Research Article

Marcinkiewicz Integrals on Weighted Weak Hardy Spaces

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We prove that, under the condition $\Omega \in \text{Lip}_\alpha$, Marcinkiewicz integral μ_Ω is bounded from weighted weak Hardy space $WH_w^p(\mathbb{R}^n)$ to weighted weak Lebesgue space $WL_w^p(\mathbb{R}^n)$ for $\max\{n/(n+1/2), n/(n+\alpha)\} < p \leq 1$, where w belongs to the Muckenhoupt weight class. We also give weaker smoothness condition assumed on Ω to imply the boundedness of μ_Ω from $WH_w^1(\mathbb{R}^n)$ to $WL_w^1(\mathbb{R}^n)$.

1. Introduction and Results

Suppose that S^{n-1} is the unit sphere in $\mathbb{R}^n$ ($n \geq 2$) equipped with Lebesgue measure $d\sigma(x')$. Let Ω be homogeneous of degree zero on $\mathbb{R}^n$ satisfying $\Omega \in L^1(S^{n-1})$ and

$$\int_{S^{n-1}} \Omega(x') d\sigma(x') = 0, \quad (1)$$

where $x' = x/|x|$ for any $x \neq 0$. Then, the Marcinkiewicz integral operator μ_Ω is defined by

$$\mu_\Omega(f)(x) = \left(\int_0^\infty |F_{\Omega,t}(f)(x)|^2 \frac{dt}{t^3} \right)^{1/2}, \quad (2)$$

where

$$F_{\Omega,t}(f)(x) = \int_{|x-y| \leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y) dy. \quad (3)$$

The operator μ_Ω was originally introduced by Marcinkiewicz [1] in 1938 for $n = 1$ and $\Omega(t) = \text{sign } t$. In 1958, Stein [2] defined the Marcinkiewicz integral of higher dimensions and proved that if $\Omega \in \text{Lip}_\alpha(S^{n-1})$, $0 < \alpha \leq 1$, then μ_Ω is of strong type (p, p) for $1 < p \leq 2$ and of weak type $(1, 1)$. We say that $\Omega \in \text{Lip}_\alpha(S^{n-1})$ if

$$|\Omega(x') - \Omega(y')| \leq |x' - y'|^\alpha, \quad x', y' \in S^{n-1}. \quad (4)$$

In 1962, Benedek et al. [3] showed that if Ω is continuously differentiable on S^{n-1} , then μ_Ω is of strong type (p, p) for all

$1 < p < \infty$. In 1990, Torchinsky and Wang [4] considered the weighted case and proved that if $\Omega \in \text{Lip}_\alpha(S^{n-1})$, $0 < \alpha \leq 1$, then, for all $w \in A_p$ (Muckenhoupt weight class), μ_Ω is bounded on $L_w^p(\mathbb{R}^n)$.

In 1999, Ding et al. [5] improved Torchinsky and Wang's result, by ridding Ω of the smoothness condition, as follows.

Theorem A. Let $\Omega \in L^q(S^{n-1})$, $1 < q \leq \infty$, and satisfy (1). If $w^{q'} \in A_p$, $1 < p < \infty$, then there is $C > 0$ independent of f such that

$$\|\mu_\Omega(f)\|_{L_w^p} \leq C \|f\|_{L_w^p}, \quad (5)$$

where q' is the dual of q such that $1/q + 1/q' = 1$.

In 2003, Ding et al. [6] discussed the boundedness of Marcinkiewicz integral on weighted Hardy space, getting the following results.

Theorem B. Let $0 < \alpha \leq 1$, $\beta = \min\{\alpha, 1/2\}$, and $n/(n+\beta) < p < 1$. Let $\Omega \in \text{Lip}_\alpha(\mathbb{R}^n)$ satisfy (1). If $w \in A_{p(1+\beta/n)}$, then there exists a constant $C > 0$ independent of f such that $\|\mu_\Omega(f)\|_{L_w^p} \leq C \|f\|_{H_w^p}$.

Theorem C. Let Ω satisfy (1) and L^q condition ($q > 1$). If $w^{q'} \in A_1$, then there is a $C > 0$ independent of f such that $\|\mu_\Omega(f)\|_{L_w^p} \leq C \|f\|_{H_w^1}$.

In 2007, C. C. Lin and Y. C. Lin [7] showed that under weaker condition assumed on Ω , which is called L^q -Dini type condition of order α , Marcinkiewicz integral operator μ_Ω is bounded from H_w^p to L_w^p .

We say that Ω satisfies the L^q -Dini condition if $\Omega \in L^q(S^{n-1})$, $q \geq 1$, is homogenous of degree zero on R^n , and

$$\int_0^\infty \frac{w_q(\delta)}{\delta} d\delta < \infty, \quad (6)$$

where $w_q(\delta)$ denotes the integral modulus of continuity q of Ω defined by

$$w_q(\delta) = \sup_{|\rho| < \delta} \left(\int_{S^{n-1}} |\Omega(\rho x') - \Omega(x')|^q d\sigma(x') \right)^{1/q}, \quad (7)$$

and ρ is a rotation in R^n with $|\rho| = \|\rho - I\|$. If $q = 1$, we let $w(\delta) = w_q(\delta)$.

In 2002, Ding et al. [8] considered the boundedness of Marcinkiewicz integral on weak Hardy space; they obtained the following result.

