Research Article
Existence of Nontrivial Solutions for a Critical Perturbed Quasilinear Elliptic System

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Abstract

We consider a perturbed quasilinear elliptic system involving the \( p \)-Laplacian with critical growth terms in \( \mathbb{R}^N \). Under proper conditions, we establish the existence of nontrivial solutions by using the variational methods.

1. Introduction

In this paper, we are concerned with the following perturbed quasilinear elliptic system involving the \( p \)-Laplacian:

\[
-\varepsilon \Delta_p u + V(x)|u|^{p-2}u = f(x)|u|^{q-2}u + \alpha \frac{\beta}{\alpha + \beta} K(x)|u|^\alpha |V|^{\beta}, \quad x \in \mathbb{R}^N,
\]
\[
-\varepsilon \Delta_p V + V(x)|V|^{p-2}V = \gamma \delta \frac{\gamma}{\alpha + \beta} K(x)|u|^\alpha |V|^{\beta}, \quad x \in \mathbb{R}^N,
\]

where \( 2 < p < N, \Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u) \) is the \( p \)-Laplacian operator, \( \alpha > 1, \beta > 1 \) satisfy \( \alpha + \beta = p^* \), \( p^* = \frac{Np}{N-p} \) is the critical Sobolev exponent for \( N \geq 3 \), \( p < q < p^* \) and the functions \( f(x), g(x), \gamma, \gamma \delta \) satisfy some suitable conditions.

Set \( \alpha = \beta, f(x) = g(x), \) and \( u = v \). The problem (1) reduces to the semilinear scalar quasilinear elliptic equation

\[
-\varepsilon \Delta_p u + V(x)|u|^{p-2}u = f(x)|u|^{q-2}u + \frac{\alpha}{\alpha + \beta} K(x)|u|^\alpha |v|^{\beta}, \quad x \in \mathbb{R}^N.
\]

The type of problem (2) including \( p = 2 \) and \( p \neq 2 \) has been extensively studied in many papers involving bounded domain and unbounded domain. See, for example, [1–13] and the references therein.

In recent years, much attention has been paid to the existence of solutions for problem (1) with \( \varepsilon = 1 \) and \( p = 2 \) in bounded domain. Wu [14] was concerned with the following semilinear elliptic system with subcritical nonlinearity of concave-convex type and sign-changing weights:

\[
-\Delta u = \lambda f(x)|u|^{q-2}u + \frac{\alpha}{\alpha + \beta} h(x)|u|^{a-2}u |v|^{\beta}, \quad x \in \Omega,
\]
\[
-\Delta v = \mu g(x)|v|^{q-2}v + \frac{\beta}{\alpha + \beta} h(x)|u|^{a}|v|^{\beta-2}v, \quad x \in \Omega.
\]

where \( 1 < q < 2, \alpha > 1, \beta > 1 \) satisfy \( 2 < \alpha + \beta < 2^* \) and the functions \( f(x), g(x), h(x) \) satisfy some suitable conditions. He established the existence of at least two positive solutions for the problem (3) when the pair of the parameters \((\lambda, \mu)\)
belongs to a certain subset of $\mathbb{R}^2$. Hsu and Lin [15] considered a similar problem and proved that the problem (3) has at least two positive solutions in $W^{1,2}_0(\Omega) \times W^{1,2}_0(\Omega)$ involving critical exponents. Subsequently, Hsu [16] extended the results of [15] to the quasilinear case $p > 1$. The paper [16] was devoted to the following quasilinear elliptic system:

$$
\begin{align*}
-\Delta_p u &= \lambda |u|^{p-2} u + \frac{2\alpha}{\alpha + \beta} |u|^{\alpha-2} u |v|^\beta, & x \in \Omega, \\
-\Delta_p v &= \mu |v|^{p-2} v + \frac{2\beta}{\alpha + \beta} |u|^\alpha |v|^{\beta-2} v, & x \in \Omega,
\end{align*}
$$

(4)

where $\lambda, \mu > 0$, $\alpha > 1$, $\beta > 1$ satisfy $\alpha + \beta = p^*$ and $p^* = \frac{Np}{N-p}$ denotes the critical Sobolev exponent. He proved that the problem (4) has at least two positive solutions in $W^{1,p}_0(\Omega) \times W^{1,p}_0(\Omega)$.

However, as far as we know, there are almost no results on problem (1) involving critical exponents in whole space. In our work, we consider the problem (1) and use variational methods to get positive solutions. Our main arguments use similar ideas found in [8, 15]. The main difficulty is that the coupled system (5) shows that the coupled terms are due to unbounded domain but not for some estimates and results hold for the Laplacian operator. At the same time, the differences between these two papers.

