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Research Article

An Odd Rearrangement of $L^1(\mathbf{R}^n)$

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We introduce an odd rearrangement f_* defined by $\pi(f)(x) = f_*(x) = \operatorname{sgn}(x_1) f^*(\nu_n |x|^n)$, $x \in \mathbf{R}^n$, where f^* is a decreasing rearrangement of the measurable function f. With the help of this odd rearrangement, we show that for each $f \in L^1(\mathbf{R}^n)$, there exists a $g \in H^1(\mathbf{R}^n)$ such that $d_f = d_g$, where d_f is an distribution function of f. Moreover, we study the boundedness of singular integral operators when they are restricted to odd rearrangement of $L^1(\mathbf{R}^n)$, and we give some results on Hilbert transform.

1. Introduction

From pages 39 to 40 in [1], we know the following relations between the Hardy space $H^1(\mathbf{R}^n)$ and the Lebesgue space $L^1(\mathbf{R}^n)$:

$$H^1(\mathbf{R}^n) \subset L^1(\mathbf{R}^n), \qquad \|f\|_{L^1} \le \|f\|_{H^1}, \tag{1}$$

where $H^1(\mathbf{R}^n)$ does not coincide with $L^1(\mathbf{R}^n)$ since the elements of $H^1(\mathbf{R}^n)$ have integral zero. This means that if $f \in L^1(\mathbf{R}^n)$ and $\int_{\mathbf{R}^n} f(x)dx \neq 0$, then $f \notin H^1(\mathbf{R}^n)$. On the other hand, it is well known that both Lebesgue spaces and Lorentz spaces can be characterized by distribution functions or decreasing rearrangements. This naturally raises the following question.

Question 1. For each $f \in L^1$, does there exist a function $g \in H^1(\mathbf{R}^n)$ which is an appropriate modification of f satisfying $d_f = d_g$? Here d_f is a distribution function of measurable function f (see Definition 1).

In this paper, we will present a positive answer to Question 1 by introducing an odd rearrangement. We will first address this problem in the case of $n = 1$. Then in a similar way, motivated by the decreasing rearrangement defined on $[0, \infty)$ in [2, pp. 44–46] that keeps the distribution function, we introduce an odd rearrangement to solve the problem on $\mathbf{R}^n$. To obtain applications of our result, the singular integral characterization of Hardy space attracts our attention. It is known that we can characterize $H^1(\mathbf{R})$ by the Hilbert transform and characterize $H^1(\mathbf{R}^n)$, when $n \ge 2$, by the Riesz transforms (see [1]). As it is pointed out in [2], although it is important to study the boundedness of the singular integral operators on $L^1(\mathbf{R}^n)$, it may be impossible for us to give some conditions on singular integral operators to ensure they map $L^1(\mathbf{R}^n)$ to $L^1(\mathbf{R}^n)$. In this paper, we will show that some singular integral operators are of strong type $(1, 1)$ when some singular integral operators is restricted to odd rearrangements of $L^1(\mathbf{R}^n)$.

The distribution function and decreasing rearrangement are two important tools for us to study Lebesgue space and Lorentz space. Below, we briefly review their definition on the Euclidean space.

Definition 1. For a measurable function f on $\mathbf{R}^n$, the distribution function of f is the function d_f defined on $[0, \infty)$ by

$$d_f(\alpha) = \mu(\{x \in \mathbf{R}^n : |f(x)| > \alpha\}), \tag{2}$$

where μ is the Lebesgue measure.

Definition 2. Let f be a complex-valued function defined on $\mathbf{R}^n$. The decreasing rearrangement of f is the function defined on $[0, \infty)$ by

$$f^*(t) = \inf\{s > 0 : d_f(s) \le t\}. \tag{3}$$

Here, we adopt the convention $\inf \emptyset = \infty$; thus $f^* = \infty$ whenever $d_f(\alpha) > t$ for all $\alpha \geq 0$.

We easily observe that when two measurable functions have the same distribution function or decreasing rearrangement, then they have the same L^p-norm.

The following definition of odd rearrangement is the main issue in our study.

Definition 3. Let f be a complex-valued function defined on $\mathbf{R}^n$. The odd rearrangement of f is the function f_* defined on $\mathbf{R}^n$ by

$$\pi(f)(x) = f_*(x) = \operatorname{sgn}(x_1) f^*(\nu_n|x|^n), \quad x \in \mathbf{R}^n, \tag{4}$$

where f^* is a decreasing rearrangement of the measurable function f and π is a projection: $f(x) \to f_*(x)$.

In this paper, we will show that f and f_* have the same distribution function and decreasing rearrangement. This indicates that the L^p-norm of f remains unchanged under the action of projection π. For convenience, we note that $\mathscr{A}(\mathbf{R}^n)$ by the subset of $L^1(\mathbf{R}^n)$ consists of all odd rearrangements of L^1-functions.

As a first step, we study the properties of the odd rearrangement in Proposition 4. Next, we give criteria to determine the boundedness of the singular integral operators when it is restricted to odd rearrangements of $L^1(\mathbf{R}^n)$ via Theorem 6. Thirdly, we consider three kinds of singular operators: singular integral operators with rough kernels, general singular integrals, and singular integrals of nonconvolutional type. Using the boundedness of Riesz transforms and Hilbert transform, we establish the relationships between the Hardy space and the Lebesgue space. Finally, we use the rearrangement which is an extension of the odd rearrangement to obtain some results on Hilbert transform.

This paper is organized as follows. In Section 2, we list some properties of odd rearrangement and seek sufficient conditions that ensure singular integral operators be of strong type $(1,1)$ when restricted to odd rearrangements of $L^1(\mathbf{R}^n)$. In Section 3, with the help of the rearrangement, we construct an odd function in $\bigcap_{0<p\leq\infty} L^p(\mathbf{R})$ but not in $H^1(\mathbf{R})$ and show that for each $f \in L^1(\mathbf{R})$ and $c \in (0,\infty)$ there exists a function $g \in H^1(\mathbf{R})$ such that $d_f = d_g$ and $\|H(g)\|_{L^1}/\|g\|_{L^1} = c$.

Throughout, the letter C always denotes (possibly different) constants which are independent of all essential variables.

2. The Odd Rearrangement and Singular Integrals

From pages 46 to 48 in [2], we know many properties of the decreasing rearrangement. We find that the odd rearrangement has some similar properties with the decreasing rearrangement. The following are some properties of the odd rearrangement f_*.

Proposition 4. (1) $d_{f_*} = d_{f^*}$,
(2) $(f_*)^* = f^*$,
(3) $(f_*)_* = f_*$,
(4) $(f_* + g_*)_* = f_* + g_*$,
(5) $\|f\|_{L^{p,q}} = \|f_*\|_{L^{p,q}}$,
(6) $\|f_* + g_*\|_{L^1} = \|f_*\|_{L^1} + \|g_*\|_{L^1}$,
(7) $(kf)_* = |k| f^*$,
(8) $|f_n| \nearrow |f|$ *a.e. implies* $|(f_n)_*| \nearrow |f_*|$,
(9) $|f| \leq \liminf_{n\to\infty} |f_n|$ *a.e. implies* $|f_*| \leq \liminf_{n\to\infty} |(f_n)_*|$ *a.e.*
(10) $(|f|^p)_* = \operatorname{sgn}(x_1)|f_*|^p$,
(11) $\|f\|_{L^\infty} = \lim \operatorname{sgn}_{x\to 0, x_1\neq 0}(x_1) f_*(x)$.

Proof. We only need to prove that $d_{f_*} = d_{f^*}$; since other properties can be showed similar with the proof of Proposition 1.4.5 in [2], we know that $f_*(x) = \operatorname{sgn}(x_1)\, f^*(\nu_n|x|^n)$, $\forall \alpha > 0$.

(i) If $d_{f^*}(\alpha) = 0$, then $f^*(t) \leq \alpha$, $\forall t > 0$. So we have

$$|f_*(x)| \leq f^*(\nu_n|x|^n) \leq \alpha, \quad \forall x \neq 0. \tag{5}$$

This means that $d_{f_*}(\alpha) = 0$.