Theorem D. Suppose that $\Omega \in L^1(S^{n-1})$ satisfies (1) and

$$\int_0^1 \frac{w(\delta)}{\delta} (1 + |\log \delta|)^\sigma d\delta < \infty, \quad \text{for some } \sigma > 1. \quad (8)$$

Then, there exists a constant $C > 0$ such that

$$\|\mu_\Omega(f)\|_{WL^1} \leq C \|f\|_{WH^1}. \quad (9)$$

In this paper, we will investigate the boundedness of Marcinkiewicz integral on weighted weak Hardy space. We have the following conclusions.

Theorem 1. Suppose that $\Omega \in \text{Lip}_\alpha(R^n)$, $0 < \alpha \leq 1$, satisfies (1) and $\beta = \min\{\alpha, 1/2\}$. If $n/(n + \beta) < p \leq 1$ and $w \in A_{p(1+\beta/n)}$, then there exists a constant $C > 0$ independent of f such that $\|\mu_\Omega(f)\|_{WL_w^p} \leq C \|f\|_{WH_w^p}$.

Theorem 2. Let $\Omega \in L^q(S^{n-1})$, $1 < q \leq \infty$, satisfy (1) and

$$\int_0^1 \frac{w_q(\delta)}{\delta} (1 + |\log \delta|)^\sigma d\delta < \infty, \quad \text{for some } \sigma > 1. \quad (10)$$

If $w^{q'} \in A_1$, then there is a $C > 0$ independent of f such that $\|\mu_\Omega(f)\|_{WL_w^1} \leq C \|f\|_{WH_w^1}$.

Remark 3. From [8, Remark 2], Condition (10) is weaker than Condition (4).

2. Some Preliminaries and Notations

We begin this section with some properties of A_p weights which play an important role in the proofs of our main results.

A weight w is a nonnegative, locally integrable function on R^n . Let $B = B(x_0, r_B)$ denote the ball with the center x_0 and radius r_B . For any ball B and $\lambda > 0$, λB denotes the ball concentric with B whose radius is λ times as long. For a given weight function w and a measurable set E , we also denote the Lebesgue measure of E by $|E|$ and set weighted measure $w(E) = \int_E w(x) dx$.

Definition 4. A weight w is said to belong to A_p for $1 < p < \infty$, if there exists a constant C such that for every ball, $B \subset R^n$,

$$\left(\frac{1}{|B|} \int_B w(x) dx \right) \left(\frac{1}{|B|} \int_B w(x)^{1-p'} dx \right)^{p-1} \leq C. \quad (11)$$

The class A_1 is defined by replacing the above inequality with

$$\frac{1}{|B|} \int_B w(y) dy \leq C \cdot \text{ess inf}_{x \in B} w(x) \quad (12)$$

for every ball $B \subset R^n$.

The classical A_p weight theory was first introduced by Muckenhoupt in the study of weighted L^p boundedness of Hardy-Littlewood maximal functions in [9].

We know that if $w \in A_p$ with $1 < p < \infty$, then $w \in A_r$ for all $r > p$ and $w \in A_q$ for some $1 < q < p$. Therefore, we will use the notation $q_w \equiv \inf\{q > 1 : w \in A_q\}$ to denote the critical index of w . Obviously, if $w \in A_q$, $q > 1$, then we have $1 \leq q_w < q$.

We state the following results about A_p class that will be used later on.

Lemma 5 (see [10]). Let $w \in A_p$, $p \geq 1$. Then, for any ball B and any $\lambda > 1$, there exists an absolute constant $C > 0$ such that

$$w(\lambda B) \leq C \lambda^{np} w(B), \quad (13)$$

where C depends neither on B nor on λ .

Lemma 6 (see [10]). Let $w \in A_q$ with $q \geq 1$. Then, for any $r > 0$, there exists an absolute $C > 1$ independent of r such that

$$\int_{|x-x_0| \geq r} \frac{w(x)}{|x-x_0|^{nq}} dx \leq C r^{-nq} w(B(x_0, 2r)). \quad (14)$$

We also need the following result about L^q -Dini condition.

Lemma 7 (see [11]). Suppose that $q > 1$ and Ω satisfies L^q -Dini condition. Then, for $|y| \leq R/2$,

$$\begin{aligned} & \left(\int_{R < |x| < 2R} \left| \frac{\Omega(x-y)}{|x-y|^{n-1}} - \frac{\Omega(x)}{|x|^{n-1}} \right|^q dx \right)^{1/q} \\ & \leq C R^{n/q-n+1} \left(\frac{|y|}{R} + \int_{|y|/2R < \delta < |y|/R} \frac{w_q(\delta)}{\delta} d\delta \right). \end{aligned} \quad (15)$$

Given a Muckenhoupt weight function w on R^n , for $0 < p < \infty$, we denote by L_w^p the space of all functions satisfying

$$\|f\|_{L_w^p} = \left(\int_{R^n} |f(x)|^p w(x) dx \right)^{1/p} < \infty. \quad (16)$$

When $q = \infty$, L_w^∞ will be taken to mean L^∞ , and we set $\|f\|_{L_w^\infty} = \|f\|_{L^\infty}$. And also the weighted weak Lebesgue spaces is defined by

$$\|f\|_{WL_w^p} = \sup_{\lambda > 0} \lambda w(\{x \in R^n : |f(x)| > \lambda\})^{1/p} < \infty. \quad (17)$$

Let us now turn to the weighted weak Hardy spaces. The (unweighted) weak H^p spaces have first appeared in the work of Fefferman et al. [12]. The atomic decomposition theory of weak H^1 space on R^n was given by Fefferman and Soria in [13]. In 2000, Quek and Yang [14] introduced the weighted weak Hardy spaces WH_w^p and established their atomic decompositions. Moreover, by using the atomic decomposition theory of WH_w^p , Quek and Yang also obtained the boundedness of Calderón-Zygmund type operators on these weighted spaces.