In particular, we will mention our own work [17] and further discuss the differences between these two papers. In [17], we are concerned with the following system:

$$
\begin{align*}
-\Delta_p u + \lambda V(x) |u|^{p-2} u &= \lambda K(x) |u|^{p-2} u + \lambda H_u(u, v), & x \in \mathbb{R}^N, \\
-\Delta_p v + \lambda V(x) |v|^{p-2} v &= \lambda K(x) |v|^{p-2} v + \lambda H_v(u, v), & x \in \mathbb{R}^N,
\end{align*}
$$

(5)

$$
\begin{align*}
&u(x), v(x) > 0, & x \in \mathbb{R}^N, \\
&u(x), v(x) \rightarrow 0, & \text{as } |x| \rightarrow \infty.
\end{align*}

The coupled system (5) shows that the coupled terms are $H_u(u, v)$ and $H_v(u, v)$. The energy functional associated with (5) is defined by

$$
I_\lambda(u, v) = \frac{1}{p} \int_{\mathbb{R}^N} (|\nabla u|^p + \lambda V(x) |u|^p + |\nabla v|^p + \lambda V(x) |v|^p) - \frac{\lambda}{p^*} \int_{\mathbb{R}^N} K(x) (|u|^{p^*} + |v|^{p^*}) - \lambda \int_{\mathbb{R}^N} H(u, v).
$$

(6)

We can prove that $I_\lambda(u, v)$ satisfies the (PS)$_c$ condition at some energy level $c$ and possesses the mountain-pass structure. By using the mountain-pass theorem, we obtain the existence of nontrivial weak solutions for the system (5). In [17], we mostly focus on discussing the properties of the functions $H_u(u, v)$, $H_v(u, v)$ and the associated primitive function $H(u, v)$ which bring some difficulties in proving that $I_\lambda(u, v)$ satisfies the compactness condition. The difficulty is not mainly due to the critical nonlinearities $K(x)|u|^{p^*-2}u$ and $K(x)|v|^{p^*-2}v$.

But, in the current paper, the coupled terms of the system (1) are the critical nonlinearities $K(x)|u|^{p^*-2}u$ and $K(x)|v|^{p^*-2}v$ ($\alpha + \beta = p^*$). By using the variational methods, we can establish the existence of nontrivial weak solutions. Although we use similar ideas found in [17], the difficulty is mostly due to the effect of the coupled critical nonlinearities. By means of best Sobolev embedding constant $S_{\alpha, \beta}$ and Holder inequality, we find some energy level $c$ and prove that the corresponding energy functional associated with the system (1) satisfies the (PS)$_c$ condition for all $c < \alpha_0 \lambda^{1-N/p}$. Comparing with the procedure in [17], the one in this paper is complex.

Let $\lambda = \varepsilon^{-p}$. Problem (1) is equivalent to the following problem:

$$
\begin{align*}
-\Delta_p u + \lambda V(x) |u|^{p-2} u &= \lambda f(x)|u|^{p-2} u + \frac{\lambda \alpha}{\alpha + \beta} K(x) |u|^{\alpha-2} u |v|^{\beta}, & x \in \mathbb{R}^N, \\
-\Delta_p v + \lambda V(x) |v|^{p-2} v &= \lambda g(x) |v|^{p-2} v + \frac{\lambda \beta}{\alpha + \beta} K(x) |u|^\alpha |v|^{\beta-2} v, & x \in \mathbb{R}^N,
\end{align*}
$$

(7)

We will prove that problem (7) has at least one nontrivial solution under the suitable conditions on $V(x)$, $K(x)$, $f(x)$, $g(x)$.

Set the space

$$
E_\lambda = \left\{ u \in W^{1,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N} \lambda V(x) |u|^p < \infty, \lambda > 0 \right\}
$$

and the associated norm

$$
\|u\|_E^p = \int_{\mathbb{R}^N} (|\nabla u|^p + \lambda V(x) |u|^p)
$$

(9)

for any $u \in E_\lambda$. Let $E = E_\lambda \times E_{\lambda}$. Thus $\|u, v\|_E^p = \|u\|_E^p + \|v\|_E^p$. The problem (7) is posed in the framework of the Sobolev space $E$ with the norm

$$
\|(u, v)\|_E = \left( \int_{\mathbb{R}^N} (|\nabla u|^p + \lambda V(x) |u|^p + |\nabla v|^p + \lambda V(x) |v|^p) \right)^{1/p}.
$$

(10)
We will show the existence of nontrivial solutions of (7) by searching for critical points of the associated functional:

\[ I_p(u, v) = \frac{1}{p} \int_{\mathbb{R}^N} \left( |\nabla u|^p + \lambda V(x) |u|^p + |\nabla v|^p + \lambda V(x) |v|^p \right) \]

\[ + \frac{\lambda}{q} \int_{\mathbb{R}^N} \left( f(x) |u|^q + g(x) |v|^q \right) + \frac{\lambda}{\alpha + \beta} \int_{\mathbb{R}^N} K(x) |u|^{\alpha} |v|^{\beta}. \]

(11)