(ii) If $d_{f^*}(\alpha) > 0$, $\forall \epsilon > 0$ $(\epsilon < d_{f^*}(\alpha))$, for convenience, we choose

$$\begin{aligned} t_1 &= d_{f^*}(\alpha) - \epsilon, \\ t_2 &= d_{f^*}(\alpha) + \epsilon. \end{aligned} \tag{6}$$

Since f^* is a nonnegative decreasing function defined on $[0,\infty)$, it follows that

$$\begin{aligned} f^*(t_1) &> \alpha, \\ f^*(t_2) &\leq \alpha; \end{aligned} \tag{7}$$

then we have

$$\begin{aligned} f^*\left(\nu_n\left(\nu_n^{-1/n} t_1^{1/n}\right)^n\right) &> \alpha, \\ f^*\left(\nu_n\left(\nu_n^{-1/n} t_2^{1/n}\right)^n\right) &\leq \alpha. \end{aligned} \tag{8}$$

So if $x \in B(0, \nu_n^{-1/n} t_1^{1/n}) \bigcap \{x \in \mathbf{R}^n; x_1 \neq 0\}$, then we obtain that

$$|f_*(x)| = f^*(\nu_n|x|^n) \geq f^*(t_1) > \alpha; \tag{9}$$

this means that

$$d_{f_*}(\alpha) \geq \left|B\left(0, \nu_n^{-1/n} t_1^{1/n}\right)\right| = t_1 = d_{f^*}(\alpha) - \epsilon. \tag{10}$$

And if $x \notin B(0, \nu_n^{-1/n} t_2^{1/n})$, then we obtain that

$$|f_*(x)| \leq f^*(\nu_n|x|^n) \leq f^*(t_2) \leq \alpha; \tag{11}$$

this means that

$$d_{f_*}(\alpha) \leq \left|B\left(0, \nu_n^{-1/n} t_2^{1/n}\right)\right| = t_2 = d_{f^*}(\alpha) + \epsilon. \tag{12}$$

Let $\epsilon \to 0$; we conclude that $d_{f_*} = d_{f^*}$. □

Firstly, we show that for each $f \in \mathscr{A}(\mathbf{R}^n)$, we can find one sequence of simple functions that converge to f.

Lemma 5. *For each $f \in \mathscr{A}(\mathbf{R}^n)$, there exists one sequence $\{f_k\}_k \subset \mathscr{A}(\mathbf{R}^n)$ satisfying*

$$f_k(x) = \frac{1}{2^k}\sum_{j=1}^{2^{2k}} \operatorname{sgn}(x_1)\chi_{B(0,R_{k,j})}(x), \qquad |f_k(x)| \nearrow |f(x)| \quad (k \longrightarrow \infty), \tag{13}$$

where $B(0, R_{k,j+1}) \subset B(0, R_{k,j})$.

Proof. We take

$$f_k = \operatorname{sgn}(x_1)\frac{1}{2^k}\sum_{j=1}^{2^{2k}} \chi_{\{x\in\mathbf{R}^n : f^*(\nu_n|x|^n) > j/2^k\}}. \tag{14}$$

Since $f^*(t)$ is a nonnegative decreasing right continuous function defined on $[0,\infty)$, it follows that $\{f^*(t) > \alpha\}$ is an open set of $[0,\infty)$, which means that $\{x \in \mathbf{R}^n : f^*(\nu_n|x|^n) > j/2^k\}$ is an open ball of $\mathbf{R}^n$. We take $B(0, R_{k,j}) = \{x \in \mathbf{R}^n : f^*(\nu_n|x|^n) > j/2^k\}$; then the theorem can be showed easily. □

Note that $\mathscr{B}(\mathbf{R}^n) = \{\operatorname{sgn}(x_1)f(x) : f(x)$ is a radical function in $L^1(\mathbf{R}^n)\}$. It is easy for us to show that $\mathscr{B}(\mathbf{R}^n)$ is a subspace of $L^1(\mathbf{R}^n)$. Now we give criteria to determine the boundedness of the singular integral operators when they are restricted to odd rearrangements of $L^1(\mathbf{R}^n)$ via Theorem 6, Corollary 7, and Corollary 8.

Theorem 6. *Suppose that T is a singular integral operator satisfying the following.*

(1) *There exists a constant C independent of R such that*

$$\|T(\operatorname{sgn}(x_1)\chi_{B(0,R)})\|_{L^1} \le C\|\operatorname{sgn}(x_1)\chi_{B(0,R)}\|_{L^1}, \quad \forall R > 0. \tag{15}$$

(2) *T is weak type of $(1,1)$ when restricted to the functions of $\mathscr{B}(\mathbf{R}^n)$.*

Then $T \circ \pi$ is strong type of $(1,1)$ with norm at most C.

Proof. For each $f \in \mathscr{A}(\mathbf{R}^n)$, according to Lemma 5, there exists one sequence $\{f_k\}_k$ satisfying

$$\{f_k\}_k \subset \mathscr{A}(\mathbf{R}^n), \qquad f_k(x) = \frac{1}{2^k}\sum_{j=1}^{2^{2k}} \operatorname{sgn}(x_1)\chi_{B(0,R_{k,j})}(x); \qquad |f_k(x)| \nearrow |f(x)| \quad (k \longrightarrow \infty), \tag{16}$$

where $B(0, R_{k,j+1}) \subset B(0, R_{k,j})$. We can easily obtain that

$$\|f_k - f\|_{L^1} \longrightarrow 0 \quad (k \longrightarrow \infty). \tag{17}$$

Due to the fact that T is weak type of $(1,1)$, we obtain that

$$\|T(f_k) - T(f)\|_{L^{1,\infty}} \longrightarrow 0 \quad (k \longrightarrow \infty). \tag{18}$$

Then $T(f_k)$ converges to $T(f)$ in measure, so there exists some subsequence $\{T(f_{k_i})\}_{i>0}$ converges to $T(f)$ a.e. Via Fatou's lemma, we conclude that

$$\begin{aligned}\|T(f)\|_{L^1} &= \left\|\lim_{i\to\infty} T(f_{k_i})\right\|_{L^1} \\ &\le \liminf_{i\to\infty}\left\|T\left(f_{k_i}\right)\right\|_{L^1} \\ &\le \liminf_{i\to\infty} C\left\|f_{k_i}\right\|_{L^1} \le C\|f\|_{L^1}.\end{aligned} \tag{19}$$

□

Theorem 6 is a criterion to determine the boundedness of the singular integral operators when they are restricted to odd rearrangements of $L^1(\mathbf{R}^n)$. By Theorem 6, we can find two simpler criteria. Corollary 7 is an immediate consequence of Theorem 6, while Corollary 8 can be showed later.

Corollary 7. *Suppose that T is a singular integral operator satisfying*

(1) *for any $a > 0$, $T(\delta^a f) = \delta^a T(f)$);*

(2) *$\|T(\operatorname{sgn}(x_1)\chi_{B(0,1)})\|_{L^1} < \infty$;*

(3) *T is weak type of $(1,1)$ when restricted to the functions of $\mathscr{B}(\mathbf{R}^n)$.*

Then $T \circ \pi$ is strong type of $(1,1)$.

Corollary 8. *Suppose that T is a singular integral operator satisfying the following:*

(1) *T is a bounded operator from $H^1(\mathbf{R}^n)$ to $L^1(\mathbf{R}^n)$;*

(2) *T is weak type of $(1,1)$.*

Then $T \circ \pi$ is strong type of $(1,1)$.

Now let us study the boundedness of singular integral operators with the help of Theorem 6 and Corollary 7. Firstly, we consider the boundedness of singular integral operators with rough kernels. The method of rotations is used here; thus we need to consider the boundedness of Hilbert transform firstly.