We write $\mathcal{S}(R^n)$ to denote the Schwartz space of all rapidly decreasing smooth functions and $\mathcal{S}'(R^n)$ to denote the space of all tempered distributions, that is, the topological dual of $\mathcal{S}(R^n)$. Let $w \in A_\infty$, $0 < p \leq 1$, and $N = [n(q_w/p - 1)]$. Define

$$\mathcal{A}_{N,w} = \left\{ \varphi \in \mathcal{S}(R^n) : \sup_{x \in R^n} \sup_{|\alpha| \leq N+1} (1 + |x|)^{N+n+1} \times |D^\alpha \varphi(x)| \leq 1 \right\}, \quad (18)$$

where $\alpha = (\alpha_1, \dots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n$, $|\alpha| = \alpha_1 + \dots + \alpha_n$, and $D^\alpha \varphi = \partial^{|\alpha|} \varphi / (\partial x_1^{\alpha_1}, \dots, \partial x_n^{\alpha_n})$.

For $\mathcal{S}'(R^n)$ the grand maximal function of f is defined by

$$G_w(f)(x) = \sup_{\varphi \in \mathcal{A}_{N,w}} \sup_{|x-y| < t} |(\varphi_t * f)(y)|. \quad (19)$$

Then, we can define the weighted weak Hardy space $WH_w^p(R^n)$ by

$$WH_w^p(R^n) = \{ \mathcal{S}'(R^n) : G_w f \in WL_w^p(R^n) \}. \quad (20)$$

Moreover, we set $\|f\|_{WH_w^p} = \|G_w f\|_{WL_w^p}$.

Lemma 8 (see [14]). *Let $0 < p \leq 1$ and $w \in A_\infty$. For every f in $WH_w^p(R^n)$, there exists a sequence of bounded measurable functions $\{f_k\}_{k=-\infty}^\infty$ such that*

- (i) $f = \sum_{k=-\infty}^\infty f_k$ in the sense of distributions.
- (ii) Each f_k can be further decomposed into $f_k = \sum_i b_i^k$, where $\{b_i^k\}$ satisfies the following:
 - (a) each b_i^k is supported in a ball B_i^k with $\sum_i w(B_i^k) \leq c2^{-kp}$ and $\sum_i \chi_{B_i^k}(x) \leq c$. Here, χ_E denotes the characteristic function of the set E and $c \sim \|f\|_{WH_w^p}$;
 - (b) $\|b_i^k\|_{L^\infty} \leq C2^k$, where $C > 0$ is independent of i and k ;
 - (c) $\int_{R^n} b_i^k(x) x^\alpha dx = 0$ for every multi-index α with $|\alpha| \leq [n(q_w/p - 1)]$.

Conversely, if $f \in \mathcal{S}'(R^n)$ has a decomposition satisfying (i) and (ii), then $f \in WH_w^p(R^n)$. Moreover, we have $\|f\|_{WH_w^p} \sim c$.

3. Proof of Theorem 1

It suffices to show that there exists a constant $C > 0$, such that, for any $f \in WH_w^p$ and $\beta > 0$,

$$\beta^p w(\{x \in R^n : |\mu_\Omega(f)(x)| > \beta\}) \leq C \|f\|_{WH_w^p}^p. \quad (21)$$

For any given $\beta > 0$, we choose $k_0 \in \mathbb{Z}$ such that $2^{k_0} \leq \beta < 2^{k_0+1}$. Then, by Lemma 8, for every $f \in WH_w^p$, we can write

$$f = \sum_{k=-\infty}^\infty f_k = \sum_{k=-\infty}^{k_0} f_k + \sum_{k=k_0+1}^\infty f_k = F_1 + F_2, \quad (22)$$

where

$$F_1 = \sum_{k=-\infty}^{k_0} f_k = \sum_{k=-\infty}^{k_0} \sum_i b_i^k, \quad F_2 = \sum_{k=k_0+1}^\infty f_k = \sum_{k=k_0+1}^\infty \sum_i b_i^k \quad (23)$$

and $\{b_i^k\}$ satisfies (a)–(c) in Lemma 8. Then,

$$\begin{aligned} & \beta^p w(\{x \in R^n : |\mu_\Omega(f)(x)| > \beta\}) \\ & \leq \beta^p w\left(\left\{x \in R^n : |\mu_\Omega(F_1)(x)| > \frac{\beta}{2}\right\}\right) \\ & \quad + \beta^p w\left(\left\{x \in R^n : |\mu_\Omega(F_2)(x)| > \frac{\beta}{2}\right\}\right) \\ & = M_1 + M_2. \end{aligned} \quad (24)$$

We claim that the following inequality holds:

$$\|F_1\|_{L_w^2} \leq C\beta^{1-p/2} \|f\|_{WH_w^p}^{p/2}. \quad (25)$$

In fact, it follows from Minkowski's integral inequality and the bounded overlapping property of the cubes $\{B_i^k\}$ that

$$\begin{aligned} \|F_1\|_{L_w^2} & \leq \sum_{k=-\infty}^{k_0} \sum_i \|b_i^k\|_{L_w^2} \\ & \leq \sum_{k=-\infty}^{k_0} \sum_i \|b_i^k\|_{L^\infty} w(B_i^k)^{1/2} \\ & \leq C \sum_{k=-\infty}^{k_0} 2^k \left(\sum_i w(B_i^k) \right)^{1/2} \\ & \leq C \sum_{k=-\infty}^{k_0} 2^{k(1-p/2)} \|f\|_{WH_w^p}^{p/2} \\ & \leq C \sum_{k=-\infty}^{k_0} 2^{(k-k_0)(1-p/2)} \beta^{1-p/2} \|f\|_{WH_w^p}^{p/2} \\ & \leq C\beta^{1-p/2} \|f\|_{WH_w^p}^{p/2}. \end{aligned} \quad (26)$$

