Research Article

Periodic Solutions for Semilinear Fourth-Order Differential Inclusions via Nonsmooth Critical Point Theory

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Received 31 May 2013; Revised 30 September 2013; Accepted 28 October 2013; Published 21 January 2014

Academic Editor: Alberto Fiorenza

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Three periodic solutions with prescribed wavelength for a class of semilinear fourth-order differential inclusions are obtained by using a nonsmooth version critical point theorem. Some results of previous related literature are extended.

1. Introduction

The Fisher-Kolmogorov (FK for short) equation was proposed as a model for phase transitions and other bistable phenomena and is one of the most fundamental models in mathematical biology and ecology; for example, see Zimmerman [1], Couillet et al. [2], and Dee and SaarLoos [3].

Recently, many authors were interested in extended FK equation of the form

$$u^{(iv)} - pu'' + a(x)u + b(x)u^3 = 0, \quad 0 < x < L, \quad (1)$$

$$u(0) = u''(0) = u(L) = u''(L) = 0,$$

where p is a positive constant and a and b are continuous positive $2L$ -periodic functions on $\mathbb{R}$. Equation (1) arises as the mesoscopic model of a phase transition in a binary system near the Lipschitz point [4, 5], and (1) is frequently used as a model for the study of pattern formation in an unstable spatially homogeneous state [2, 3]. It has been attracting more and more attention due to its significant value in theory and practical application [6–9].

Especially, in [10], by using the symmetric mountain-pass theorem, Ma and Dai considered the nonlocal semilinear fourth-order differential equation

$$\begin{aligned} &u^{(iv)} - pu'' + a(x)u - u(x) \\ &\quad \times \int_0^L f(x, y)u^2(y)dy = 0, \quad 0 < x < L, \\ &u(0) = u''(0) = u(L) = u''(L) = 0. \end{aligned} \quad (2)$$

They obtained the existence of infinitely many distinct pairs of solutions of the above problem, where p is a positive constant, a is a positive continuous even and $2L$ -periodic function, and $f(x, y) \equiv f(|x - y|) : [0, L] \rightarrow \mathbb{R}^+$ is continuous and monotone decreasing.

It is worth mentioning that the method used in [10] is not valid for more general nonlinearity. Furthermore, to the best of our knowledge, there is no author using the nonsmooth version critical point theory to consider the extended FK equation. In view of this, this paper is concerned

with the existence of three periodic solutions of the following semilinear fourth-order differential inclusion:

$$u^{(iv)} - pu'' + a(x)u \in \lambda \partial F(u) + \mu \partial G_u(x, u), \quad 0 < x < L, \quad (3)$$

$$u(0) = u''(0) = u(L) = u''(L) = 0,$$

where

(H1) $\lambda, \mu > 0$ are real parameters, p is a positive constant, and a is a positive continuous even $2L$ -periodic function on $\mathbb{R}$; F is a locally Lipschitz function defined on $\mathbb{R}$ satisfying

(F1) $F(0) = 0$, $\partial F(u) = -\partial F(-u)$ for $u \in \mathbb{R}$,

(F2) there exist constants $c_1 > 0$ and $s \in [1, 2)$, such that

$$|\xi| \leq c_1 (1 + |u|^{s-1}), \quad \xi \in \partial F(u), \quad u \in \mathbb{R}, \quad (4)$$

(F3)

$$R \int_0^L F(x^2 - Lx) dx > L \sup_{s \in [-c_2, c_2]} F(s), \quad (5)$$

where R and c_2 are two given positive constants.

G is defined on $\mathbb{R} \times \mathbb{R}$ satisfying

(G1) $G(\cdot, u) : \mathbb{R} \rightarrow \mathbb{R}$ is measurable for each $u \in \mathbb{R}$, $G(x, \cdot) : \mathbb{R} \rightarrow \mathbb{R}$ is locally Lipschitz for $x \in \mathbb{R}$, and $G(x, 0) = 0$ and $-\partial G_u(-x, -u) = \partial G_u(x, u) = \partial G_u(x + 2L, u)$ for a.e. $x \in \mathbb{R}$ and $u \in \mathbb{R}$;

(G2) there exists constant $b > 0$, such that

$$|\xi| \leq b (1 + |u|^{s-1}), \quad u \in \mathbb{R}, \quad \xi \in \partial G_u(x, u), \quad (6)$$

where s is defined in (F2).