Lemma 9. *Consider the characteristic function $\chi_{(-1,1)}$ of an interval $(-1,1)$,*

$$H(\operatorname{sgn}(\cdot)\chi_{(-1,1)})(x) = \frac{1}{\pi}\log\frac{x^2}{|x^2-1|}, \qquad \|H(\operatorname{sgn}(\cdot)\chi_{(-1,1)})(x)\|_{L^1} = \frac{8}{\pi}\log\left(1+\sqrt{2}\right). \tag{20}$$

Proof. As in [2, pp. 251-252], for the characteristic function $\chi_{[a,b]}$ of an interval $[a,b]$,

$$H(\chi_{[a,b]})(x) = \frac{1}{\pi}\log\frac{|x-a|}{|x-b|}. \tag{21}$$

So it is not difficult for us to get the above results. □

Proposition 10. *For each $f \in L^1(\mathbf{R})$, we have $H(f_*) \in L^1(\mathbf{R})$,*

$$\|H \circ \pi(f)\|_{L^1} \leq \frac{4}{\pi} \log\left(1+\sqrt{2}\right) \|f\|_{L^1}. \tag{22}$$

Equality can occur if $f = \operatorname{sgn}(x)\chi_{(-1,1)}$.

Proof. Since Hilbert transform H is bounded linear operator mapping $L^1(\mathbf{R})$ to $L^{1,\infty}(\mathbf{R})$ and satisfying

$$T\left(\delta^a f\right) = \delta^a T\left(f\right), \quad \forall a > 0. \tag{23}$$

Using Lemma 9, it is an immediate consequence of Corollary 7. □

With the help of boundedness of Hilbert transform, we establish the relationship between $L^1(\mathbf{R})$ and $H^1(\mathbf{R})$ in the following corollary.

Corollary 11. *For all $g \in \mathscr{A}(\mathbf{R})$,*

$$\|g\|_{H^1} \sim \|g\|_{L^1}. \tag{24}$$

Proof. We use singular integral characterization of $H^1(R)$ in [2]; following Proposition 10, we get that

$$\|g\|_{H^1} = \|g\|_{L^1} + \|H(g)\|_{L^1} \sim \|g\|_{L^1}, \tag{25}$$

for all $g \in \mathscr{A}(\mathbf{R})$. □

Since the boundedness of Hilbert transform has been known, we can use the method of rotations to study singular integral operator T_Ω where Ω is odd and integrable over $\mathbf{S}^{n-1}$.

Lemma 12. *If Ω is odd and integrable over $\mathbf{S}^{n-1}$, then*

$$\begin{aligned}\|T_\Omega\left(\operatorname{sgn}\left(x_1\right)\chi_{B(0,1)}\right)\|_{L^1} &\leq 4\log\left(1+\sqrt{2}\right)\|\Omega\|_{L^1(\mathbf{S}^{n-1})} \\ &\quad \times \|\operatorname{sgn}\left(x_1\right)\chi_{B(0,1)}\|_{L^1}.\end{aligned} \tag{26}$$

Proof. Via the method of rotation we can obtain

$$\begin{aligned}&T_\Omega\left(\operatorname{sgn}\ \left(x_1\right)\chi_{B(0,1)}\right)(y) \\ &\quad = \frac{\pi}{2}\int_{S^{n-1}} \Omega\left(\theta\right) H_\theta\left(\operatorname{sgn}\left(x_1\right)\chi_{B(0,1)}\right)(y)\, d\theta,\end{aligned} \tag{27}$$

where H_θ are the directional Hilbert transforms defined by

$$H_\theta\left(f\right)(x) = \frac{1}{\pi}\,\text{p.v.}\int_{-\infty}^{+\infty} f\left(x - t\theta\right)\frac{dt}{t}. \tag{28}$$

Since the following identity is valid for all matrices, $A \in O(n)$:

$$H_{A(e_1)}\left(f\right)(x) = H_{e_1}\left(f \circ A\right)\left(A^{-1}x\right), \tag{29}$$

we take $g = \operatorname{sgn}(x_1)\chi_{B(0,1)}$; then

$$\begin{aligned}&\int_{\mathbf{S}^{n-1}} \Omega\left(\theta\right) H_\theta\left(g\right)(x)\, d\theta \\ &\quad = \int_{\mathbf{S}^{n-1}\bigcap\{\theta:\theta_1>0\}} \Omega\left(\theta\right) H_\theta\left(g\right)(x)\, d\theta \\ &\qquad + \int_{\mathbf{S}^{n-1}\bigcap\{\theta:\theta_1<0\}} \Omega\left(\theta\right) H_\theta\left(g\right)(x)\, d\theta \\ &\quad = \int_{\mathbf{S}^{n-1}\bigcap\{\theta:\theta_1>0\}} \Omega\left(\theta\right) H_\theta\left(g\right)(x)\, d\theta \\ &\qquad - \int_{\mathbf{S}^{n-1}\bigcap\{\theta:\theta_1>0\}} \Omega\left(\theta\right) H_{-\theta}\left(g\right)(x)\, d\theta \\ &\quad = \int_{\mathbf{S}^{n-1}\bigcap\{\theta:\theta_1>0\}} \Omega\left(\theta\right)\left(H_\theta - H_{-\theta}\right)\left(g\right)(x)\, d\theta.\end{aligned} \tag{30}$$

Since for any $a, b > 0$ the following inequality holds:

$$\begin{aligned}&\|H(\chi_{(0,a)} - \chi_{(-b,0)})(x) + H(\chi_{(0,a)} - \chi_{(-b,0)})(-x)\|_{L^1(\mathbf{R})} \\ &\quad \leq 2\|H\left(\chi_{(0,(a+b)/2)} - \chi_{(-(a+b)/2,0)}\right)(x)\|_{L^1(\mathbf{R})},\end{aligned} \tag{31}$$

we obtain

$$\begin{aligned}&\|\left(H_\theta - H_{-\theta}\right)\left(g\right)(x)\|_{L^1} \\ &\quad = \left\|\left(H_{A(e_1)} - H_{-A(e_1)}\right)\left(g\right)(x)\right\|_{L^1} \\ &\quad = \left\|H_{e_1}(g\circ A)(A^{-1}x) - H_{e_1}(g\circ(-A))(-A^{-1}x)\right\|_{L^1} \\ &\quad = \left\|H_{e_1}(g\circ A)(x) + H_{e_1}(g\circ A)(-x)\right\|_{L^1} \\ &\quad \leq 2\left\|H_{e_1}(g)(x)\right\|_{L^1} = \frac{8}{\pi}\log\left(1+\sqrt{2}\right)\|g(x)\|_{L^1}.\end{aligned} \tag{32}$$

Then, via Minkowski's integral inequality, we obtain

$$\begin{aligned}&\|T_\Omega(\operatorname{sgn}\left(x_1\right)\chi_{B(0,1)})\|_{L^1} \\ &\quad = \frac{\pi}{2}\left\|\int_{\mathbf{S}^{n-1}\bigcap\{\theta:\theta_1>0\}} \Omega(\theta)(H_\theta - H_{-\theta})(g)(x)d\theta\right\|_{L^1} \\ &\quad \leq \frac{\pi}{2}\int_{\mathbf{S}^{n-1}\bigcap\{\theta:\theta_1>0\}} |\Omega\left(\theta\right)|\cdot\|(H_\theta - H_{-\theta})(g)(x)\|_{L^1} d\theta \\ &\quad = 4\log\left(1+\sqrt{2}\right)\|\Omega\|_{L^1(\mathbf{S}^{n-1})}\|\operatorname{sgn}\ \left(x_1\right)\chi_{B(0,1)}\|_{L^1}.\end{aligned} \tag{33}$$

□

According to [3], if $\Omega \in L\log^+L(\mathbf{S}^{n-1})$ with $\int_{\mathbf{S}^{n-1}} \Omega(\theta)d\theta = 0$, then T_Ω is weak type of $(1,1)$. With the help of Corollary 8 and Lemma 12, we easily get the following proposition.

Proposition 13. *If Ω is an odd function defined on $\mathbf{S}^{n-1}$ and $\Omega \in L\log^+L(\mathbf{S}^{n-1})$, then*

$$\|T_\Omega \circ \pi(f)\|_{L^1} \leq 4\log\left(1+\sqrt{2}\right)\|\Omega\|_{L^1(\mathbf{S}^{n-1})}\|f\|_{L^1}. \tag{34}$$

We know that Riesz transforms satisfy the conditions in Proposition 13, so $R_j \circ \pi$ maps $L^1(\mathbf{R}^n)$ to $L^1(\mathbf{R}^n)$. Then we establish one relationship between $L^1(\mathbf{R}^n)$ and $H^1(\mathbf{R}^n)$ in the following corollary.