Notice that $w \in A_{p(1+\beta/n)} \subset A_2$; then from Chebyshev's inequality and the boundedness of μ_Ω on L_w^2 we have

$$M_1 \leq \beta^p \frac{C}{\beta^2} \|\mu_\Omega(F_1)\|_{L_w^2}^2 \leq C \|f\|_{\text{WH}_w^p}^p \quad (27)$$

We set

$$A_{k_0} = \bigcup_{k=k_0+1}^{\infty} \bigcup_i \widetilde{B}_i^k, \quad (28)$$

where $\widetilde{B}_i^k = B(x_i^k, (3/2)^{(k-k_0)/(n+\beta)} \sqrt{n} d_i^k)$. Then, we get

$$\begin{aligned} M_2 &\leq \beta^p w \left(\left\{ x \in A_{k_0} : |\mu_\Omega(F_2)(x)| > \frac{\beta}{2} \right\} \right) \\ &\quad + \beta^p w \left(\left\{ x \in (A_{k_0})^c : |\mu_\Omega(F_2)(x)| > \frac{\beta}{2} \right\} \right) \quad (29) \\ &= M_{21} + M_{22}. \end{aligned}$$

Since $w \in A_{p(1+\beta/n)}$, then, by Lemmas 5 and 8 and the fact $2^{k_0} \leq \beta < 2^{k_0+1}$, we get

$$\begin{aligned} M_{21} &\leq \beta^p w(A_{k_0}) \\ &\leq \beta^p \sum_{k=k_0+2}^{\infty} \sum_i w(\widetilde{B}_i^k) \\ &\leq C \beta^p \sum_{k=k_0+2}^{\infty} \left(\frac{3}{2} \right)^{(k-k_0)p} \sum_i w(B_i^k) \quad (30) \\ &\leq C \|f\|_{\text{WH}_w^p}^p \sum_{k=k_0+2}^{\infty} \left(\frac{3}{4} \right)^{(k-k_0)p} \\ &\leq C \|f\|_{\text{WH}_w^p}^p. \end{aligned}$$

An application of Chebyshev's inequality gives us that

$$\begin{aligned} M_{22} &\leq C \int_{(A_{k_0})^c} |\mu_\Omega(F_2)|^p w(x) dx \\ &\leq C \sum_{k=k_0+1}^{\infty} \sum_i \int_{(\widetilde{B}_i^k)^c} |\mu_\Omega(b_i^k)(x)|^p w(x) dx. \quad (31) \end{aligned}$$

Below, we will give the estimate of integral $I = \int_{(\widetilde{B}_i^k)^c} |\mu_\Omega(b_i^k)(x)|^p w(x) dx$. We have

$$\begin{aligned} I &\leq \int_{(\widetilde{B}_i^k)^c} \left(\int_0^{|x-x_i^k|+\sqrt{n}d_i^k} \left| \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} b_i^k(y) dy \right|^2 \frac{dt}{t^3} \right)^{p/2} \\ &\quad \times w(x) dx \\ &\quad + \int_{(\widetilde{B}_i^k)^c} \left(\int_{|x-x_i^k|+\sqrt{n}d_i^k}^{\infty} \left| \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} b_i^k(y) dy \right|^2 \frac{dt}{t^3} \right)^{p/2} \\ &\quad \times w(x) dx \\ &= I_1 + I_2. \quad (32) \end{aligned}$$

For $y \in B_i^k$ and $x \in (\widetilde{B}_i^k)^c$, we have $|x-y| \approx |x-x_i^k| \approx |x-x_i^k| + \sqrt{n}d_i^k$. Thus,

$$\left| \frac{1}{|x-y|^2} - \frac{1}{(|x-x_i^k| + \sqrt{n}d_i^k)^2} \right| \leq \frac{Cd_i^k}{|x-y|^3}. \quad (33)$$

By Minkowski's inequality for integral,

$$\begin{aligned} &\left(\int_0^{|x-x_i^k|+\sqrt{n}d_i^k} \left| \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} b_i^k(y) dy \right|^2 \frac{dt}{t^3} \right)^{1/2} \\ &\leq \int_{B_i^k} \frac{|\Omega(x-y)|}{|x-y|^{n-1}} |b_i^k(y)| \left(\int_{|x-y|}^{|x-x_i^k|+\sqrt{n}d_i^k} \frac{dt}{t^3} \right)^{1/2} dy \\ &\leq \frac{C2^k (d_i^k)^{n+1/2}}{|x-x_i^k|^{n+1/2}}. \quad (34) \end{aligned}$$

From the fact that $q = (1+\beta/n)p \leq (1+1/2n)p$ and Lemma 6, we have

$$\begin{aligned} I_1 &\leq C2^{kp} (d_i^k)^{(n+1/2)p} \int_{(\widetilde{B}_i^k)^c} \frac{w(x)}{|x-x_i^k|^{(n+1/2)p}} dx \\ &\leq C2^{kp} (d_i^k)^{(n+1/2)p} \left(\left(\frac{3}{2} \right)^{(k-k_0)/(n+\beta)} \sqrt{n} d_i^k \right)^{-(n+1/2)p} \quad (35) \\ &\quad \times w(\widetilde{B}_i^k). \end{aligned}$$