In fact, the weak solutions of (7) are the critical points of the functional \( I_p \). As to the weak solution \((u, v)\) of (7), we mean that \((u, v) \in E\) which satisfies

\[ \int_{\mathbb{R}^N} \left( |\nabla u|^{p-2} \nabla u \nabla \varphi + \lambda V(x) |u|^{p-2} u \varphi \right) + |\nabla v|^{p-2} \nabla v \nabla \psi + \lambda V(x) |v|^{p-2} v \psi \]

\[ = \lambda \int_{\mathbb{R}^N} \left( f(x) |u|^q + g(x) |v|^q \right) + \frac{\lambda \alpha}{\alpha + \beta} \int_{\mathbb{R}^N} K(x) |u|^{\alpha} \varphi + \frac{\lambda \beta}{\alpha + \beta} \int_{\mathbb{R}^N} K(x) |v|^{\beta} \psi, \quad \forall (\varphi, \psi) \in E. \]

(12)

The main result of this paper reads as follows.

**Theorem 1.** Let \((H_1)-(H_3)\) be satisfied. Then, for any \( \sigma > 0 \), there is \( \Lambda_\sigma > 0 \) such that if \( \lambda > \Lambda_\sigma \), the problem (7) has at least one solution \((u_\Lambda, v_\Lambda)\) which satisfies

\[ \frac{q - p}{pq} \int_{\mathbb{R}^N} \left( |\nabla u_\Lambda|^p + |\nabla v_\Lambda|^p + \lambda V(x) \left( |u_\Lambda|^p + |v_\Lambda|^p \right) \right) \leq \sigma \lambda^{1-N/p}. \]

(13)

This paper is organized as follows. In Section 2, we give some notations and preliminaries. Section 3 is devoted to the proof of the main result.

### 2. Notations and Preliminaries

In this section, we will show the range of \( c \) where the \( (PS)_c \) condition holds for the functional \( I_p \) and prove that \( I_p \) possesses the mountain-pass structure. First, we make use of the following notations.

- Let \( C_0^\infty(\mathbb{R}^N) \) denote the collection of smooth functions with compact support.
- Let \( D^{1,p}(\mathbb{R}^N) \) be the completion of \( C_0^\infty(\mathbb{R}^N) \) under the norm

\[ \| u \| = \int_{\mathbb{R}^N} |\nabla u|^p. \]

(14)

\( L^1(\mathbb{R}^N) \), \( 1 \leq s < \infty \), denote Lebesgue spaces and the norm of \( L^1 \) is denoted by \( \| u \|_1 \) for \( 1 \leq s < \infty \). The dual space of a Banach space \( E \) will be denoted by \( E^* \). \( B_r := \{ x \in \mathbb{R}^N : |x| \leq r \} \) is the ball in \( \mathbb{R}^N \). \( o(1) \) denotes \( o(1) \to 0 \) as \( n \to \infty \). \( c, \epsilon \) represent various positive constants, the exact values of which are not important.

- \( S_{\alpha, \beta} \) is the best Sobolev embedding constant defined by

\[ S_{\alpha, \beta} = \inf_{u \in W^{1,p}(\mathbb{R}^N)} \left( \int_{\mathbb{R}^N} |\nabla u|^p + |V u|^p \right)^{\alpha/(\alpha + \beta)}. \]

(15)

By use of a similar proof of Theorem 5 in [18], we can obtain that

\[ S_{\alpha, \beta} = \left( \left( \frac{\alpha}{\beta} \right)^{\beta/(\alpha + \beta)} + \left( \frac{\beta}{\alpha} \right)^{\alpha/(\alpha + \beta)} \right) S, \]

(16)

where \( S \) is the best Sobolev embedding constant defined by

\[ S = \inf_{u \in W^{1,p}(\mathbb{R}^N)} \left( \int_{\mathbb{R}^N} |\nabla u|^p \right)^{1/p}. \]

(17)

which is achieved by the function

\[ U_\epsilon(x) = C_N \left( \frac{\epsilon^{1/p}}{\epsilon + |x|^{(N-p)/p}} \right)^{(N-p)/p}, \quad \epsilon > 0. \]

(18)

**Definition 2.** Let \( I \in C^1(\mathbb{R}, \mathbb{R}) \).

- (1) A sequence \( \{(u_n, v_n)\} \subset E \) which satisfies \( I(u_n, v_n) = c + o(1) \) and \( I'(u_n, v_n) = o(1) \) strongly in \( E^* \) as \( n \to \infty \) is called a \((PS)_c \) sequence in \( E \) for \( I \).
- (2) We say that \( I \) satisfies \((PS)_c \) condition if and only if any \((PS)_c \) sequence \( \{(u_n, v_n)\} \) in \( E \) for \( I \) has a convergent subsequence.

The main result of Section 2 is the following compactness result.

**Proposition 3.** Assume that \((H_1)-(H_3)\) are satisfied. Then, for any \((PS)_c \) sequence \( \{(u_n, v_n)\} \) for \( I_p \), there exists a constant \( \alpha_0 > 0 \) (independent of \( \lambda \)) such that either \( (u_n, v_n) \to (u, v) \) or \( c - I_p(u, v) \geq \alpha_0 \lambda^{1-N/p} \); that is to say, the functional \( I_p \) satisfies the \((PS)_c \) condition for all \( c \leq \alpha_0 \lambda^{1-N/p} \).