Corollary 14. *(i) For all $f \in L^1(\mathbf{R}^n)$, then*

$$\left\|R_j \circ \pi(f)\right\|_{L^1} \leq 4\log\left(1+\sqrt{2}\right)\omega_n\|f\|_{L^1}. \tag{35}$$

Equality can occur if and only if $f = 0$ a.e.

(ii) For all $g \in \mathscr{A}(\mathbf{R}^n)$,

$$\|g\|_{H^1} \sim \|g\|_{L^1}. \tag{36}$$

Remark 15. Corollary 14 is the high dimensional extension of Corollary 11.

Here we can prove Corollary 8.

Proof of Corollary 8. Since T is a bounded operator from $H^1(\mathbf{R}^n)$ to $L^1(\mathbf{R}^n)$ and $\operatorname{sgn}(x_1)\chi_{B(0,R)} \in \mathscr{A}(\mathbf{R}^n)$, then via Corollary 14, we obtain that

$$\begin{aligned}\left\|T(\operatorname{sgn}(x_1)\chi_{B(0,R)})\right\|_{L^1} &\lesssim \left\|\operatorname{sgn}(x_1)\chi_{B(0,R)}\right\|_{H^1}\\ &\lesssim \left\|\operatorname{sgn}(x_1)\chi_{B(0,R)}\right\|_{L^1}.\end{aligned} \tag{37}$$

So Corollary 8 is an immediate consequence of Theorem 6 and Corollary 14. □

Remark 16. When Ω is odd and integrable over $\mathbf{S}^{n-1}$, whether T_Ω is weak type of $(1,1)$ is an open problem. So we suppose $\Omega \in L\log^+L(\mathbf{S}^{n-1})$ in Proposition 13. As showed in [4], T_Ω is weak type $(1,1)$ when restricted to radial functions in $L^1(\mathbf{R}^n)$. But we can try to show that T_Ω can be weak type $(1,1)$ when restricted to functions in $\mathscr{B}(\mathbf{R}^n)$, and then the same conclusion of Proposition 13 can be obtained. On the other hand, we try to omit the "odd" condition in the following proposition.

Proposition 17. *Let $\Omega \in L\log^+L(\mathbf{S}^{n-1})$ with $\int_{\mathbf{S}^{n-1}}\Omega(\theta)d\theta = 0$ and suppose for some $r > 0$ that*

$$\int_{B(0,2r)^c}\inf_{l_x\in\mathbf{C}}\int_{B(x,r)}\left|\frac{\Omega(y)}{|y|^n}-l_x\right|dy\,dx < \infty. \tag{38}$$

Then

$$\left\|T_\Omega(\operatorname{sgn}(x_1)\chi_{B(0,1)})\right\|_{L^1} < \infty, \tag{39}$$

and there exists a constant C, such that for all $f \in L^1(\mathbf{R}^n)$,

$$\left\|T_\Omega \circ \pi(f)\right\|_{L^1} \leq C\|f\|_{L^1}. \tag{40}$$

Proof. Since $\Omega \in L\log^+L(\mathbf{S}^{n-1})$, according to [2, pp. 274-275], the singular integral operator T_Ω is bounded on $L^2(\mathbf{R}^n)$. Then we obtain

$$\begin{aligned}&\left\|T_\Omega(\operatorname{sgn}(x_1)\chi_{B(0,1)})\right\|_{L^1(\mathbf{R}^n)}\\ &\quad= \left\|\delta^{r^{-1}}T_\Omega(\operatorname{sgn}(x_1)\chi_{B(0,r)})\right\|_{L^1(\mathbf{R}^n)}\\ &\quad= r^{-n}\left\|T_\Omega(\operatorname{sgn}(x_1)\chi_{B(0,r)})\right\|_{L^1(\mathbf{R}^n)}\\ &\quad= r^{-n}\left\|T_\Omega(\operatorname{sgn}(x_1)\chi_{B(0,r)})\right\|_{L^1(B(0,r))}\\ &\qquad+ r^{-n}\left\|T_\Omega\left(\operatorname{sgn}(x_1)\chi_{B(0,r)}\right)\right\|_{L^1(B(0,2r)^c)}\\ &\quad= r^{-n}I_1 + r^{-n}I_2,\\ &I_1 = \left\|T_\Omega(\operatorname{sgn}(x_1)\chi_{B(0,r)})\right\|_{L^1(B(0,r))}\\ &\quad= \left\|T_\Omega(\operatorname{sgn}(x_1)\chi_{B(0,r)})\right\|_{L^2}\\ &\qquad\times\left\|\chi_{B(0,2r)}\right\|_{L^2} < \infty.\end{aligned} \tag{41}$$

For each $l_x \in \mathbf{C}$,

$$\begin{aligned}&\left|T_\Omega\left(\operatorname{sgn}(x_1)\chi_{B(0,r)}\right)\right|\\ &\quad= \left|\int_{\mathbf{R}^n}\frac{\Omega(y/|y|)}{|y|^n}\operatorname{sgn}(x_1-y_1)\chi_{B(0,r)}(x-y)\,dy\right|\\ &\quad= \left|\int_{B(x,r)}\frac{\Omega(y/|y|)}{|y|^n}\operatorname{sgn}(x_1-y_1)\,dy\right|\\ &\quad= \left|\int_{B(x,r)}\left(\frac{\Omega(y/|y|)}{|y|^n}-l_x\right)\operatorname{sgn}(x_1-y_1)\,dy\right|\\ &\quad\leq \int_{B(x,r)}\left|\frac{\Omega(y/|y|)}{|y|^n}-l_x\right|dy;\end{aligned} \tag{42}$$

this means that

$$\begin{aligned}&\left|T_\Omega\left(\operatorname{sgn}(x_1)\chi_{B(0,r)}\right)\right| \leq \inf_{l_x\in\mathbf{C}}\int_{B(x,r)}\left|\frac{\Omega(y/|y|)}{|y|^n}-l_x\right|dy,\\ &I_2 = \int_{B(0,2r)^c}\left|T_\Omega\left(\operatorname{sgn}(x_1)\chi_{B(0,r)}\right)\right|dx\\ &\quad\leq \int_{B(0,2r)^c}\inf_{l_x\in\mathbf{C}}\int_{B(x,r)}\left|\frac{\Omega(y/|y|)}{|y|^n}-l_x\right|dy\,dx < \infty,\end{aligned} \tag{43}$$

According to [3], T_Ω is weak type of $(1,1)$, so Proposition 17 is an immediate consequence of Corollary 7. □

Now let us apply Corollary 8 to the general singular integral operators and show their boundedness on $L^1(\mathbf{R}^n)$ when restricted to odd rearrangement.

The following is the definition of the general singular operators.

The function K defined $\mathbf{R}^n\backslash\{0\}$ that satisfies size estimates,

$$\sup_{0<R<\infty}\frac{1}{R}\int_{|x|\leq R}|K(x)|\,|x|\,dx \leq A_1, \tag{44}$$

the smooth estimate, expressed in terms of Hömander's condition,

$$\sup_{|y|\neq 0}\int_{|x|>2|y|}\left|K(x-y)-K(x)\right|dx\leq A_2, \tag{45}$$

and cancelation condition,

$$\sup_{0<R_1<R_2}\left|\int_{R_1<|x|<R_2}K(x)\,dx\right|\leq A_3. \tag{46}$$

Condition (3) implies that there exists a sequence $\delta_j\searrow 0$ as $j\to 0$ such that the following limit exists:

$$\lim_{j\to\infty}\int_{\delta_j\leq|x|\leq 1}K(x)\,dx. \tag{47}$$

The tempered distribution W has the form

$$W(\varphi)=\lim_{j\to\infty}\int_{|x|\geq\delta_j}K(x)\varphi(x)\,dx,\quad \varphi\in\mathbf{S}(\mathbf{R}^n). \tag{48}$$

Proposition 18. *Let the function K be defined on $\mathbf{R}^n\setminus\{0\}$ that satisfies* (45), (46), *and* (47). *Suppose that the operator T is given by convolution with W. Then $\|T\circ\pi(f)\|_{L^1}\leq C_n(A_1+A_2+A_3)\|f\|_{L^1}$.*

Proof. By the results of [1, pp. 93-94], T admits an extension that is L^p bounded for $1<p<\infty$ and is of weak type $(1,1)$. Also, there exists a constant C_n such that for all $f\in H^1(\mathbf{R}^n)$,

$$\|T(f)\|_{L^1}\leq C_n(A_1+A_2+A_3)\|f\|_{H^1}. \tag{49}$$

Thus Proposition 18 is an immediate consequence of Corollary 8. □

Finally, we give results of C-Z operators. Corollary 8 is used again.