Since $w \in A_{(1+1/2n)p}$, then there exists a $\epsilon > 0$ such that $w \in A_{(1+1/2n)p-\epsilon}$. Thus, we have

$$w(\widetilde{B}_i^k) \leq \left(\frac{3}{2} \right)^{((k-k_0)/(n+\beta))((n+1/2)p-\epsilon)} w(B_i^k). \quad (36)$$

Hence,

$$I_1 \leq C2^{kp} \left(\frac{3}{2} \right)^{-((k-k_0)/(n+\beta))ne} w(B_i^k). \quad (37)$$

For $n/(n+\beta) < p \leq 1$ and $w \in A_q$ with $q = (1+\beta/n)p > 1$, we can see that $[n(q_w/p-1)] = 0$. Since $t \geq |x-x_i^k| + \sqrt{n}d_i^k$, we have $B_i^k \subset \{y : |x-y| \leq t\}$. By the vanishing moment condition of b_i^k ,

$$\begin{aligned} I_2 &= \int_{(\widetilde{B}_i^k)^c} \left(\int_{|x-x_i^k|+\sqrt{n}d_i^k}^{\infty} \left| \int_{B_i^k} \left(\frac{\Omega(x-y)}{|x-y|^{n-1}} - \frac{\Omega(x-x_i^k)}{|x-x_i^k|^{n-1}} \right) \right. \right. \\ &\quad \left. \left. \times b_i^k(y) dy \right|^2 \frac{dt}{t^3} \right)^{p/2} w(x) dx. \quad (38) \end{aligned}$$

But

$$\begin{aligned}
& \left| \int_{B_i^k} \left(\frac{\Omega(x-y)}{|x-y|^{n-1}} - \frac{\Omega(x-x_i^k)}{|x-x_i^k|^{n-1}} \right) b_i^k(y) dy \right| \\
& \leq \int_{B_i^k} \left| \frac{1}{|x-y|^{n-1}} - \frac{1}{|x-x_i^k|^{n-1}} \right| |\Omega(x-y) b_i^k(y)| dy \\
& \quad + \int_{B_i^k} \frac{|\Omega(x-y) - \Omega(x-x_i^k)|}{|x-x_i^k|^{n-1}} |b_i^k(y)| dy \\
& = K_1 + K_2.
\end{aligned} \tag{39}$$

Note that for any $y \in B_i^k$ and $x \in (\widetilde{B_i^k})^c$, $|x-y| \approx |x-x_i^k|$. Thus,

$$\begin{aligned}
K_1 & \leq C \|\Omega\|_{L^\infty} \|b_i^k\|_{L^\infty} \int_{B_i^k} \left| \frac{1}{|x-y|^{n-1}} - \frac{1}{|x-x_i^k|^{n-1}} \right| dy \\
& \leq C \|b_i^k\|_{L^\infty} \frac{(d_i^k)^{n+1}}{|x-x_i^k|^n}.
\end{aligned} \tag{40}$$

From the fact that $\Omega \in \text{Lip}_\alpha(S^{n-1})$ and Ω is homogeneous of degree zero, we get

$$\begin{aligned}
& |\Omega(x-y) - \Omega(x-x_i^k)| \\
& \leq \left| \Omega\left(\frac{x-y}{|x-y|}\right) - \Omega\left(\frac{x-x_i^k}{|x-x_i^k|}\right) \right| \\
& \leq C \left| \frac{x-y}{|x-y|} - \frac{x-x_i^k}{|x-x_i^k|} \right|^\alpha \\
& \leq C \left(\frac{|y-x_i^k|}{|x-x_i^k|} \right)^\alpha.
\end{aligned} \tag{41}$$

Then,

$$K_2 \leq C \|b_i^k\|_{L^\infty} \frac{(d_i^k)^{n+\alpha}}{|x-x_i^k|^{n-1+\alpha}}. \tag{42}$$

For $x \in (\widetilde{B_i^k})^c$, we have $|x-x_i^k| \approx |x-x_i^k| + \sqrt{n}d_i^k$. Then,

$$\int_{|x-x_i^k|+\sqrt{n}d_i^k}^\infty \frac{dt}{t^3} \leq \frac{C}{|x-x_i^k|^2}. \tag{43}$$

Hence,

$$\begin{aligned}
I_2 & \leq C 2^{kp} (d_i^k)^{(n+1)p} \int_{(\widetilde{B_i^k})^c} \frac{w(x)}{|x-x_i^k|^{(n+1)p}} dx \\
& \quad + C 2^{kp} (d_i^k)^{(n+\alpha)p} \int_{(\widetilde{B_i^k})^c} \frac{w(x)}{|x-x_i^k|^{(n+\alpha)p}} dx \\
& = L_1 + L_2.
\end{aligned} \tag{44}$$

Since $w \in A_{p(1+\beta/n)}$ and $\beta = \min\{\alpha, 1/2\}$, then $w \in A_{p(1+1/n)}$ and $w \in A_{p(1+1/n)-\epsilon}$ for some $\epsilon > 0$. From Lemma 6,

$$\begin{aligned}
L_1 & \leq C 2^{kp} \left(\frac{3}{2}\right)^{-((k-k_0)/(n+\beta))p(n+1)} w(\widetilde{B_i^k}) \\
& \leq C 2^{kp} \left(\frac{3}{2}\right)^{-((k-k_0)/(n+\beta))n\epsilon} w(B_i^k).
\end{aligned} \tag{45}$$