By applying a nonsmooth version critical point theorem [11, 12], we prove that, when λ and μ are in given interval, (3) admits at least three solutions. Moreover, we achieve an estimate of the solutions norms independent of G , λ , and μ . Concretely, we get the following main result.

Theorem 1. Assume that (H1) and (F1)–(F3) hold. Then there exist a nondegenerate interval $[\alpha, \beta] \subset (0, \infty)$ and $\rho > 0$ with the following property: for any $\lambda \in [\alpha, \beta]$ and any function G satisfying (G1) and (G2), there exists $\delta > 0$, such that, for $\mu \in (0, \delta)$, the problem (3) admits at least three solutions with norm in X less than ρ .

Remark 2. Note that the coefficient a is even and $2L$ -periodic, if u is a solution of (3) and $\bar{u}(x)$ is its antisymmetric extension with respect to $x = 0$:

$$\bar{u}(x) = \begin{cases} u(x), & 0 \leq x \leq L, \\ -u(-x), & -L \leq x \leq 0. \end{cases} \quad (7)$$

Combining with (F1) and (G1), we can see that $2L$ -periodic extension of $\bar{u}$ over $\mathbb{R}$ is a $2L$ -periodic solution of the equation

$$u^{(iv)} - pu'' + a(x)u \in \lambda \partial F(u(x)) + \mu \partial G_u(x, u(x)), \quad x \in \mathbb{R}. \quad (8)$$

The rest of this paper is arranged as follows. Section 2 contains some preliminaries of nonsmooth analysis and abstract results which are needed later. Section 3 concerns a variational method for problem (3); in the final section, we give the proof of the main result.

2. Preliminaries

We collect some basic notions and results of nonsmooth analysis, namely, the calculus for locally Lipschitz functionals developed by Clarke [13] and Motreanu and Panagiotopoulos [14].

Let $(X, \|\cdot\|_X)$ be a Banach space, $(X^*, \|\cdot\|_{X^*})$ be its topological dual, and $\varphi : X \rightarrow \mathbb{R}$ a functional. We recall that φ is locally Lipschitz (l.L.), if, for each $u \in X$, there exist a neighborhood U of u and a real number $K > 0$ such that

$$|\varphi(v) - \varphi(w)| \leq K\|v - w\|_X \quad \text{for } v, w \in U. \quad (9)$$

If φ is l.L. and $u \in X$, the generalized directional derivative of φ at u along the direction $v \in X$ is

$$\varphi^0(u; v) = \overline{\lim}_{\substack{\omega \rightarrow u \\ t \rightarrow 0^+}} \frac{\varphi(\omega + tv) - \varphi(\omega)}{t}. \quad (10)$$

The generalized gradient of φ at u is the set

$$\partial\varphi(u) = \{u^* \in X^* : \langle u^*, v \rangle \leq \varphi^0(u; v) \text{ for } v \in X\}. \quad (11)$$

Then for $u \in X$, $\partial\varphi(u) \in 2^{X^*}$ is a nonempty, convex, and w^* -compact subset [13, Proposition 1]. We call that φ has compact gradient if $\partial\varphi$ maps bounded subsets of X into relatively compact subsets of X^* . We say that $u \in X$ is a critical point of l.L. functional φ if $0 \in \partial\varphi(u)$.

Lemma 3 (see [14, Proposition 1.1]). If $\varphi \in C^1(X, \mathbb{R})$, then φ is l.L. and

$$\varphi^0(u; v) = \langle \varphi'(u), v \rangle, \quad \partial\varphi(u) = \{\varphi'(u)\}, \quad u, v \in X. \quad (12)$$

Lemma 4 (see [14, Proposition 1.3]). Let $\varphi : X \rightarrow \mathbb{R}$ be an l.L. functional. Then for $u \in X$, $\varphi^0(u; \cdot)$ is subadditive and positively homogeneous and

$$\varphi^0(u; v) \leq K\|v\|, \quad v \in X \quad (13)$$

with $K > 0$ being a Lipschitz constant for φ around u .