Proposition 19. *Assume that $K(x,y)$ is in $SK(\delta,A)$, and T is an element of $CZO(\delta,A,B)$ that is associated with the kernel K. Then for all $f\in L^1(\mathbf{R}^n)$,*

$$\|T\circ\pi(f)\|_{L^1}\leq C_{n,\delta}(A+B)\|f\|_{L^1}, \tag{50}$$

where C_n is a constant that depends on the dimension.

Proof. By the results of [1, pp. 171–190], T admits an extension that is L^p bounded for $1<p<\infty$ and is of weak type $(1,1)$. Also, there exists a constant $C_{n,\delta}$ such that for all $f\in H^1(\mathbf{R}^n)$,

$$\|T(f)\|_{L^1}\leq C_{n,\delta}(A+B)\|f\|_{H^1}. \tag{51}$$

Thus Proposition 19 is an immediate consequence of Corollary 8 too. □

3. Some Results on Hilbert Transform

In Section 2, we find that if $f=\chi_E\in L^1(\mathbf{R})$, by odd rearrangement, we obtain $d_f=d_{f_*}$ and $\|H(f_*)\|_{L^1}/\|f_*\|=C_1=(4/\pi)\log(1+\sqrt{2})$. In this section, we use the rearrangement, which is like the odd rearrangement, to show that for given $f\in L^1(\mathbf{R})$ and $c\in(0,\infty)$, there exists a function $g\in H^1(\mathbf{R})$ satisfying $d_f=d_g$ and $\|H(g)\|_{L^1}/\|g\|_{L^1}=c$.

Proposition 20. *For given $f=\chi_E\in L^1(\mathbf{R})$ and $c\in(0,\infty)$, there exists a function $g\in H^1(\mathbf{R})$ satisfying $d_f=d_g$ and $\|H(g)\|_{L^1}/\|g\|_{L^1}=c$.*

Proof. Since $\|H(f_*)\|_{L^1}/\|f_*\|_{L^1}=C_1=(4/\pi)\log(1+\sqrt{2})$, we discuss it in two cases.

Case 1. Given that $f=\chi_E\in L^1(\mathbf{R})$ and $c\in(C_1,\infty)$, then there exists a function $g\in H^1(\mathbf{R})$ satisfying $d_f=d_g$ and $\|H(g)\|_{L^1}/\|g\|_{L^1}=c$.

Case 2. Given that $f=\chi_E\in L^1(\mathbf{R})$ and $c\in(0,C_1)$, then there exists a function $g\in H^1(\mathbf{R})$ satisfying $d_f=d_g$ and $\|H(g)\|_{L^1}/\|g\|_{L^1}=c$.

Proof of Case 1. We take $F(t)=t\log t-t$, $t>0$, and $F(0)=0$. Then $F(t)$ is a continuous function defined on $[0,\infty)$. And we consider the function

$$h(t,x)=\chi_{(1,t)}(x)-\chi_{(-t,-1)}(x),\quad t>1. \tag{52}$$

Then we obtain $\|H(h(t,\cdot))(x)\|_{L^1(\mathbf{R})}=(4/\pi)[F(t+\sqrt{(t^1+1)/2})-F(t-\sqrt{(t^1+1)/2})-F(\sqrt{(t^1+1)/2}-1)-F(\sqrt{(t^1+1)/2}+1)]$. We take

$$\begin{aligned}G(t)=&F\left(t+\sqrt{\frac{t^1+1}{2}}\right)-F\left(t-\sqrt{\frac{t^1+1}{2}}\right)\\&-F\left(\sqrt{\frac{t^1+1}{2}}-1\right)-F\left(\sqrt{\frac{t^1+1}{2}}+1\right);\end{aligned} \tag{53}$$

then

$$\frac{\|H(h(t,\cdot))(x)\|_{L^1(\mathbf{R})}}{\|h(t,x)\|_{L^1(\mathbf{R})}}=\frac{2}{\pi}\frac{G(t)}{t-1}. \tag{54}$$

Here, we use three steps to prove $\{(2/\pi)(G(t)/(t-1));t>1\}=(C_1,\infty)$.

Step 1. Consider

$$\lim_{t\to 1^+}\frac{G(t)}{t-1}=+\infty. \tag{55}$$

Proof. We take $\sqrt{(1+t^2)/2}=h$; then $t=\sqrt{2h^2-1}$. Consider

$$\begin{aligned}&\lim_{t\to 1^+}\frac{G(t)}{t-1}\\&=\lim_{h\to 1^+}\left(F\left(\sqrt{2h^2-1}+h\right)-F\left(\sqrt{2h^2-1}-h\right)\right.\\&\qquad\left.-F(h-1)-F(h+1)\right)\\&\quad\times\left(\sqrt{2h^2-1}-1\right)^{-1}\\&=\lim_{h\to 1^+}\frac{1}{2h/\sqrt{2h^2-1}}\end{aligned}$$

$$\times\left[\left(\frac{2h}{\sqrt{2h^2-1}}+1\right)\log\left(\sqrt{2h^2-1}+h\right)\right.$$
$$-\left(\frac{2h}{\sqrt{2h^2-1}}-1\right)\log\left(\sqrt{2h^2-1}-h\right)$$
$$\left.-\log(h-1)-\log(h+1)\right]$$
$$=\lim_{h\to 1^+}\frac{1}{2h/\sqrt{2h^2-1}}$$
$$\times\left[\frac{2h}{\sqrt{2h^2-1}}\log\frac{\sqrt{2h^2-1}+h}{\sqrt{2h^2-1}-h}\right.$$
$$+\log\left(\sqrt{2h^2-1}+h\right)$$
$$+\log\left(\sqrt{2h^2-1}-h\right)$$
$$\left.-\log(h-1)-\log(h+1)\right]$$
$$=\lim_{h\to 1^+}\frac{\left(2h/\sqrt{2h^2-1}\right)\log\left(\sqrt{2h^2-1+h}/\sqrt{2h^2-1-h}\right)}{2h/\sqrt{2h^2-1}}$$
$$=\lim_{h\to 1^+}\log\frac{\sqrt{2h^2-1}+h}{\sqrt{2h^2-1}-h}=+\infty. \tag{56}$$

Step 2. $\lim_{t\to\infty}G(t)/(t-1)=2\log(1+\sqrt{2})$. Consider

$$\lim_{t\to+\infty}\frac{G(t)}{t-1}$$
$$=\lim_{h\to+\infty}\left(F\left(\sqrt{2h^2-1}+h\right)-F\left(\sqrt{2h^2-1}-h\right)\right.$$
$$\left.-F(h-1)-F(h+1)\right)$$
$$\times\left(\sqrt{2h^2-1}-1\right)^{-1} \tag{57}$$
$$=\lim_{h\to+\infty}\log\frac{\sqrt{2h^2-1}+h}{\sqrt{2h^2-1}-h}$$
$$=\log\frac{\sqrt{2}+1}{\sqrt{2}-1}=\log\left(\sqrt{2}+1\right)^2=2\log\left(\sqrt{2}+1\right).$$

Step 3. $G(t)/(t-1)$ is decreasing on the interval $(1,\infty)$. Consider

$$\frac{dh}{dt}=\frac{t}{2\sqrt{(1+t^2)/2}}>0,\quad t>1,$$
$$\frac{d}{dt}\left(\frac{G(t)}{t-1}\right)$$
$$=\frac{d}{dh}\left(\left(F\left(\sqrt{2h^2-1}+h\right)-F\left(\sqrt{2h^2-1}-h\right)\right.\right.$$
$$\left.-F(h-1)-F(h+1)\right)$$
$$\left.\times\left(\sqrt{2h^2-1}-1\right)^{-1}\right)\frac{dh}{dt}$$
$$=\frac{4h}{\left(\sqrt{2h^2-1}-1\right)^2\sqrt{2h^2-1}}$$
$$\times\log\frac{h+1}{\sqrt{2h^2-1}+h}<0,\quad (h>1). \tag{58}$$

So we obtain that

$$\frac{d}{dt}\left(\frac{G(t)}{t-1}\right)<0. \tag{59}$$

Thus $G(t)/(t-1)$ is decreasing on the interval $(1,\infty)$. Since $G(t)/(t-1)$ is continuous on the interval $(0,\infty)$, it is easy to conclude that $\{(2/\pi)(G(t)/(t-1));t>1\}=(C_1,\infty)$. Now let us discuss Case 1.