Analogously, we have

$$L_2 \leq C 2^{kp} \left(\frac{3}{2}\right)^{-((k-k_0)/(n+\beta))n\epsilon} w(B_i^k). \tag{46}$$

Then,

$$I_2 \leq C 2^{kp} \left(\frac{3}{2}\right)^{-((k-k_0)/(n+\beta))n\epsilon} w(B_i^k). \tag{47}$$

Hence,

$$\begin{aligned}
M_{22} & \leq C \sum_{k=k_0+1}^\infty \sum_i 2^{kp} \left(\frac{3}{2}\right)^{-((k-k_0)/(n+\beta))n\epsilon} w(B_i^k) \\
& \leq C \|f\|_{\text{WH}_w^p}^p \sum_{k=k_0+1}^\infty \left(\frac{3}{2}\right)^{-((k-k_0)/(n+\beta))n\epsilon} \\
& \leq C \|f\|_{\text{WH}_w^p}^p.
\end{aligned} \tag{48}$$

Combining the estimate of M_{21} and M_{22} , we get $M_2 \leq C \|f\|_{\text{WH}_w^p}^p$. This completes the proof of the Theorem 1.

4. Proof of Theorem 2

We follow the strategy of the proof of Theorem 1. For any given $\lambda > 0$, we may choose $k_0 \in \mathbb{Z}$ such that $2^{k_0} \leq \lambda < 2^{k_0+1}$. For every $f \in \text{WH}_w^1$, and then by Lemma 8, we can write

$$f = \sum_{k=-\infty}^\infty f_k = \sum_{k=-\infty}^{k_0} f_k + \sum_{k=k_0+2}^\infty f_k = F_1 + F_2, \tag{49}$$

where $F_1 = \sum_{k=-\infty}^{k_0} f_k = \sum_{k=-\infty}^{k_0} \sum_i b_i^k$, $F_2 = \sum_{k=k_0+2}^\infty f_k = \sum_{k=k_0+2}^\infty \sum_i b_i^k$, and $\{b_i^k\}$ satisfies (a)–(c) in Lemma 8. Then,

$$\begin{aligned}
& \lambda w(\{x \in R^n : |\mu_\Omega(f)(x)| > \lambda\}) \\
& \leq \lambda w\left(\left\{x \in R^n : |\mu_\Omega(F_1)(x)| > \frac{\lambda}{2}\right\}\right) \\
& \quad + \lambda w\left(\left\{x \in R^n : |\mu_\Omega(F_2)(x)| > \frac{\lambda}{2}\right\}\right) = P_1 + P_2.
\end{aligned} \tag{50}$$

Similar to the estimate of M_1 in the proof of Theorem 1, we obtain

$$P_1 \leq C \|f\|_{\text{WH}_w^1}. \tag{51}$$

We set

$$A_{k_0} = \bigcup_{k=k_0+1}^{\infty} \bigcup_i \widetilde{B}_i^k, \quad (52)$$

where $\widetilde{B}_i^k = B(x_i^k, (3/2)^{(k-k_0)/n} \sqrt{n} d_i^k)$. Then, we get

$$\begin{aligned} P_2 &\leq \lambda w \left(\left\{ x \in A_{k_0} : |\mu_{\Omega}(F_2)(x)| > \frac{\lambda}{2} \right\} \right) \\ &\quad + \lambda w \left(\left\{ x \in (A_{k_0})^c : |\mu_{\Omega}(F_2)(x)| > \frac{\lambda}{2} \right\} \right) \quad (53) \\ &= P_{21} + P_{22}. \end{aligned}$$

Since $w \in A_1$, and then by Lemmas 5 and 8 and the fact $2^{k_0} \leq \lambda < 2^{k_0+1}$, similar to the estimate in the proof of Theorem 1, we get

$$P_{21} \leq C \|f\|_{\text{WH}_w^1}. \quad (54)$$

An application of Chebyshev's inequality gives us that

$$\begin{aligned} P_{22} &\leq C \int_{(A_{k_0})^c} |\mu_{\Omega}(F_2)| w(x) dx \\ &\leq C \sum_{k=k_0+2}^{\infty} \sum_i \int_{(\widetilde{B}_i^k)^c} |\mu_{\Omega}(b_i^k)(x)| w(x) dx. \end{aligned} \quad (55)$$

Below, we give the estimate of integral $J = \int_{(\widetilde{B}_i^k)^c} |\mu_{\Omega}(b_i^k)(x)| w(x) dx$. We have

$$\begin{aligned} J &\leq \int_{(\widetilde{B}_i^k)^c} \left(\int_0^{|x-x_i^k|+\sqrt{n}d_i^k} \left| \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} b_i^k(y) dy \right|^2 \frac{dt}{t^3} \right)^{1/2} \\ &\quad \times w(x) dx \\ &\quad + \int_{(\widetilde{B}_i^k)^c} \left(\int_{|x-x_i^k|+\sqrt{n}d_i^k}^{\infty} \left| \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} b_i^k(y) dy \right|^2 \frac{dt}{t^3} \right)^{1/2} \\ &\quad \times w(x) dx \\ &= J_1 + J_2. \end{aligned} \quad (56)$$

For $y \in B_i^k$ and $x \in (\widetilde{B}_i^k)^c$, we have $|x-y| \approx |x-x_i^k| \approx |x-x_i^k| + \sqrt{n}d_i^k$. Thus,

$$\left| \frac{1}{|x-y|^2} - \frac{1}{(|x-x_i^k| + \sqrt{n}d_i^k)^2} \right| \leq \frac{Cd_i^k}{|x-y|^3}. \quad (57)$$