Lemma 5 (see [14, Proposition 1.6]). Let $\varphi, \psi : X \rightarrow \mathbb{R}$ be l.L. functionals. Then

$$\begin{aligned} \partial(\lambda\varphi)(u) &= \lambda\partial\varphi(u), & \partial(\varphi + \psi)(u) &\subset \partial\varphi(u) + \partial\psi(u), \\ \lambda &\in \mathbb{R}, \quad u \in X. \end{aligned} \quad (14)$$

Lemma 6 (see [12, Lemma 6]). Let $\varphi : X \rightarrow \mathbb{R}$ be a l.L. functional with compact gradient. Then φ is sequentially weakly continuous.

In the sequel we need the following lemmas to study the existence of solutions for the problem (3). Firstly, we present a definition.

Definition 7 (see [15, Definition 2.1]). An operator $A : X \rightarrow X^*$ is of type $(S)_+$, if, for any sequence $\{u_n\}$ in X , $u_n \rightharpoonup u$ and $\lim_{n \rightarrow \infty} \langle A(u_n), u_n - u \rangle \leq 0$ imply that $u_n \rightarrow u$.

Lemma 8 (a particular case of [12, Theorem 14]). Let $(X, \|\cdot\|)$ be a reflexive Banach space, $I \subset \mathbb{R}$ an interval, $\mathcal{N} \in C^1(X, \mathbb{R})$ a sequentially weakly lower semicontinuous functional whose derivative is of type $(S)_+$, and $\mathcal{F} : X \rightarrow \mathbb{R}$ an l.l. functional with compact gradient, and let $\rho \in \mathbb{R}$. Assume that

$$\lim_{\|u\| \rightarrow \infty} [\mathcal{N}(u) - \lambda \mathcal{F}(u)] = +\infty \quad \text{for } \lambda \in I; \quad (15)$$

$$\begin{aligned} & \sup_{\lambda \in I} \inf_{u \in X} [\mathcal{N}(u) + \lambda(\rho - \mathcal{F}(u))] \\ & < \inf_{u \in X} \sup_{\lambda \in I} [\mathcal{N}(u) + \lambda(\rho - \mathcal{F}(u))]. \end{aligned} \quad (16)$$

Then, there exist $\alpha, \beta \in I$ ($\alpha < \beta$) and $r > 0$ with the following property: for $\lambda \in [\alpha, \beta]$ and any l.l. functional $\mathcal{G} : X \rightarrow \mathbb{R}$ with compact gradient, there exists $\delta > 0$ such that, for every $\mu \in (0, \delta)$, the functional $\varphi_{\lambda, \mu} = \mathcal{N} - \lambda \mathcal{F} - \mu \mathcal{G}$ admits at least three critical points in X , with norms less than r .

The main hypothesis of Lemma 8 is the minimax inequality (16). An easy way to have it satisfied is illustrated by the following result obtained by Ricceri [16].

Lemma 9 (see [16, Proposition 3.1]). Let X be a nonempty set and $\mathcal{N}, \mathcal{F} : X \rightarrow \mathbb{R}$ functionals, and let $\tilde{u}, \hat{u} \in X$ and $\tau > 0$, such that

$$\mathcal{N}(\tilde{u}) = \mathcal{F}(\tilde{u}) = 0; \quad (17)$$

$$\mathcal{N}(\hat{u}) > \tau; \quad (18)$$

$$\sup_{u \in \mathcal{N}^{-1}(-\infty, \tau]} \mathcal{F}(u) < \frac{\tau \mathcal{F}(\hat{u})}{\mathcal{N}(\hat{u})}. \quad (19)$$

Then for each ρ satisfying

$$\sup_{u \in \mathcal{N}^{-1}(-\infty, \tau]} \mathcal{F}(u) < \rho < \frac{\tau \mathcal{F}(\hat{u})}{\mathcal{N}(\hat{u})}, \quad (20)$$

one has that (16) with $I \subset [0, \infty)$ holds.

3. Variational Approach for the Problem (3)

We introduce the Banach space $X = H^2(0, L) \cap H_0^1(0, L)$ endowed with the norm $\|u\| = (\int_0^L (u''^2(x) + u'^2(x) + u^2(x)) dx)^{1/2}$. Obviously, X is a reflexive Banach space and compactly embedded in $C[0, L]$. So there exists constant c , such that $c = \sup_{u \in X, \|u\| \neq 0} \|u\|_\infty / \|u\|$.