$\forall c\in(C_1,\infty)$, then there exists $t_c>1$ satisfying

$$\frac{\|H(h(t_c,\cdot))(x)\|_{L^1(\mathbf{R})}}{\|h(t_c,x)\|_{L^1(\mathbf{R})}}=c. \tag{60}$$

Taking $g(x)=h(t_c,(2(t_c-1)/|E|)x)$, then we obtain that

$$\frac{\|H(g)\|_{L^1}}{\|g\|_{L^1}}=\frac{\|H(h(t_c,\cdot))(x)\|_{L^1(\mathbf{R})}}{\|h(t_c,x)\|_{L^1(\mathbf{R})}}=c. \tag{61}$$

At the same time, we can easily verify that $d_f=d_g$.

Proof of Case 2. We know that $\|H(\chi_{(0,1)}-\chi_{(-1,0)})(x)\|_{L^1}/\|\chi_{(0,1)}-\chi_{(-1,0)}\|_{L^1}=C_1$ in Section 2. If we take $f_1(x)=\chi_{(0,1)}-\chi_{(-1,0)}$, then $\|H(f_1)(x)\|_{L^1}/\|f_1\|_{L^1}=C_1$; we note that

$$f_n(x)=\sum_{k=1}^{2^{n-1}}\chi\left(\frac{k-1}{2^{n-1}},\frac{k}{2^{n-1}}\right)-\chi\left(-\frac{k}{2^{n-1}},-\frac{k-1}{2^{n-1}}\right),$$
$$\frac{\|H(f_n)(x)\|_{L^1}}{\|f_n\|_{L^1}}=C_n,$$
$$f_{n+1,1}(x)=\sum_{k=1}^{2^{n-1}}\chi\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)-\chi\left(-\frac{k}{2^n},-\frac{k-1}{2^n}\right),$$
$$f_{n+1,2}(x)=\sum_{k=2^{n-1}+1}^{2^n}\chi\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)-\chi\left(-\frac{k}{2^n},-\frac{k-1}{2^n}\right). \tag{62}$$

Here, we prove that $C_n\searrow 0$ $(n\to 0)$.

We know that Hilbert transform satisfies

$$H\left(\tau^h(f)\right)=\tau^h H(f),\quad (h\in\mathbf{R}),$$
$$H\left(\delta^a(f)\right)=\delta^a H(f),\quad (a>0). \tag{63}$$

Thus we obtain

$$\frac{\|H(f_{n+1,i})(x)\|_{L^1}}{\|f_{n+1,i}\|_{L^1}}=C_n,\quad i=1,2. \tag{64}$$

Then

$$\begin{aligned} C_{n+1} &= \frac{\|H(f_{n+1})(x)\|_{L^1}}{\|f_{n+1}\|_{L^1}} \\ &= \frac{\|H(f_{n+1,1}+f_{n+1,2})(x)\|_{L^1}}{\|f_{n+1}\|_{L^1}} \\ &= \frac{\|H(f_{n+1,1})(x)+H(f_{n+1,2})(x)\|_{L^1}}{\|f_{n+1}\|_{L^1}} \\ &< \frac{\|H(f_{n+1,1})(x)\|_{L^1}+\|H(f_{n+1,2})(x)\|_{L^1}}{\|f_{n+1}\|_{L^1}} \\ &= C_n. \end{aligned} \tag{65}$$

Since $\|f_n\|_{L^1} \equiv 2, n \geq 1$, so we just need an estimate of $\|H(f_n)\|_{L^1}$:

$$\begin{aligned} &\|H(f_n)\|_{L^1} \\ &= \left\| \sum_{k=1}^{2^{n-1}} \log \frac{\left(x-\left((k-1)/2^{n-1}\right)\right)^2}{\left|\left(x-\left((k-1)/2^{n-1}\right)\right)^2-\left(1/2^{n-1}\right)^2\right|} \right\|_{L^1} \\ &= \frac{1}{2^{n-1}} \left\| \sum_{k=1}^{2^{n-1}} \log \frac{(x-k+1)^2}{|(x-k+1)^2-1|} \right\|_{L^1} \\ &= \frac{1}{2^{n-1}} \left\| \log \prod_{k=1}^{2^{n-1}} \frac{(x-k+1)^2}{|(x-k+1)^2-1|} \right\|_{L^1} \\ &= \frac{1}{2^{n-1}} \left\| \log \frac{|x|}{|x+1|} \frac{|x-(2^{n-1}-1)|}{|x-2^{n-1}|} \right\|_{L^1} \\ &= \frac{1}{2^{n-1}} \left\| \log \frac{\left|x+2^{n-2}-(1/2)\right|}{\left|x+2^{n-2}+(1/2)\right|} \right. \\ &\qquad \left. \times \frac{\left|x-\left(2^{n-2}-(1/2)\right)\right|}{\left|x-2^{n-2}-(1/2)\right|} \right\|_{L^1} \\ &= \frac{1}{2} \left\| H\left(\chi_{(1-(1/2^{n-1}),1+(1/2^{n-1}))} \right. \right. \\ &\qquad \left. \left. -\chi_{(-1-(1/2^{n-1}),-1+(1/2^{n-1}))}\right)\right\|_{L^1}. \end{aligned} \tag{66}$$

In order to estimate the limit of $\|H(f_n)\|_{L^1}$, we consider the following function:

$$h(t) = \left\|H(\chi_{(1-t,1+t)}-\chi_{(-1-t,-1+t)})\right\|_{L^1}, \quad t \in [0,1]. \tag{67}$$

We obtain that

$$\begin{aligned} h(t) = 4\Big[&F\left(1+t+\sqrt{1+t^2}\right)-F\left(1+t-\sqrt{1+t^2}\right) \\ &-F\left(\sqrt{1+t^2}-1+t\right)-F\left(\sqrt{1+t^2}+1-t\right)\Big]; \end{aligned} \tag{68}$$

thus

$$\lim_{n\to\infty}\|H(f_n)\|_{L^1} = \lim_{t\to 0}\frac{1}{2}h(t) = 0. \tag{69}$$

This means that $C_n \searrow 0, (n \to \infty)$. Now we construct the following functions:

$$\begin{aligned} g_{n,t}(x) = &\sum_{k=1}^{2^{n-1}} \chi_{((k-1)/2^{n-1},\, k/2^{n-1})} - \chi_{(-k/2^{n-1},-(k-1)/2^{n-1})} \\ &-2\left[\chi_{((k-1)/2^{n-1},(k-1)/2^{n-1}+t)}\right. \\ &\left.-\chi_{(-t-(k-1)/2^{n-1},-(k-1)/2^{n-1})}\right]. \end{aligned} \tag{70}$$

When $t = 0$, we have $g_{n,t} = f_n$; and when $t = 1/2^n$, we have

$$\begin{aligned} \frac{\|H(g_{n,0})\|_{L^p}}{\|g_{n,0}\|_{L^p}} &= C_n, \\ \frac{\|H(g_{n,1/2^n})\|_{L^p}}{\|g_{n,1/2^n}\|_{L^p}} &= C_{n+1}. \end{aligned} \tag{71}$$

Using the continuity of $\|H(g_{n,t})\|_{L^1}/\|g_{n,t}\|_{L^1} = \|H(g_{n,t})\|_{L^1}/2$ on the interval $[0,1/2^n]$, we obtain that for each $c \in (C_{n+1}, C_n)$, there exists a constant $t_c \in (0,1/2^n)$ satisfying

$$\frac{\|H(g_{n,t_c})\|_{L^1}}{\|g_{n,t_c}\|_{L^1}} = c. \tag{72}$$

So Case 2 is true. □

Here, we can get some corollaries.