To prove Proposition 3, we need the following lemmas.

**Lemma 4.** Assume that \((H_1)-(H_3)\) hold. Let the sequence \( \{(u_n, v_n)\} \subset E \) be a \((PS)_c \) sequence for \( I_p \); then we get that \( c \geq 0 \) and \( \{(u_n, v_n)\} \) is bounded in the space \( E \).
Proof. By direct computation, we have

\[ I_p(u_n, v_n) - I_p'(u_n, v_n)(u_n, v_n) \]

\[ = \frac{1}{p} \| (u_n, v_n) \|_E^p - \frac{1}{q} \int \left( f(x) |u_n|^p + g(x) |v_n|^p \right) \]

\[ = \frac{1}{p} \| (u_n, v_n) \|_E^p - \frac{1}{q} \int \left( f(x) |u_n|^p + g(x) |v_n|^p \right) \]

\[ - \frac{\lambda}{\alpha + \beta} \int K(x) |u_n|^\alpha |v_n|^\beta \]

\[ = \left( \frac{1}{p} - \frac{1}{q} \right) \| (u_n, v_n) \|_E^p \]

\[ + \left( \frac{1}{q} - \frac{1}{\alpha + \beta} \right) \lambda \int K(x) |u_n|^\alpha |v_n|^\beta. \]  

(19)

Together with \((H_2)\) and \(p < q < p^*\), we get

\[ I_p(u_n, v_n) - I_p'(u_n, v_n)(u_n, v_n) \]

\[ \geq \left( \frac{1}{p} - \frac{1}{q} \right) \| (u_n, v_n) \|_E^p. \]  

(20)

By the fact that \(I_p(u_n, v_n) = c + o(1)\) and \(I_p'(u_n, v_n) = o(1)\), we easily obtain the desired conclusion. \(\square\)

Lemma 4 shows that a (PS)_c sequence \(\{(u_n, v_n)\}\) is bounded. Hence, we may assume that \((u_n, v_n) \rightharpoonup (u, v) \) in \(E\), \(u_n \to u, v_n \to v\) a.e. in \(\mathbb{R}^N\) and \((u_n, v_n) \to (u, v)\) in \(L_1^q(\mathbb{R}^N) \times L_1^{q'}(\mathbb{R}^N)\) for any \(p < q < p^*\).

**Lemma 5.** We can choose a subsequence \(\{(u_n, v_n)\}\) such that, for any \(\epsilon > 0\), there is \(r_\epsilon > 0\) with \(r \geq r_\epsilon\):

\[ \lim_{j \to \infty} \sup \int_{B_j \setminus B_j} \left( |u_n|^q + |v_n|^q \right) \leq \epsilon, \]  

where \(p < q < p^*\).

**Proof.** Note that, for each \(j \in \mathbb{N}\), we have

\[ \int_{B_j} \left( |u_n|^q + |v_n|^q \right) \to \int_{B_j} (|u|^q + |v|^q). \]  

(22)

So there exists \(n_0 \in \mathbb{N}\) such that

\[ \int_{B_j} \left( |u_n|^q + |v_n|^q - |u|^q - |v|^q \right) < \frac{1}{j}, \]  

(23)

for all \(n \geq n_0 + 1\). We may choose \(n_j = n_0 + j\) such that

\[ \int_{B_j} \left( |u_n|^q + |v_n|^q - |u|^q - |v|^q \right) < \frac{1}{j}. \]  

(24)

It is obvious that there is \(r_\epsilon\) satisfying

\[ \int_{\mathbb{R}^N \setminus B_r} \left( |u|^q + |v|^q \right) \leq \epsilon \]  

\(\forall r \geq r_\epsilon. \)  

(25)

Furthermore

\[ \int_{B_r \setminus B_r} \left( |u_n|^q + |v_n|^q \right) \leq \frac{1}{j} + \int_{\mathbb{R}^N \setminus B_r} \left( |u|^q + |v|^q \right) \]

\[ < \frac{1}{j} + \int_{B_r} \left( |u|^q + |v|^q \right) \]

\[ + \int_{B_r} \left( |u|^q - |u_n|^q + |v|^q - |v_n|^q \right), \]  

the lemma follows. \(\square\)

Let \(\eta : \mathbb{R}^+ \to [0, 1]\) be a smooth function satisfying \(\eta(t) = 1\) if \(t \leq 1\) and \(\eta(t) = 0\) if \(t \geq 3\). Define \(\bar{u}_j(x) = \eta(3|x/j|)u(x)\) and \(\bar{v}_j(x) = \eta(3|x/j|)v(x)\). It is obvious that

\[ (\bar{u}_j, \bar{v}_j) \to (u, v) \]  

in \(E\) as \(j \to \infty. \)  

(27)

