\lambda V(x)\left(\left|u_n^1\right|^p+\left|v_n^1\right|^p\right)\\
&\quad=\int_{\mathbb{R}^N}\left(K(x)\left|u_n^1\right|^\alpha\left|v_n^1\right|^\beta+f(x)\left|u_n^1\right|^q+g(x)\left|v_n^1\right|^q\right)\\
&\qquad-\lambda\int_{\mathbb{R}^N}V_b(x)\left(\left|u_n^1\right|^p+\left|v_n^1\right|^p\right)+o(1)\\
&\quad\le\lambda C_b\int_{\mathbb{R}^N}\left|u_n^1\right|^\alpha\left|v_n^1\right|^\beta+o(1).
\end{aligned}\tag{41}$$

Together with (38), we have

$$\begin{aligned}
S_{\alpha,\beta}&\le\lambda C_b\left(\int_{\mathbb{R}^N}\left|u_n^1\right|^\alpha\left|v_n^1\right|^\beta\right)^{1-p/(\alpha+\beta)}+o(1)\\
&\le\lambda C_b\left(\frac{N\left(c-I_p(u,v)\right)}{\lambda K_{\min}}\right)^{p/N}+o(1).
\end{aligned}\tag{42}$$

Set $\alpha_0 = S_{\alpha,\beta}^{N/p}C_b^{-N/p}N^{-1}K_{\min}$. This implies $\alpha_0\lambda^{1-N/p} \le c - I_p(u,v) + o(1)$. In the following, we give the proof of Proposition 3.

Proof of Proposition 3. For any $(\mathrm{PS})_c$ sequence $\{(u_n,v_n)\}\subset E$ with $(u_n,v_n)\rightharpoonup(u,v)$, it follows that either $(u_n,v_n)\to(u,v)$ or $c-I_p(u,v)\ge\alpha_0\lambda^{1-N/p}$. On the contrary, if $(u_n,v_n)\nrightarrow(u,v)$, this shows

$$\begin{aligned}
\liminf_{n\to\infty}\left\|(u_n^1,v_n^1)\right\|_E&>0,\\
c-I_p(u,v)&>0.
\end{aligned}\tag{43}$$

In connection with the above-mentioned analysis, we get that the functional $I_p(u,v)$ satisfies the $(\mathrm{PS})_c$ condition for all $c<\alpha_0\lambda^{1-N/p}$. □

3. Proof of the Main Result

First, we will prove that the functional I_p possesses the mountain-pass structure.

Lemma 8. *Assume that (H_1)–(H_3) be satisfied. There exist α_λ, $\rho_\lambda>0$ such that*

$$\begin{aligned}
I_p(u,v)&>0\quad\text{if}\ 0<\|(u,v)\|_E<\rho_\lambda,\\
I_p(u,v)&\ge\alpha_\lambda\quad\text{if}\ \|(u,v)\|_E=\rho_\lambda.
\end{aligned}\tag{44}$$

Proof. Observe that, for each $s\in[p,p^*]$, there is c_s such that if $\lambda\ge1$,

$$\|u\|_s\le c_s\|u\|_{E_\lambda}\quad\forall u\in E_\lambda.\tag{45}$$

Together with Young inequality, we get

$$\int_{\mathbb{R}^N}K(x)|u|^\alpha|v|^\beta\le c_1\left(\|u\|_{p^*}^{p^*}+\|v\|_{p^*}^{p^*}\right)\le c_1c_{p^*}\|(u,v)\|_E^{p^*}.\tag{46}$$

Combining (H_3) and (46), there is a constant c_δ such that

$$\begin{aligned}
I_p(u,v)&\ge\frac{1}{2p}\|(u,v)\|_E^p-c_\delta\|(u,v)\|_E^{p^*}\\
&=\frac{1}{2p}\|(u,v)\|_E^p\left(1-2pc_\delta\|(u,v)\|_E^{p^*-p}\right).
\end{aligned}\tag{47}$$

Set $\rho_\lambda=(1/4pc_\delta)^{1/(p^*-p)}$; it implies

$$I_p(u,v)\ge\frac{1}{4p}\rho_\lambda^p\triangleq\alpha_\lambda>0\quad\text{if}\ \|(u,v)\|_E=\rho_\lambda.\tag{48}$$