Corollary 21. *Suppose that I_1, I_2 are two disjoint open intervals with given same length; note that $d(I_1, I_2)$ is the distance between I_1 and I_2; then $\|H(\chi_{I_1}-\chi_{I_2})\|_{L^1}/\|\chi_{I_1}-\chi_{I_2}\|_{L^1}$ depends only on $d(I_1, I_2)$. Precisely*

(i) $\|H(\chi_{I_1}-\chi_{I_2})\|_{L^1}/\|\chi_{I_1}-\chi_{I_2}\|_{L^1} = C_1 = 4/\pi \log(1+\sqrt{2})$ *when $d = 0$;*

(ii) $\|H(\chi_{I_1}-\chi_{I_2})\|_{L^1}/\|\chi_{I_1}-\chi_{I_2}\|_{L^1} \nearrow \infty, (d \to \infty)$;

(iii) $\{\|H(\chi_{I_1}-\chi_{I_2})\|_{L^1}/\|\chi_{I_1}-\chi_{I_2}\|_{L^1} :$ *I_1 and I_2 are disjoint open intervals*$\} = [(4/\pi)\log(1+\sqrt{2}), \infty)$.

Proof. Since Hilbert transform, H, commutes with translations and dilations, we can suppose that

$$\begin{aligned} \chi_{I_1}-\chi_{I_2} &= \chi_{(a,a+1)}-\chi_{(-a-1,-a)}, \quad a \geq 0, \\ d(I_1, I_2) &= 2a. \end{aligned} \tag{73}$$

Because Hilbert transform, H, commutes with dilations, we obtain that

$$\begin{aligned} &\frac{\|H(\chi_{I_1}-\chi_{I_2})\|_{L^1}}{\|\chi_{I_1}-\chi_{I_2}\|_{L^1}} \\ &= \frac{\left\|\log\left|x^2-a^2\right|/|x^2-(a+1)^2|\right\|_{L^1}}{2} \\ &= \frac{\left\|\log\left|x^2-1\right|/|x^2-(1+(1/a))^2|\right\|_{L^1}}{2(1/a)}. \end{aligned} \tag{74}$$

We take $t = 1 + (1/a)$; then we obtain that

$$\frac{\left\|H(\chi_{I_1} - \chi_{I_2})\right\|_{L^1}}{\left\|\chi_{I_1} - \chi_{I_2}\right\|_{L^1}} = \frac{\|H(h(t,\cdot))(x)\|_{L^1(\mathbf{R})}}{\|h(t,x)\|_{L^1(\mathbf{R})}} = \frac{2}{\pi}\frac{G(t)}{t-1}, \tag{75}$$

where $h(t,x) = \chi_{(1,t)}(x) - \chi_{(-t,-1)}(x)$, $t = 1 + (1/d) > 1$, and $G(t)$ is defined in the proof of Proposition 20. According to the proof of Proposition 20, we conclude the following:

(i) $\lim_{t\to 1^+} G(t)/(t-1) = +\infty$;

(ii) $\lim_{t\to\infty} G(t)/(t-1) = 2\log(1+\sqrt{2})$;

(iii) $G(t)/(t-1)$ is decreasing and continuous on the interval $(1,\infty)$.

And the corollary is consequence of the three items above. □

As in [1], we know that the elements of $H^1(\mathbf{R}^n)$ have integral zero. In Exercise 6.4.3 of [1], we learn that, for some $p > 1$, if $g \in L^p$ is compactly supported with integral zero, then $g \in H^1$. In the following example, with the help of rearrangement, we construct an odd function in $\bigcap_{0<p\le\infty} L^p(\mathbf{R})$ while not in $H^1(\mathbf{R})$. Thus "compact" condition cannot be removed without modifying other conditions properly.

Example 22. According to Corollary 21, we construct two sequences of intervals $\{I_{1,n}\}$ and $\{I_{2,n}\}$ satisfying the following:

(i) $|I_{1,n}| = |I_{2,n}| = 1/2^n$ and $I_{1,n}$ and $I_{2,n}$ are disjoint intervals;

(ii) $\|H(\chi_{I_{1,n+1}} - \chi_{I_{2,n+1}})\|_{L^1} \ge 2\sum_{k=1}^{n}\|H(\chi_{I_{1,k}} - \chi_{I_{2,k}})\|_{L^1}$, $\forall n \ge 1$;

(iii) $I_{1,n} = -I_{2,n}$ and $I_{1,n} \subset (0,\infty)$, $\forall n \ge 1$;

(iv) $d(I_{1,n+1}, I_{2,n+1}) \ge d(I_{1,n}, I_{2,n}) + 1$, $\forall n \ge 1$.

We take $f_n = \chi_{I_{1,n}} - \chi_{I_{2,n}}$ and $f = \sum_{n\ge1} f_n$, since $d_f = d_{\chi_{(-1,1)}}$; we obtain that

$$\|f\|_{L^p} = \|\chi_{(-1,1)}\|_{L^p} < \infty, \quad 0 < p \le \infty, \qquad \|H(f)\|_{L^2} = \|f\|_{L^2} < \infty. \tag{76}$$

Since $\sum_{1\le k\le n} f_k \to f$ in $L^2(\mathbf{R})$ as $n \to \infty$, we obtain that $\sum_{1\le k\le n} H(f_k) \to H(f)$ in $L^2(\mathbf{R})$ as $n \to \infty$. This means that there is a sequence $\{n_k\}$ satisfying

$$\sum_{1\le i\le n_k} H(f_i) \longrightarrow H(f) \text{ a.e.} \tag{77}$$

By noting that $I_{1,n} = (a_n, b_n)$, then we obtain that

$$0 < a_n < b_n < a_{n+1} < b_{n+1}, \qquad \sqrt{\frac{a_n^2 + b_n^2}{2}} < \sqrt{\frac{a_{n+1}^2 + b_{n+1}^2}{2}}. \tag{78}$$

Since $H(f_n) = \log|(x^2 - a_n^2)/(x^2 - b_n^2)|$, we obtain that $H(f_n)(x) < 0$ if and only if $|x| < \sqrt{(a_n^2 + b_n^2)/2}$. And we easily obtain that

$$\left(\sum_{k=1}^{n} H(f_k)\right)_- \le \left(\sum_{k=1}^{n+1} H(f_k)\right)_-. \tag{79}$$

Since $\int_{|x|<\sqrt{(a_n^2+b_n^2)/2}} H(f_n)(x)dx = -\int_{|x|>\sqrt{(a_n^2+b_n^2)/2}} H(f_n)(x)dx$, we obtain that

$$\int_{\mathbf{R}} H(f_n)(x)\,dx = 0, \qquad \int_{\mathbf{R}} \sum_{k=1}^{n} H(f_k)(x)\,dx = 0. \tag{80}$$

Then

$$\begin{aligned}
\|H(f)\|_{L^1} &\ge \|H(f)_-\|_{L^1} \\
&= \left\|\lim_{k\to\infty}\left[\sum_{1\le i\le n_k} H(f_i)\right]_-\right\|_{L^1} \\
&= \lim_{k\to\infty}\left\|\left[\sum_{1\le i\le n_k} H(f_i)\right]_-\right\|_{L^1} \\
&= \lim_{k\to\infty}\left\|\left[\sum_{1\le i\le n_k} H(f_i)\right]_+\right\|_{L^1} \\
&\ge \lim_{k\to\infty}\left\|\left[H(f_{n_k})\right]_+\right\|_{L^1} = \frac{1}{2}\lim_{k\to\infty}\|H(f_{n_k})\|_{L^1} \\
&\ge \lim_{k\to\infty}\sum_{i=1}^{n_k-1}\|H(f_i)\|_{L^1} \\
&\ge \lim_{k\to\infty}(n_k - 1)\|H(f_1)\|_{L^1} = \infty.
\end{aligned} \tag{81}$$

Since $\|f\|_{H^1} \sim \|f\|_{L^1} + \|H(f)\|_{L^1}$, we get that $f \notin H^1(\mathbf{R})$.