From the positivity of p and a , it is easy to see that

$$\|u\|_X = \left(\int_0^L (u''^2 + pu'^2 + a(x)u^2) dx \right)^{1/2} \quad (21)$$

is also a norm of X . Therefore, there exist two constants C_0 and C' such that

$$C_0 \|u\| \leq \|u\|_X \leq C' \|u\|. \quad (22)$$

Definition 10. A function $u \in X$ is said to be a weak solution of the problem (3), it is understood an element $u \in X$ for which there corresponds to a mapping $[0, 1] \ni x \mapsto u^*(x)$ with $u^*(x) \in \lambda \partial F(u(x)) + \mu \partial G_u(x, u(x))$ for a.e. $x \in [0, L]$, and having the property that for every $v \in X$, $u^*v \in L^1[0, 1]$ and

$$\int_0^L (u''v'' + pu'v' + a(x)uv - u^*v) dx = 0. \quad (23)$$

Definition 11. A function u is called a solution of the problem (3) if $u \in AC^3[0, 1]$ and

$$\begin{aligned} u^{(iv)} - pu'' + a(x)u &= u^*, \quad \text{a.e. } x \in [0, L], \\ u(0) = u''(0) = u(L) = u''(L) &= 0, \end{aligned} \quad (24)$$

where $u^*(x) \in \lambda \partial F(u(x)) + \mu \partial G_u(x, u(x))$ for a.e. $x \in [0, L]$.

Lemma 12. If $u \in X$ is a weak solution of (3), then u is a solution of (3).

Proof. Let $u \in X$ be a weak solution of (3). Then there exists u^* satisfying (23) for all $v \in X$ and $u^*(x) \in \lambda \partial F(u(x)) + \mu \partial G_u(x, u(x))$ for a.e. $x \in [0, L]$. Using integration by parts, (23) becomes

$$\int_0^L u''v'' dx = \int_0^L (pu'' - a(x)u + u^*)v(x) dx, \quad (25)$$

so $u \in H^4(0, L)$; especially (25) holds for $v \in X$ with $v'(0) = v'(L) = 0$; via integration by parts, we obtain

$$\begin{aligned} u^{(iv)}(x) - pu''(x) + a(x)u(x) - u^*(x) &= 0, \\ \text{a.e. } x \in [0, L]. \end{aligned} \quad (26)$$

Due to $u \in X$, similar to [7, Section 2], we can get $u \in AC^3[0, L]$.

Now we show that the boundary conditions are satisfied. From (25) and integration by parts, we can get

$$\begin{aligned} u''(L)v'(L) - u''(0)v'(0) \\ + \int_0^L (u^{(iv)} - pu'' + a(x)u - u^*)v dx &= 0. \end{aligned} \quad (27)$$

Since u satisfies (26), we have that $u''(L)v'(L) - u''(0)v'(0) = 0$ holds for all $v \in X$, which implies that $u''(L) = u''(0) = 0$; in view of $u \in X$, we get $u(0) = u(L) = 0$. Thus u is a solution of (3). $\square$

4. Proof of the Main Result

In order to prove the main result, we shall show some related lemmas. Firstly, define functionals

$$\mathcal{N}(u) = \frac{1}{2} \int_0^L (u''^2 + pu'^2 + a(x)u^2) dx, \quad u \in X, \quad (28)$$

$$\mathcal{F}(u) = \int_0^L F(u(x)) dx, \quad \mathcal{G}(u) = \int_0^L G(x, u(x)) dx, \quad u \in X,$$

$$I_{\lambda, \mu}(u) = \mathcal{N}(u) - \lambda \mathcal{F}(u) - \mu \mathcal{G}(u), \quad u \in X. \quad (29)$$

Next, we will give some properties of those functionals.

Lemma 13. Assume (H1) holds; then $\mathcal{N} \in C^1(X, \mathbb{R})$ and $\mathcal{N}' : X \rightarrow X^*$ is of type $(S)_+$.