We complete the proof. □

Lemma 9. *Under the assumptions of Lemma 8, for any finite-dimensional subspace $F\subset E$, we have*

$$I_p(u,v)\longrightarrow-\infty\quad\text{as}\ \|(u,v)\|_E\longrightarrow\infty\quad\text{for}\ (u,v)\in F.\tag{49}$$

Proof. Together with the assumptions (H_2) and (H_3), it follows that

$$I_p(u,v)\le\frac{1}{p}\|(u,v)\|_E^p-\lambda c_3\|(u,v)\|_q^q\quad\forall(u,v)\in E,\tag{50}$$

where $c_3=\min_{x\in\mathbb{R}^N}[f(x),g(x)]$.

Noting that all norms in a finite-dimensional space are equivalent and $q>p$, Lemma 9 follows. □

Next, we will search for special finite-dimensional subspaces by which we establish sufficiently small minimax levels.

Define the functionals

$$\begin{aligned}
\Phi_p(u,v)&=\frac{1}{p}\int_{\mathbb{R}^N}\left(|\nabla u|^p+\lambda V(x)|u|^p+|\nabla v|^p+\lambda V(x)|v|^p\right)\\
&\quad-\lambda c_3\int_{\mathbb{R}^N}\left(|u|^q+|v|^q\right),\\
\Psi_p(u,v)&=\frac{1}{p}\int_{\mathbb{R}^N}\left(|\nabla u|^p+|\nabla v|^p+V\left(\lambda^{-1/p}x\right)\left(|u|^p+|v|^p\right)\right)\\
&\quad-c_3\int_{\mathbb{R}^N}\left(|u|^q+|v|^q\right).
\end{aligned}\tag{51}$$

It is apparent that $\Phi_p\in C^1(E)$ and $I_p(u,v)\le\Phi_p(u,v)$ for all $(u,v)\in E$.

Observe that

$$\inf\left\{\int_{\mathbb{R}^N}|\nabla\phi|^p : \phi\in C_0^\infty\left(\mathbb{R}^N,\mathbb{R}\right), \|\phi\|_q=1\right\}=0. \quad (52)$$

For any $\delta>0$, there are $\phi_\delta,\ \psi_\delta\in C_0^\infty(\mathbb{R}^N,\mathbb{R})$ with $\|\phi_\delta\|_q=\|\psi_\delta\|_q=1$ such that

$$\operatorname{supp}\left(\phi_\delta,\psi_\delta\right)\subset B_{r_\delta}(0), \qquad |\nabla\phi_\delta|_p^p, |\nabla\psi_\delta|_p^p<\delta. \quad (53)$$

Let $e_\lambda(x)=(\phi_\delta(\sqrt[p]{\lambda}x),\psi_\delta(\sqrt[p]{\lambda}x))$; then $\operatorname{supp} e_\lambda\subset B_{\lambda^{-1/p}r_\delta}(0)$. For $t\geq 0$, we get

$$\begin{aligned}
&\Phi_p\left(te_\lambda\right)=\lambda^{1-N/p}\Psi_p\left(t\phi_\delta,t\psi_\delta\right),\\
&\max_{t\geq 0}\Psi_p\left(t\phi_\delta,t\psi_\delta\right)\\
&\quad\leq\frac{q-p}{pq\left(qc_3\right)^{p/(q-p)}}\left\{\int_{\mathbb{R}^N}\left(|\nabla\phi_\delta|^p+V\left(\lambda^{-1/p}x\right)|\phi_\delta|^p\right)\right\}^{q/(q-p)}\\
&\qquad+\frac{q-p}{pq\left(qc_3\right)^{p/(q-p)}}\\
&\qquad\times\left\{\int_{\mathbb{R}^N}\left(|\nabla\psi_\delta|^p+V\left(\lambda^{-1/p}x\right)|\psi_\delta|^p\right)\right\}^{q/(q-p)}.
\end{aligned} \quad (54)$$

Combining $V(0)=0$ and $\operatorname{supp}(\phi_\delta,\psi_\delta)\subset B_{r_\delta}(0)$, this implies that there is $\Lambda_\delta>0$ such that, for all $\lambda\geq\Lambda_\delta$, we have

$$\max_{t\geq 0}\Phi_p\left(t\phi_\delta,t\psi_\delta\right)\leq\lambda^{1-N/p}\frac{2\left(q-p\right)}{pq\left(qc_3\right)^{p/(q-p)}}\left(2\delta\right)^{q/(q-p)}. \quad (55)$$

Thus, for $\lambda\geq\Lambda_\delta$,

$$\max_{t\geq 0}I_p\left(te_\lambda\right)\leq\lambda^{1-N/p}\frac{2\left(q-p\right)}{pq\left(qc_3\right)^{p/(q-p)}}\left(2\delta\right)^{q/(q-p)}. \quad (56)$$

It follows from (56) that we have the following.