Then, Minkowski's inequality for integral gives us that

$$\begin{aligned} J_1 &\leq \int_{(\widetilde{B}_i^k)^c} \int_{B_i^k} \frac{|\Omega(x-y)|}{|x-y|^{n-1}} |b_i^k(y)| \\ &\quad \times \left(\int_{|x-y|}^{|x-x_i^k|+\sqrt{n}d_i^k} \frac{dt}{t^3} \right)^{1/2} dy w(x) dx \quad (58) \\ &\leq C(d_i^k)^{1/2} \int_{B_i^k} \left(\int_{(\widetilde{B}_i^k)^c} \frac{|\Omega(x-y)|}{|x-y|^{n+1/2}} w(x) dx \right) \\ &\quad \times |b_i^k(y)| dy. \end{aligned}$$

Since $w^{q'} \in A_1$, then $w \in A_1 \subset A_{1+1/2n}$. From Lemmas 6 and 5, we have

$$\begin{aligned} &\left(\int_{(\widetilde{B}_i^k)^c} \frac{w(x)^{q'}}{|x-y|^{n+1/2}} dx \right)^{1/q'} \\ &\leq C \left(\left(\frac{3}{2} \right)^{(k-k_0)/n} d_i^k \right)^{-n/q' - 1/2q'} \left(w^{q'}(\widetilde{B}_i^k) \right)^{1/q'} \\ &\leq C \left(\left(\frac{3}{2} \right)^{(k-k_0)/n} d_i^k \right)^{-n/q' - 1/2q'} \\ &\quad \times \left(\left(\frac{3}{2} \right)^{(k-k_0)/n} \right)^{n/q'} \left(w^{q'}(B_i^k) \right)^{1/q'} \\ &\leq C \left(\left(\frac{3}{2} \right)^{(k-k_0)/n} d_i^k \right)^{-1/2q'} \inf_{x \in B_i^k} w(x). \end{aligned} \quad (59)$$

Hence, for $y \in B_i^k$, Hölder's inequality and $\Omega \in L^q(S^{n-1})$ imply that

$$\begin{aligned} &\int_{(\widetilde{B}_i^k)^c} \frac{|\Omega(x-y)|}{|x-y|^{n+1/2}} w(x) dx \\ &\leq \left(\int_{(\widetilde{B}_i^k)^c} \frac{|\Omega(x-y)|^q}{|x-y|^{n+1/2}} dx \right)^{1/q} \\ &\quad \times \left(\int_{(\widetilde{B}_i^k)^c} \frac{w(x)^{q'}}{|x-y|^{n+1/2}} dx \right)^{1/q'} \\ &\leq \left(\int_{(3/2)^{(k-k_0)/n} \sqrt{n}d_i^k}^{\infty} \int_{S^{n-1}} \frac{|\Omega(x')|^q}{r^{n+1/2}} r^{n-1} d\sigma(x') dr \right)^{1/q} \\ &\quad \times \left(\left(\frac{3}{2} \right)^{(k-k_0)/n} d_i^k \right)^{-1/2q'} \inf_{x \in B_i^k} w(x) \\ &\leq C \left(\left(\frac{3}{2} \right)^{(k-k_0)/n} d_i^k \right)^{-1/2q} \end{aligned}$$

$$\begin{aligned}
& \times \left(\left(\frac{3}{2} \right)^{-(k-k_0)/n} d_i^k \right)^{-1/2q'} \inf_{x \in B_i^k} w(x) \\
& \leq C \left(\left(\frac{3}{2} \right)^{(k-k_0)/n} d_i^k \right)^{-1/2} \inf_{x \in B_i^k} w(x).
\end{aligned} \tag{60}$$

Then,

$$\begin{aligned}
J_1 & \leq C 2^k (d_i^k)^{1/2} \left(\left(\frac{3}{2} \right)^{(k-k_0)/n} d_i^k \right)^{-1/2} \inf_{x \in B_i^k} w(x) |B_i^k| \\
& \leq C 2^k w(B_i^k) \left(\frac{2}{3} \right)^{(k-k_0)/2n}.
\end{aligned} \tag{61}$$

We now estimate J_2 . Since $t \geq |x - x_i^k| + \sqrt{n}d_i^k$, we have $B_i^k \subset \{y : |x - y| \leq t\}$ and hence

$$\begin{aligned}
J_2 & = \int_{(\widetilde{B_i^k})^c} \left(\int_{|x-x_i^k|+\sqrt{n}d_i^k}^{\infty} \left| \int_{B_i^k} \left(\frac{\Omega(x-y)}{|x-y|^{n-1}} - \frac{\Omega(x-x_i^k)}{|x-x_i^k|^{n-1}} \right) b_i^k(y) dy \right|^2 \frac{dt}{t^3} \right)^{1/2} \\
& \quad \times w(x) dx.
\end{aligned} \tag{62}$$