Proof. Clearly, by (H1) and the definition of $\mathcal{N}$, we know $\mathcal{N} \in C^1(X, \mathbb{R})$ and

$$\langle \mathcal{N}'(u), v \rangle = \int_0^L (u''v'' + pu'v' + a(x)uv) dx, \quad u, v \in X. \quad (30)$$

Let $\{u_n\} \subset X$ such that $u_n \rightarrow u$ and $\lim_{n \rightarrow \infty} \langle \mathcal{N}'(u_n), u_n - u \rangle \leq 0$. According to Definition 7, we shall show that $u_n \rightarrow u$. In fact

$$\begin{aligned} 0 &= \langle \mathcal{N}'(u), u_n - u \rangle \\ &= \int_0^L (u''(u_n - u)'' + pu'(u_n - u)') \\ &\quad + a(x)u(u_n - u)) dx. \end{aligned} \quad (31)$$

On the other hand, owing to

$$\begin{aligned} &\langle \mathcal{N}'(u_n), u_n - u \rangle \\ &= \int_0^L (u_n''(u_n - u)'' + pu_n'(u_n - u)') \\ &\quad + a(x)u_n(u_n - u)) dx. \end{aligned} \quad (32)$$

Equation (32) subtracting (31), we get

$$\begin{aligned} &\overline{\lim}_{n \rightarrow \infty} \int_0^L ((u_n - u)''^2 + p((u_n - u)')^2 \\ &\quad + a(x)(u_n - u)^2) dx \\ &= \overline{\lim}_{n \rightarrow \infty} \langle \mathcal{N}'(u_n), u_n - u \rangle \leq 0, \end{aligned} \quad (33)$$

so

$$\begin{aligned} &\lim_{n \rightarrow \infty} \int_0^L ((u_n - u)''^2 + p((u_n - u)')^2 \\ &\quad + a(x)(u_n - u)^2) dx = 0. \end{aligned} \quad (34)$$

It follows from the equivalence of $\|u\|_X$ and $\|u\|$ that $u_n \rightarrow u$ as $n \rightarrow \infty$. $\square$

Lemma 14. Assume (H1), (F1), (F2), (G1), and (G2) hold. Then the functional $I_{\lambda, \mu} : X \rightarrow \mathbb{R}$ is l.L. Moreover, for each critical point $u \in X$ of $I_{\lambda, \mu}$, u is a weak solution of (3).

Proof. Let $I_{\lambda, \mu}(u) = \mathcal{N}(u) + I_1(u)$, where $I_1(u) = -\lambda \mathcal{F}(u) - \mu \mathcal{G}(u)$. Due to $\mathcal{N} \in C^1(X, \mathbb{R})$, by Lemma 3, $\mathcal{N}$ is l.L. on X . From (F2) and (G2), we know that I_1 is l.L. on $L^s[0, L]$. Moreover X is compactly embedded into $L^s[0, L]$, so I_1 is also l.L. on X ([17, Theorem 2.2] and [18]). Furthermore, according to Lemma 5, we get

$$\partial I_1(u) \subset -\lambda \int_0^L \partial F(u) dx - \mu \int_0^L \partial G_u(x, u) dx. \quad (35)$$

The interpretation of (35) is as follows: for every $u^* \in \partial I_1(u)$, there corresponds a mapping $u^*(x) \in -\lambda \partial F(u) - \mu \partial G_u(x, u)$ for a.e. $x \in [0, L]$ having the property that, for every $v \in X$, the function $u^*v \in L^1[0, L]$ and $\langle u^*, v \rangle = \int_0^L u^*(x)v(x)dx$. Therefore $I_{\lambda, \mu}$ is l.L. on X .

Now we show that each critical point u of $I_{\lambda, \mu}$ is a weak solution of (3). Let $u \in X$ be a critical point of $I_{\lambda, \mu}$. So

$$\begin{aligned} 0 &\in \partial I_{\lambda, \mu}(u) \\ &= \{u^* \in X^* : \langle u^*, v \rangle \leq I_{\lambda, \mu}^0(u; v) \text{ for } v \in X\}. \end{aligned} \quad (36)$$

By Lemma 3 and (36), we get

$$\mathcal{N}'(u) + u^* = 0 \quad \text{with } u^* \in \partial I_1(u), \quad (37)$$

and hence $u^*(x) \in -\lambda \partial F(u(x)) - \mu \partial G_u(x, u(x))$ a.e. on $[0, L]$; it follows from (30) and (37), that for every $v \in X$, we have

$$\int_0^L (u''v'' + pu'v' + a(x)uv + u^*v) dx = 0. \quad (38)$$

Thus by Definition 10, u is a weak solution of (3). $\square$

Lemma 15. Suppose (F1) and (F2) are fulfilled. Then $\partial \mathcal{F} : X \rightarrow 2^{X^*}$ is compact.