Lemma 10. *For any $\sigma>0$, there is $\Lambda_\sigma>0$ such that, for each $\lambda\geq\Lambda_\sigma$, there exists $\bar{e}_\lambda\in E$ with $\|\bar{e}_\lambda\|_E>\rho_\lambda$, $I_p(\bar{e}_\lambda)\leq 0$ and*

$$\max_{t\geq 0}I_p\left(t\bar{e}_\lambda\right)\leq\sigma\lambda^{1-N/p}, \quad (57)$$

where ρ_λ is defined in Lemma 8.

Proof. For any $\sigma>0$, we can choose $\delta>0$ so small that

$$\frac{2\left(q-p\right)}{pq\left(qc_3\right)^{p/(q-p)}}\left(2\delta\right)^{q/(q-p)}\leq\sigma. \quad (58)$$

Set $e_\lambda(x)=(\phi_\delta(\sqrt[p]{\lambda}x),\psi_\delta(\sqrt[p]{\lambda}x))$. Taking $\Lambda_\delta=\Lambda_\sigma$, there is $\bar{t}_\lambda>0$ such that $\|\bar{t}_\lambda e_\lambda\|_E>\rho_\lambda$ and $I_p(te_\lambda)\leq 0$ for all $t\geq\bar{t}_\lambda$. By (56), we choose $\bar{e}_\lambda=\bar{t}_\lambda e_\lambda$ which satisfies the requirements. □

Finally, we will give the proof of the main result.

Proof of Theorem 1. Denote by

$$c_\lambda=\inf_{\gamma\in\Gamma_\lambda}\max_{t\in[0,1]}I_p\left(\gamma\left(t\right)\right), \quad (59)$$

where $\Gamma_\lambda=\{\gamma\in C([0,1],E):\gamma(0)=0,\gamma(1)=\bar{e}_\lambda\}$.

By Lemma 10, for any $\sigma>0$ with $0<\sigma<\alpha_0$, there is $\Lambda_\sigma>0$ such that, for $\lambda\geq\Lambda_\sigma$, we choose c_λ satisfying $c_\lambda\leq\sigma\lambda^{1-N/p}$.

It is clear that the functional I_p satisfies $(\mathrm{PS})_{c_\lambda}$ condition and has the mountain-pass structure if $c_\lambda\leq\sigma\lambda^{1-N/p}$. Hence, by the mountain-pass theorem, there is $(u_\lambda,v_\lambda)\in E$ such that

$$I_p\left(u_\lambda,v_\lambda\right)=c_\lambda, \qquad I_p'\left(u_\lambda,v_\lambda\right)=0. \quad (60)$$

Namely, (u_λ,v_λ) is a weak solution of (7). Similar to the arguments in [8], we also get that (u_λ,v_λ) is a positive least energy solution. Furthermore,

$$\begin{aligned}
I_p\left(u_\lambda,v_\lambda\right)&=I_p\left(u_\lambda,v_\lambda\right)-\frac{1}{q}I_p'\left(u_\lambda,v_\lambda\right)\left(u_\lambda,v_\lambda\right)\\
&\geq\left(\frac{1}{p}-\frac{1}{q}\right)\|(u_\lambda,v_\lambda)\|_E^p.
\end{aligned} \quad (61)$$

This shows that

$$\frac{q-p}{pq}\|(u_\lambda,v_\lambda)\|_E^p\leq I_p\left(u_\lambda,v_\lambda\right)=c_\lambda\leq\sigma\lambda^{1-N/p}. \quad (62)$$

The proof is complete. □

Conflict of Interests

The authors declare that they have no competing interests.

Authors' Contribution

The authors contributed equally to this paper. They read and approved the final paper.

Acknowledgments

The authors would like to thank the referees for their precious comments and suggestions about the original paper. This research was supported by the Fundamental Research Funds for the Central Universities (2013XK03).