**Lemma 6.** One has

\[ \lim_{j \to \infty} \int_{\mathbb{R}^N} f(x) \left( |u_n|^q - |u_n|^q - |u_n - \bar{u}_j|^q - |u_n - \bar{u}_j|^q \right) \]

\[ = \lim_{j \to \infty} \int_{\mathbb{R}^N} f(x) \left( |u_n|^q - |u_n|^q - |u_n - \bar{u}_j|^q - |u_n - \bar{u}_j|^q \right) \]

\[ = -\int_{\mathbb{R}^N} f(x) \left( |u_n|^q - |u_n|^q - |u_n - \bar{u}_j|^q - |u_n - \bar{u}_j|^q \right) \]

\[ = 0, \]  

(28)

\[ \lim_{j \to \infty} \int_{\mathbb{R}^N} g(x) \left( |v_n|^q - |v_n|^q - |v_n - \bar{v}_j|^q - |v_n - \bar{v}_j|^q \right) \]

\[ = 0, \]  

(29)

uniformly in \((\varphi, \psi) \in E\) with \(|(\varphi, \psi)|_E \leq 1\).

**Proof.** Because the proof is similar to the one of Lemma 3.4 [8], we omit it. \(\square\)

**Lemma 7.** One has along a subsequence

\[ I_p(u_n - \bar{u}_n, v_n - \bar{v}_n) \rightharpoonup c - I_p(u, v), \]

\[ I_p'(u_n - \bar{u}_n, v_n - \bar{v}_n) \to 0 \text{ in } E^* \]  

the dual space of \(E\).  

(30)

**Proof.** Together with the fact that \((\bar{u}_j, \bar{v}_j) \to (u, v) \) in \(E\), we get

\[ I_p(u_n - \bar{u}_n, v_n - \bar{v}_n) \]

\[ = I_p(u_n, v_n) - I_p(u, v) \]

\[ + \frac{\lambda}{p^*} \int_{\mathbb{R}^N} K(x) \left( |u_n|^\alpha |v_n|^\beta - |u_n - \bar{u}_n|^\alpha |v_n - \bar{v}_n|^\beta \right) \]

\[ - |\bar{u}_n|^\alpha |\bar{v}_n|^\beta \]  

(31)

\[ \leq \int_{\mathbb{R}^N} \left( |u_n|^p + |v_n|^p \right) \leq \epsilon \]  

\(\forall r \geq r_\epsilon. \)  

(25)
By using similar ideas of proving the Brézis-Lieb Lemma [19], we easily get

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} K(x) \left[ |u_n|^p |v_n|^\beta - |u_n - \bar{u}_n|^p |v_n - \bar{v}_n|^\beta \right. \\
\left. - \beta |\bar{u}_n|^\beta |\bar{v}_n|^\beta \right] = 0, \\
\lim_{n \to \infty} \int_{\mathbb{R}^N} f(x) \left[ |u_n|^p - |u_n - \bar{u}_n|^p + |\bar{u}_n|^p \right] = 0, \\
\lim_{n \to \infty} \int_{\mathbb{R}^N} g(x) \left[ |v_n|^p - |v_n - \bar{v}_n|^p - |\bar{v}_n|^p \right] = 0. 
\]

(32)

In connection with the fact that \( I_p(u_n, v_n) = c + o(1) \) and \( I_p(\bar{u}_n, \bar{v}_n) = I_p(u, v) + o(1) \), we obtain

\[
I_p \left( u_n - \bar{u}_n, v_n - \bar{v}_n \right) \to c - I_p(u, v). 
\]

(33)

For any \((\varphi, \psi) \in E\), it follows that

\[
I_p' \left( u_n - \bar{u}_n, v_n - \bar{v}_n \right) (\varphi, \psi) \\
= I_p' \left( u_n, v_n \right) (\varphi, \psi) - I_p' \left( \bar{u}_n, \bar{v}_n \right) (\varphi, \psi) \\
+ \frac{\lambda}{\alpha + \beta} \int_{\mathbb{R}^N} K(x) \left[ |u_n|^{p-2} u_n |v_n|^\beta - |u_n - \bar{u}_n|^{p-2} \right. \\
\times \left. (u_n - \bar{u}_n) |v_n - \bar{v}_n|^\beta \right. \\
\left. - \beta |\bar{u}_n|^{\beta-2} |\bar{v}_n|^\beta \right] \varphi \\
+ \frac{\lambda \beta}{\alpha + \beta} \int_{\mathbb{R}^N} K(x) \left[ |u_n|^{p-2} |v_n|^\beta - |u_n - \bar{u}_n|^{p-2} \right. \\
\times \left. (u_n - \bar{u}_n) |v_n - \bar{v}_n|^\beta \right. \\
\left. - \beta |\bar{u}_n|^{\beta-2} |\bar{v}_n|^\beta \right] \psi \\
+ \lambda \int_{\mathbb{R}^N} f(x) \left[ |u_n|^{q-2} u_n - |u_n - \bar{u}_n|^{q-2} (u_n - \bar{u}_n) \\
- \beta |\bar{u}_n|^{q-2} \bar{u}_n \right] \varphi \\
+ \lambda \int_{\mathbb{R}^N} g(x) \left[ |v_n|^{q-2} v_n - |v_n - \bar{v}_n|^{q-2} (v_n - \bar{v}_n) \\
- \beta |\bar{v}_n|^{q-2} \bar{v}_n \right] \psi + o(1). 
\]