Proof. According to Lemma 14, $\mathcal{F}$ is l.L. on X and $\partial \mathcal{F}(u) \subset \int_0^L \partial F(u(x))dx$. Let $\{u_n\} \subset X$ such that $\|u_n\| \leq M$ for every $n \in \mathbb{N}$ ($M > 0$); choose $u_n^*(x) \in \partial \mathcal{F}(u_n(x))$ for a.e. $x \in [0, L]$, $n \in \mathbb{N}$, and $\langle u_n^*, v \rangle = \int_0^L u_n^*v dx$; then by (F2), we get for every $n \in \mathbb{N}$, $v \in X$,

$$\begin{aligned} \langle u_n^*, v \rangle &\leq \int_0^L |u_n^*| |v| dx \\ &\leq c_1 \left(\int_0^L |v| dx + \int_0^L |u_n|^{s-1} |v| dx \right) \\ &\leq c_1 \|v\|_{L^1} + c_1 \|u_n\|_{L^s}^{s-1} \|v\|_{L^s} \\ &\leq c_1 (M_1 + M_2 M^{s-1}) \|v\|, \end{aligned} \quad (39)$$

where M_1 and M_2 are positive constants, so $\|u_n^*\|_{X^*} \leq c_1 (M_1 + M_2 M^{s-1})$ and the sequence $\{u_n^*\}$ is bounded, and hence, up to a subsequence if necessary, $u_n^* \rightharpoonup u^* \in X^*$.

We claim that $u_n^* \rightarrow u^*$.

Suppose the contrary; we assume there exists $\epsilon > 0$ such that $\|u_n^* - u^*\|_{X^*} > \epsilon$ (choose a subsequence if necessary). Hence for every $n \in \mathbb{N}$, we can find $v_n \in X$ with $\|v_n\| < 1$ such that

$$\langle u_n^* - u^*, v_n \rangle > \epsilon. \quad (40)$$

Passing to a subsequence if necessary, we can assume that $v_n \rightharpoonup v$ in X , so $v_n \rightarrow v$ in $L^1[0, L]$ and $L^2[0, L]$. Combining with (F2), we have

$$\begin{aligned} & \langle u_n^* - u^*, v_n \rangle \\ &= \langle u_n^*, v_n - v \rangle + \langle u_n^* - u^*, v \rangle + \langle u^*, v - v_n \rangle \\ &\leq \int_0^L |u_n^*(x)| |v_n(x) - v(x)| dx \\ &\quad + \langle u_n^* - u^*, v \rangle + \langle u^*, v - v_n \rangle \\ &\leq c_1 \left(\|v_n - v\|_{L^1} + \int_0^L |u_n|^{s-1} |v_n - v| dx \right) \\ &\quad + \langle u_n^* - u^*, v \rangle + \langle u^*, v - v_n \rangle \\ &\leq c_1 \left(\|v_n - v\|_{L^1} + \|v_n - v\|_{L^2} \left(\int_0^L |u_n|^{2(s-1)} dx \right)^{1/2} \right) \\ &\quad + \langle u_n^* - u^*, v \rangle + \langle u^*, v - v_n \rangle \rightarrow 0 \quad \text{as } n \rightarrow \infty, \end{aligned} \quad (41)$$

which contradicts (40). $\square$

Proof of Theorem 1. The equivalence of $\|\cdot\|_X$ defined in (21) with the usual norm in X implies that the functional $\mathcal{N}$ defined in (28) is sequentially weak lower semicontinuous. By Lemma 13; $\mathcal{N}'$ is of type $(S)_+$. By Lemmas 14 and 15, $\mathcal{F} : X \rightarrow \mathbb{R}$ is an l.L. functional with compact gradient.