References

[1] C. O. Alves and Y. H. Ding, "Multiplicity of positive solutions to a p-Laplacian equation involving critical nonlinearity," *Journal of Mathematical Analysis and Applications*, vol. 279, no. 2, pp. 508–521, 2003.

[2] A. Ambrosetti and P. H. Rabinowitz, "Dual variational methods in critical point theory and applications," *Journal of Functional Analysis*, vol. 14, pp. 349–381, 1973.

[3] V. Benci and G. Cerami, "Existence of positive solutions of the equation $-\Delta u+a(x)u=uN+2/N-2$ in $\mathbb{R}^N$," *Journal of Functional Analysis*, vol. 88, no. 1, pp. 90–117, 1990.

[4] H. Brézis and L. Nirenberg, "Some variational problems with lack of compactness," in *Nonlinear Functional Analysis and Its Applications*, F. E. Browder, Ed., vol. 45 of *Proceedings of Symposia in Pure Mathematics*, pp. 165–201, 1986.

[5] S. Cingolani and M. Lazzo, "Multiple positive solutions to nonlinear Schrödinger equations with competing potential functions," *Journal of Differential Equations*, vol. 160, no. 1, pp. 118–138, 2000.

[6] M. Clapp and Y. Ding, "Minimal nodal solutions of a Schrödinger equation with critical nonlinearity and symmetric potential," *Differential and Integral Equations*, vol. 16, no. 8, pp. 981–992, 2003.

[7] M. del Pino and P. L. Felmer, "Multi-peak bound states for nonlinear Schrödinger equations," *Annales de l'Institut Henri Poincaré*, vol. 15, no. 2, pp. 127–149, 1998.

[8] Y. H. Ding and F. H. Lin, "Solutions of perturbed Schrödinger equations with critical nonlinearity," *Calculus of Variations and Partial Differential Equations*, vol. 30, no. 2, pp. 231–249, 2007.

[9] A. Floer and A. Weinstein, "Nonspreading wave packets for the cubic Schrödinger equation with a bounded potential," *Journal of Functional Analysis*, vol. 69, no. 3, pp. 397–408, 1986.

[10] J. P. García Azorero and I. Peral Alonso, "Existence and nonuniqueness for the p-Laplacian: nonlinear eigenvalues," *Communications in Partial Differential Equations*, vol. 12, no. 12, pp. 1389–1430, 1987.

[11] M. Guedda and L. Veron, "Quasilinear elliptic equations involving critical Sobolev exponents," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 13, no. 8, pp. 879–902, 1989.

[12] C. Liu and Y. Zheng, "Existence of nontrivial solutions for p-laplacian equations in $\mathbb{R}^N$," *Journal of Mathematical Analysis and Applications*, vol. 380, no. 2, pp. 669–679, 2011.

[13] Y.-G. Oh, "On positive multi-lump bound states of nonlinear Schrödinger equations under multiple well potential," *Communications in Mathematical Physics*, vol. 131, no. 2, pp. 223–253, 1990.

[14] T. F. Wu, "The Nehari manifold for a semilinear elliptic system involving sign-changing weight functions," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 68, no. 6, pp. 1733–1745, 2008.

[15] T.-S. Hsu and H.-L. Lin, "Multiple positive solutions for a critical elliptic system with concave-convex nonlinearities," *Proceedings of the Royal Society of Edinburgh Section A*, vol. 139, no. 6, pp. 1163–1177, 2009.

[16] T.-S. Hsu, "Multiple positive solutions for a critical quasilinear elliptic system with concave-convex nonlinearities," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 7-8, pp. 2688–2698, 2009.

[17] H. X. Zhang and W. B. Liu, "Existence of nontrivial solutions to perturbed p-Laplacian system in $\mathbb{R}^N$ involving critical nonlinearity," *Boundary Value Problems*, vol. 2012, article 53, 2012.

[18] C. O. Alves, D. C. de Morais Filho, and M. A. Souto, "On systems of elliptic equations involving subcritical or critical Sobolev exponents," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 42, pp. 771–787, 2000.

[19] H. Brézis and E. Lieb, "A relation between pointwise convergence of functions and convergence of functionals," *Proceedings of the American Mathematical Society*, vol. 88, no. 3, pp. 486–490, 1983.