(34)

It is standard to check

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} K(x) \left[ |u_n|^{p-2} u_n |v_n|^\beta - |u_n - \bar{u}_n|^{p-2} (u_n - \bar{u}_n) \right. \\
\times \left. |v_n - \bar{v}_n|^\beta - |\bar{u}_n|^{\beta-2} |\bar{v}_n|^\beta \right] \varphi = 0, \\
\lim_{n \to \infty} \int_{\mathbb{R}^N} K(x) \left[ |u_n|^{p-2} v_n - |u_n - \bar{u}_n|^{p-2} |v_n - \bar{v}_n|^\beta \right. \\
\times \left. |v_n - \bar{v}_n| - |\bar{u}_n|^\beta |\bar{v}_n|^\beta \right] \psi = 0 
\]

(35)

uniformly in \( \| (\varphi, \psi) \|_E \leq 1 \).

Combining Lemma 6 and \( I_p'(u_n, v_n) \to 0 \), we complete the proof of Lemma 7.

(36)

Set \( u_n^1 = u_n - \bar{u}_n \) and \( v_n^1 = v_n - \bar{v}_n \); then \( u_n - u = u_n^1 + (\bar{u}_n - u) \) and \( v_n - v = v_n^1 + (\bar{v}_n - v) \). In order to check \((u_n, v_n) \to (u, v)\) in \( E \), we only prove \((u_n^1, v_n^1) \to (0, 0) \) in \( E \).

Observe that

\[
I_p \left( u_n^1, v_n^1 \right) - \frac{1}{p} I_p' \left( u_n^1, v_n^1 \right) (u_n^1, v_n^1) \\
= \left( \frac{1}{p} - \frac{1}{\alpha + \beta} \right) \lambda \int_{\mathbb{R}^N} K(x) |u_n^1|^p |v_n^1|^\beta \\
+ \left( \frac{1}{p} - \frac{1}{q} \right) \frac{\lambda}{\alpha} \int_{\mathbb{R}^N} \left[ f(x) |u_n^1|^q + g(x) |v_n^1|^q \right] \\
\geq \frac{\lambda}{N} K_{\min} \int_{\mathbb{R}^N} |u_n^1|^p |v_n^1|^\beta, 
\]

(37)

where \( K_{\min} = \inf_{x \in \mathbb{R}^N} K(x) > 0 \). By Lemma 7, we get

\[
\int_{\mathbb{R}^N} |u_n^1|^p |v_n^1|^\beta \leq \frac{N (c - I_p(u, v))}{\lambda K_{\min}} + o(1). 
\]

(38)

In addition, by \((H_1)\)

\[
\int_{\mathbb{R}^N} V(x) \left[ |u_n^1|^p + |v_n^1|^p \right] \\
= \int_{\mathbb{R}^N} V_b(x) \left[ |u_n^1|^p + |v_n^1|^p \right] + o(1), 
\]

(39)

where \( V_b(x) := \max \{ V(x), b \} \). Furthermore, by using Holder inequality, \( \alpha + \beta = p^*, p < q < p^*, (H_2) \) and \((H_3)\), for any \( b > 0 \), there is a constant \( C_b > 0 \) such that

\[
\int_{\mathbb{R}^N} \left( K(x) |u_n^1|^p |v_n^1|^\beta + f(x) |u_n^1|^q + g(x) |v_n^1|^q \right) \\
\leq b \left( \| u_n^1 \|_p^p + \| v_n^1 \|_p^p \right) + C_b \int_{\mathbb{R}^N} |u_n^1|^p |v_n^1|^\beta. 
\]
Thus
\[ S_{\alpha,\beta} \left( \int_{\mathbb{R}^N} \left( |\nabla u|^q + |\nabla v|^q \right) \right)^{p/(\alpha+\beta)} \]
\[ \leq \int_{\mathbb{R}^N} \left( |\nabla u|^q + |\nabla v|^q \right) \]
\[ = \int_{\mathbb{R}^N} \left( |\nabla u|^q + |\nabla v|^q + \lambda V(x) |u|^p + \lambda V(x) |v|^p \right) \]
\[ - \int_{\mathbb{R}^N} \lambda V(x) \left( |u|^p + |v|^p \right) \]
\[ = \int_{\mathbb{R}^N} (K(x) |u|^q + f(x) |u|^q + g(x) |v|^q) \]
\[ - \lambda \int_{\mathbb{R}^N} V_n(x) \left( |u|^p + |v|^p \right) + o(1) \]
\[ \leq \lambda C_b \int_{\mathbb{R}^N} |u|^q \left| |u|^p \right| + o(1). \]