First, we verify the condition (15) in Lemma 8. By (F1), (F2), and Lebourg's mean value theorem, we have

$$|F(u)| = |F(u) - F(0)| \leq c_1 (|u| + |u|^s), \quad u \in \mathbb{R}, \quad (42)$$

hence

$$\begin{aligned} & \mathcal{N}(u) - \lambda \mathcal{F}(u) \\ &= \frac{1}{2} \|u\|_X^2 - \lambda \int_0^L F(u(x)) dx \\ &\geq \frac{C_0}{2} \|u\|^2 - \lambda c_1 \left(\int_0^L |u| dx + \int_0^L |u|^s dx \right) \\ &\geq \frac{C_0}{2} \|u\|^2 - \lambda c_1 (M_1 \|u\| + M_2 \|u\|^s), \end{aligned} \quad (43)$$

where M_1 and M_2 are positive constants, so it follows from $s < 2$ that

$$\lim_{\|u\| \rightarrow \infty} (\mathcal{N}(u) - \lambda \mathcal{F}(u)) = +\infty. \quad (44)$$

Next, we verify the condition (16) by using Lemma 9. Set $\tilde{u}(x) = 0, \hat{u}(x) = x^2 - Lx, x \in [0, L]$. Clearly, $\tilde{u}, \hat{u} \in X$ and

$\mathcal{N}(\tilde{u}) = \mathcal{F}(\tilde{u}) = 0$. Moreover, if we choose $\tau = C_0(2L + (1/6)L^3 + (1/60)L^5) - \epsilon$, where $\epsilon > 0$ is small enough, then

$$\begin{aligned} \mathcal{N}(\hat{u}) &= \frac{1}{2} \|\hat{u}\|_X^2 \geq \frac{C_0}{2} \|\hat{u}\|^2 \\ &= \frac{C_0}{2} \left(4L + \frac{1}{3}L^3 + \frac{1}{30}L^5 \right) > \tau, \end{aligned} \quad (45)$$

so (18) holds. For $u \in X$ with $\mathcal{N}(u) < \tau$, considering the compact embedding of X in $C[0, L]$, we have

$$\begin{aligned} \|u\|_\infty &\leq c \|u\| \leq c C_0^{-1/2} \|u\|_X < c C_0^{-1/2} (2\tau)^{1/2} \\ &= c C_0^{-1/2} \left(C_0 \left(4L + \frac{1}{3}L^3 + \frac{1}{30}L^5 \right) - 2\epsilon \right)^{1/2} := c_2. \end{aligned} \quad (46)$$

So, we obtain

$$\sup_{u \in \mathcal{N}^{-1}(-\infty, \tau]} \int_0^L F(u) dx \leq L \sup_{s \in [-c_2, c_2]} F(s). \quad (47)$$

Choose $R = \tau / (C'(2L + (1/6)L^3 + (1/60)L^5))$. In view of (F3) and (47), we have

$$\begin{aligned} \frac{\tau \mathcal{F}(\hat{u})}{\mathcal{N}(\hat{u})} &\geq \frac{\tau \int_0^L F(x^2 - Lx) dx}{C'(2L + (1/6)L^3 + (1/60)L^5)} \\ &= R \int_0^L F(x^2 - Lx) dx \\ &> L \sup_{s \in [-c_2, c_2]} F(s) \geq \sup_{u \in \mathcal{N}^{-1}(-\infty, \tau]} \mathcal{F}(u), \end{aligned} \quad (48)$$

and therefore, (19) holds. Then we get (16) for some $\rho \in \mathbb{R}$ and $I < (0, \infty)$.

For function G which satisfies (G1) and (G2), it follows from Lemma 14 that the functional $\mathcal{G} : X \rightarrow \mathbb{R}$ is l.L. Similar to the argument of Lemma 15, we can obtain that $\partial \mathcal{G}$ is compact. Then according to Lemma 8, there exist $\alpha, \beta \in I$ ($\alpha < \beta$) and $r > 0$ with the property that, for $\lambda \in [\alpha, \beta]$ and the above G , there is $\delta > 0$ such that, for $\mu \in (0, \delta)$, the functional

$$I_{\lambda, \mu} = \mathcal{N} - \lambda \mathcal{F} - \mu \mathcal{G} \quad (49)$$

admits at least three critical points $u_0, u_1, u_2 \in X$ with $\|u_i\| < r$, ($i = 0, 1, 2$). So by Lemmas 14 and 12, u_0, u_1, u_2 are three solutions of the problem (3). $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This work was supported by the Program for New Century Excellent Talents in University (NECT-12-0246) and FRFCU (lzujbky-2013-k02).

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