Denote $R_j^k = 2^j(3/2)^{(k-k_0)/n}\sqrt{n}$. Minkowski's inequality for integrals and the fact $|x-x_i^k| + \sqrt{n}d_i^k \approx |x-x_i^k|$ for $x \in (\widetilde{B_i^k})^c$ give us

$$\begin{aligned}
J_2 & \leq \int_{(\widetilde{B_i^k})^c} \int_{B_i^k} \left| \frac{\Omega(x-y)}{|x-y|^{n-1}} - \frac{\Omega(x-x_i^k)}{|x-x_i^k|^{n-1}} \right| |b_i^k(y)| \\
& \quad \times \left(\int_{|x-x_i^k|+\sqrt{n}d_i^k}^{\infty} \frac{dt}{t^3} \right)^{1/2} dy w(x) dx \\
& \leq \int_{(\widetilde{B_i^k})^c} \int_{B_i^k} \left| \frac{\Omega(x-y)}{|x-y|^{n-1}} - \frac{\Omega(x-x_i^k)}{|x-x_i^k|^{n-1}} \right| \\
& \quad \times \frac{|b_i^k(y)|}{|x-x_i^k|} dy w(x) dx \\
& \leq C \int_{B_i^k} \sum_{j=0}^{\infty} \frac{1}{R_j^k d_i^k} \\
& \quad \times \int_{R_j^k d_i^k \leq |x-x_i^k| \leq R_{j+1}^k d_i^k} \left| \frac{\Omega(x-y)}{|x-y|^{n-1}} - \frac{\Omega(x-x_i^k)}{|x-x_i^k|^{n-1}} \right| \\
& \quad \times w(x) dx |b_i^k(y)| dy.
\end{aligned} \tag{63}$$

Notice that

$$\left(w^{q'}(R_{j+1}^k B_i^k) \right)^{1/q'} \leq C (R_j^k)^{n/q'} (d_i^k)^{n/q'} \inf_{x \in B_i^k} w(x). \tag{64}$$

By Hölder's inequality and Lemma 7, we have

$$\begin{aligned}
& \int_{R_j^k d_i^k \leq |x-x_i^k| \leq R_{j+1}^k d_i^k} \left| \frac{\Omega(x-y)}{|x-y|^{n-1}} - \frac{\Omega(x-x_i^k)}{|x-x_i^k|^{n-1}} \right| w(x) dx \\
& \leq \left(\int_{R_j^k d_i^k \leq |x-x_i^k| \leq R_{j+1}^k d_i^k} \left| \frac{\Omega(x-y)}{|x-y|^{n-1}} - \frac{\Omega(x-x_i^k)}{|x-x_i^k|^{n-1}} \right|^q dx \right)^{1/q} \\
& \quad \times \left(w^{q'}(R_{j+1}^k B_i^k) \right)^{1/q'} \\
& \leq C (R_j^k d_i^k)^{1-n+n/q} \\
& \quad \times \left(\frac{|y-x_i^k|}{R_j^k d_i^k} + \int_{|y-x_i^k|/R_{j+1}^k d_i^k \leq \delta \leq |y-x_i^k|/R_j^k d_i^k} \frac{\omega_q(\delta)}{\delta} d\delta \right) \\
& \quad \times \left(w^{q'}(R_{j+1}^k B_i^k) \right)^{1/q'} \\
& \leq C R_j^k d_i^k \inf_{x \in B_i^k} w(x) \\
& \quad \times \left(\frac{|y-x_i^k|}{R_j^k d_i^k} + \int_{|y-x_i^k|/R_{j+1}^k d_i^k \leq \delta \leq |y-x_i^k|/R_j^k d_i^k} \frac{\omega_q(\delta)}{\delta} d\delta \right).
\end{aligned} \tag{65}$$

Then,

$$\begin{aligned}
J_2 & \leq C 2^k \inf_{x \in B_i^k} w(x) |B_i^k| \\
& \quad \times \sum_{j=0}^{\infty} \left(\frac{1}{2^j} \left(\frac{2}{3} \right)^{(k-k_0)/n} \right. \\
& \quad \left. + \int_{|y-x_i^k|/R_{j+1}^k d_i^k \leq \delta \leq |y-x_i^k|/R_j^k d_i^k} \frac{\omega_q(\delta)}{\delta} d\delta \right) \\
& \leq C 2^k w(B_i^k) \left(\left(\frac{2}{3} \right)^{(k-k_0)/n} + \int_0^{(2/3)^{(k-k_0)/n}} \frac{\omega_q(\delta)}{\delta} d\delta \right).
\end{aligned} \tag{66}$$

But

$$\begin{aligned}
& \int_0^{(2/3)^{(k-k_0)/n}} \frac{\omega_q(\delta)}{\delta} d\delta \\
& \leq \frac{C}{[1 + (k-k_0) \log(3/2)]^\sigma} \int_0^1 \frac{\omega_q(\delta)}{\delta} (1 + |\log \delta|)^\sigma d\delta \\
& < \frac{C}{(k-k_0)^\sigma},
\end{aligned} \tag{67}$$

from which we get

$$J_2 \leq C 2^k w(B_i^k) \left(\left(\frac{2}{3} \right)^{(k-k_0)/n} + \frac{C}{(k-k_0)^\sigma} \right). \quad (68)$$

Thus,

$$\begin{aligned} P_{22} &\leq C \sum_{k=k_0+2}^{\infty} \sum_i \int_{(\overline{B_i^k})^c} |\mu_{\Omega}(b_i^k)(x)| w(x) dx \\ &\leq C \sum_{k=k_0+2}^{\infty} 2^k \sum_i w(B_i^k) \left(\left(\frac{2}{3} \right)^{(k-k_0)/n} + \frac{C}{(k-k_0)^\sigma} \right) \\ &\leq C \|f\|_{\text{WH}_w^1} \sum_{k=k_0+2}^{\infty} \left(\left(\frac{2}{3} \right)^{(k-k_0)/n} + \frac{C}{(k-k_0)^\sigma} \right) \\ &\leq C \|f\|_{\text{WH}_w^1}. \end{aligned} \quad (69)$$

This completes the proof of Theorem 2.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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