Together with (38), we have
\[ S_{\alpha,\beta} \leq \lambda C_b \left( \int_{\mathbb{R}^N} |u|^q \left| |u|^p \right| \right)^{1-p/(\alpha+\beta)} + o(1) \]
\[ \leq \lambda C_b \left( \frac{N (c - I_p(u,v))}{\lambda K_{\min}} \right)^{p/N} + o(1). \]

Set \( \alpha_0 = \frac{S^{\text{opt}}_{\alpha,\beta} C_{\text{opt}} \lambda^{-1} N K_{\min}}{c - I_p(u,v)} / \lambda K_{\min} \). This implies \( \alpha_0 \lambda^{-1} N \leq c - I_p(u,v) + o(1) \). In the following, we give the proof of Proposition 3.

Proof of Proposition 3. For any \((\text{PS})_c\) sequence \( \{(u_n, v_n)\} \subset E \) with \( (u_n, v_n) \to (u, v) \), it follows that either \( (u_n, v_n) \to (u, v) \) or \( c - I_p(u, v) \geq \alpha_0 \lambda^{-1} N \). On the contrary, if \( (u_n, v_n) \to (u, v) \), this shows
\[ \lim_{n \to \infty} \inf \| (u_n, v_n) \|_E > 0, \]
\[ c - I_p(u, v) > 0. \]

In connection with the above-mentioned analysis, we get that the functional \( I_p(u, v) \) satisfies the \((\text{PS})_c\) condition for all \( c < \alpha_0 \lambda^{-1} N \).

3. Proof of the Main Result

First, we will prove that the functional \( I_p \) possesses the mountain-pass structure.

Lemma 8. Assume that \((H_1) - (H_3)\) be satisfied. There exist \( \alpha_0, \rho_\lambda > 0 \) such that
\[ I_p(u, v) > 0 \quad \text{if} \quad 0 < \| (u, v) \|_E < \rho_\lambda, \]
\[ I_p(u, v) \geq \alpha_0 \quad \text{if} \quad \| (u, v) \|_E = \rho_\lambda. \]

\[ \Phi_p(u, v) = \frac{1}{p} \int_{\mathbb{R}^N} \left( |\nabla u|^p + \lambda V(x) |u|^p + |\nabla v|^p + \lambda V(x) |v|^p \right) \]
\[ - \lambda c_3 \int_{\mathbb{R}^N} (|u|^q + |v|^q), \]
\[ \Psi_p(u, v) = \frac{1}{p} \int_{\mathbb{R}^N} \left( |\nabla u|^p + |\nabla v|^p + V \left( \lambda^{-1/2} x \right) (|u|^p + |v|^p) \right) \]
\[ - c_3 \int_{\mathbb{R}^N} (|u|^q + |v|^q). \]

It is apparent that \( \Phi_p \in C^1(E) \) and \( I_p(u, v) \leq \Phi_p(u, v) \) for all \( (u, v) \in E \).

Proof. Observe that, for each \( s \in [p, p^*] \), there is \( c_s \) such that if \( \lambda \geq 1 \),
\[ \|u\| \leq c_s \|u\|_E \quad \forall u \in E_\lambda. \]
Observe that
\[ \inf \left\{ \int_{\mathbb{R}^N} |\nabla \phi|^p : \phi \in C_0^\infty(\mathbb{R}^N, \mathbb{R}) \text{ and } \|\phi\|_q = 1 \right\} = 0. \tag{52} \]
For any \( \delta > 0 \), there are \( \phi_{\delta}, \psi_{\delta} \in C_0^\infty(\mathbb{R}^N, \mathbb{R}) \) with \( \|\phi_{\delta}\|_q = \|\psi_{\delta}\|_q = 1 \) such that
\[ \sup \left\{ (\phi_{\delta}, \psi_{\delta}) \in B_{e_{\delta}}(0), \|\nabla \phi_{\delta}\|^p, \|\nabla \psi_{\delta}\|^p \right\} < \delta. \tag{53} \]
Let \( e_\lambda(x) = (\phi_{\lambda}(\sqrt{\lambda}x), \psi_{\lambda}(\sqrt{\lambda}x)) \); then \( \sup e_\lambda \subset B_{\lambda^{-1/2} e_{\delta}}(0) \).
For \( t \geq 0 \), we get
\[ \Phi_p(t e_\lambda) = \lambda^{1-N/p} \Psi_p(t \phi_{\lambda}, t \psi_{\lambda}), \]
\[ \max_{t \geq 0} \Psi_p(t \phi_{\lambda}, t \psi_{\lambda}) \]
\[ \leq \frac{q - p}{pq(q_\delta)^{p/(q-p)}} \left\{ \int_{\mathbb{R}^N} \left( |\nabla \phi_{\delta}|^p + V \left( \lambda^{-1/p} x \right) |\phi_{\delta}|^p \right) \right\}^{q/(q-p)} \]
\[ + \frac{q - p}{pq(q_\delta)^{p/(q-p)}} \left\{ \int_{\mathbb{R}^N} \left( |\nabla \psi_{\delta}|^p + V \left( \lambda^{-1/p} x \right) |\psi_{\delta}|^p \right) \right\}^{q/(q-p)}. \tag{54} \]
Combining \( V(0) = 0 \) and \( \sup (\phi_{\delta}, \psi_{\delta}) \subset B_{e_{\delta}}(0) \), this implies that there is \( \Lambda_\delta \geq 0 \) such that, for all \( \lambda \geq \Lambda_\delta \), we have
\[ \max_{t \geq 0} \Phi_p(t \phi_{\lambda}, t \psi_{\lambda}) \leq \lambda^{1-N/p} \frac{2 (q - p)}{pq(q_\delta)^{p/(q-p)}} (2\delta)^{q/(q-p)}. \tag{55} \]
Thus, for \( \lambda \geq \Lambda_\delta \),
\[ \max_{t \geq 0} I_p(t e_\lambda) \leq \lambda^{1-N/p} \frac{2 (q - p)}{pq(q_\delta)^{p/(q-p)}} (2\delta)^{q/(q-p)}. \tag{56} \]
It follows from (56) that we have the following.