Corollary 23. *There exist two sequences $\{f_n\} \subset H^1(\mathbf{R})$ and $\{f_n\} \subset H^1(\mathbf{R})$ satisfying*

$$\|f_n\|_{L^1} = 1, \quad (n \longrightarrow \infty), \qquad \|H(f_n)\|_{L^1} \longrightarrow 0, \quad (n \longrightarrow \infty), \qquad \|H(g_n)\|_{L^1} \longrightarrow \infty, \quad (n \longrightarrow \infty). \tag{82}$$

Proof. We take $f = \chi_{(0,1)}$; according to Proposition 20, there exist two sequences $\{g_n\}$ and $\{f_n\}$ satisfying

$$d_{f_n} = d_f, \qquad \frac{\|H(f_n)\|_{L^1}}{\|f_n\|_{L^1}} = \frac{1}{n}, \qquad \frac{\|H(g_n)\|_{L^1}}{\|g_n\|_{L^1}} = n. \tag{83}$$

So we easily obtain this corollary. □

Similarly, via the method of rotation we used in Lemma 12, we can obtain the following corollary.

Corollary 24. *There exists a sequence* $\{f_m\} \subset H^1(\mathbf{R}^n)$ *satisfying*

$$\|f_m\|_{L^1} = 1, \quad (m \geq 1),$$
$$\|R_j(f_m)\|_{L^1} \longrightarrow 0, \quad (m \longrightarrow \infty). \tag{84}$$

where R_j $(1 \leq j \leq n)$ *are Riesz transforms.*

Remark 25. Since $\|f\|_{L^1(\mathbf{R}^n)} \lesssim \|f\|_{H^1}(\mathbf{R}^n)$, according to Corollaries 23 and 24, $\|H(f)\|_{L^1(\mathbf{R})}$ and $\sum_{1\leq j\leq n}\|R_j(f)\|_{L^1(\mathbf{R}^n)}$ are not equivalent norms of Hardy spaces $H^1(\mathbf{R})$ and $H^1(\mathbf{R}^n)$.

Lemma 26. *Let* $\Lambda = \{g_t, t \in I\} \subset L^1(\mathbf{R})$ *satisfying*

$$\lim_{t\to s}\|g_t - g_s\|_{L^1} = 0, \quad \forall s \in I,$$
$$\{\|g\|_{L^1}, g \in \Lambda\} = (0,\infty), \tag{85}$$

where I *is an open interval of* $\mathbf{R}$. *Then for all* $f \in L^1(\mathbf{R})$,

$$(\|f\|_{L^1}, \infty) \subset \{\|f+g\|_{L^1}, g \in \Lambda\}. \tag{86}$$

Proof. Given that $f \in L^1(\mathbf{R})$, $\forall\epsilon > 0$, and $\forall N > 0$ $(N > \epsilon)$, since $\{\|g\|_{L^1}, g \in \Lambda\} = (0,\infty)$, there exist $t_0, t_1 \in I$ satisfying

$$\|g_{t_0}\|_{L^1} < \epsilon,$$
$$\|g_{t_1}\|_{L^1} > 2\|f\|_{L^1} + N. \tag{87}$$

Thus, we obtain that

$$\|g_{t_0} + f\|_{L^1} < \|f\|_{L^1} + \epsilon,$$
$$\|g_{t_1} + f\|_{L^1} \geq \|g_{t_0}\|_{L^1} - \|f\|_{L^1} > \|f\|_{L^1} + N. \tag{88}$$

Since $\lim_{t\to s}\|g_t - g_s\|_{L^1} = 0$, $(\forall s \in I)$, we get that $\|g_t + f\|_{L^1}$ is continuous on the interval I. Thus, $(\|f\|_{L^1} + \epsilon, \|f\|_{L^1} + N) \subset (\|g_{t_0} + f\|_{L^1}, \|g_{t_1} + f\|_{L^1}) \subset \{\|f+g\|_{L^1}, g \in \Lambda\}$.

Let $\epsilon \to 0$ and $N \to \infty$; we obtain

$$(\|f\|_{L^1}, \infty) \subset \{\|f+g\|_{L^1}, g \in \Lambda\}. \tag{89}$$

□

Proposition 27. *Given each simple function:* $f \in L^1(\mathbf{R})$ *and* $c \in (0,\infty)$, *then there exists a function* $g \in H^1(\mathbf{R})$ *satisfying*

$$d_f = d_g, \quad \frac{\|H(g)\|_{L^1}}{\|g\|} = c. \tag{90}$$

Proof. Since f is a simple function in $L^1(\mathbf{R})$, there are two sequences $\{a_i\}_{1\leq i\leq m}$ and $\{t_n\}_{1\leq i\leq m}$ satisfying

$$f_1 = \sum_{i=1}^{m} a_i\chi_{(2(\sum_{k=1}^{i} t_k)-t_i, 2\sum_{k=1}^{i} t_k)} - \sum_{i=1}^{m} a_i\chi_{(2\sum_{k=1}^{i-1} t_k, 2(\sum_{k=1}^{i-1} t_k)+t_i)}, \tag{91}$$
$$d_{f_1} = d_f.$$

We take $C'' = \|H(f_1)\|_{L^1}/\|f_1\|_{L^1}$, since

$$f_1 = \sum_{i=1}^{m-1} a_i\chi_{(2(\sum_{k=1}^{i} t_k)-t_i, 2\sum_{k=1}^{i} t_k)} - \sum_{i=1}^{m-1} a_i\chi_{(2\sum_{k=1}^{i-1} t_k, 2(\sum_{k=1}^{i-1} t_k)+t_i)} + g_1, \tag{92}$$

where $g_1 = a_m\chi_{(2(\sum_{k=1}^{m} t_k)-t_m, 2\sum_{k=1}^{m} t_k)} - a_m\chi_{(2\sum_{k=1}^{m-1} t_k, 2(\sum_{k=1}^{m-1} t_k)+t_m)}$; we construct a set $\Gamma = \{g_{1,t} : t \in [0,\infty)\} \subset L^1(\mathbf{R})$, where $g_{1,t} = a_m$.

Also $\chi_{(2(\sum_{k=1}^{m} t_k)-t_m+t, 2(\sum_{k=1}^{m} t_k)+t)} - a_m\chi_{(2\sum_{k=1}^{m-1} t_k, 2(\sum_{k=1}^{m-1} t_k)+t_m)}$. Using Corollary 21, $g_{1,0} = g_1$, and $\|g_{1,t}\|_{L^1} \equiv 2a_m t_m$, we obtain that

$$\{\|H(h)\|_{L^1}, h \in \Gamma\} = [\|g_1\|_{L^1}, \infty), \tag{93}$$

Using Lemma 26, we obtain that

$$[\|H(f_1)\|_{L^1}, \infty) \subset \{\|H(f_1 - g_1 + g_{1,t})\|_{L^1}, t \in [0,\infty)\}. \tag{94}$$

Since $\|f_1 - g_1 + g_{1,t}\|_{L^1} \equiv 2\sum_{k=1}^{m} a_k t_k$, we obtain that, for each $c \in [C'', \infty)$, there exists $g = f_1 - g_1 + g_{1,t_c}$ satisfying

$$d_g = d_f, \quad \frac{\|H(g)\|_{L^1}}{\|g\|_{L^1}} = c. \tag{95}$$

By similar argument of Case 2 of Proposition 20, for each $c \in (0, C')$, there exists g satisfying

$$d_g = d_f, \quad \frac{\|H(g)\|_{L^1}}{\|g\|_{L^1}} = c. \tag{96}$$

□

Theorem 28. *For each function* $f \in L^1(\mathbf{R})$ *and* $c \in (0,\infty)$, *then there exists a function* $g \in H^1(\mathbf{R})$ *satisfying*

$$d_f = d_g, \quad \frac{\|H(g)\|_{L^1}}{\|g\|} = c. \tag{97}$$

Proof. We can use the results of Proposition 27 and the same method of rearrangement as in the proof of Proposition 20 to get this theorem proved; we leave the details to the reader. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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