**Lemma 10.** For any \( \sigma > 0 \), there is \( \Lambda_\sigma > 0 \) such that, for each \( \lambda \geq \Lambda_\sigma \), there exists \( \bar{e}_\lambda \in E \) with \( \|\bar{e}_\lambda\|_E \geq \rho_\lambda \), \( I_p(\bar{e}_\lambda) \leq 0 \) and
\[ \max_{t \geq 0} I_p(t \bar{e}_\lambda) \leq \sigma \lambda^{1-N/p}, \tag{57} \]
where \( \rho_\lambda \) is defined in Lemma 8.

**Proof.** For any \( \sigma > 0 \), we can choose \( \delta > 0 \) so small that
\[ \frac{2 (q - p)}{pq(q_\delta)^{p/(q-p)}} (2\delta)^{q/(q-p)} \leq \sigma. \tag{58} \]
Set \( e_\lambda(x) = (\phi_{\lambda}(\sqrt{\lambda}x), \psi_{\lambda}(\sqrt{\lambda}x)) \). Taking \( \Lambda_\delta = \Lambda_\sigma \), there is \( \bar{e}_\lambda > 0 \) such that \( \|\bar{e}_\lambda e_\lambda\|_E \geq \rho_\lambda \) and \( I_p(t \bar{e}_\lambda) \leq 0 \) for all \( t \geq \bar{e}_\lambda \). By (56), we choose \( \bar{e}_\lambda = \bar{e}_\lambda e_\lambda \) which satisfies the requirements.

Finally, we will give the proof of the main result.

**Proof of Theorem 1.** Denote by
\[ c_\lambda = \inf_{r \in [1, r_0]} \max_{t \in [0, 1]} I_p(r(t)), \tag{59} \]
where \( \Gamma_\lambda = \{ r \in C([0, 1], E) : r(0) = 0, r(1) = \bar{e}_\lambda \} \).
By Lemma 10, for any \( \sigma > 0 \) with \( 0 < \sigma < \alpha_0 \), there is \( \Lambda_\sigma > 0 \) such that, for \( \lambda \geq \Lambda_\sigma \), we choose \( c_\lambda \) satisfying \( c_\lambda \leq \sigma \lambda^{1-N/p} \).
It is clear that the functional \( I_p \) satisfies (PS)\( c_\lambda \) condition and has the mountain-pass structure if \( c_\lambda \leq \sigma \lambda^{1-N/p} \).
Hence, by the mountain-pass theorem, there is \((u_{\lambda}, v_{\lambda}) \in E \) such that
\[ I_p(u_{\lambda}, v_{\lambda}) = c_\lambda, \quad I'_p(u_{\lambda}, v_{\lambda}) = 0. \tag{60} \]
Namely, \((u_{\lambda}, v_{\lambda})\) is a weak solution of (7). Similar to the arguments in [8], we also get that \((u_{\lambda}, v_{\lambda})\) is a positive least energy solution. Furthermore,
\[ I_p(u_{\lambda}, v_{\lambda}) = I_p(u_{\lambda}, v_{\lambda}) - \frac{1}{p} I_p'(u_{\lambda}, v_{\lambda})(u_{\lambda}, v_{\lambda}) \]
\[ \geq \left( \frac{1}{p} - \frac{1}{q} \right) \| (u_{\lambda}, v_{\lambda}) \|_E^p. \tag{61} \]
This shows that
\[ \frac{q - p}{pq} \| (u_{\lambda}, v_{\lambda}) \|_E^p \leq I_p(u_{\lambda}, v_{\lambda}) = c_\lambda \leq \sigma \lambda^{1-N/p}. \tag{62} \]
The proof is complete.

**Conflict of Interests**
The authors declare that they have no competing interests.

**Authors’ Contribution**
The authors contributed equally to this paper. They read and approved the final paper.

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