Research Article

The Relationship between Two Involutive Semigroups $\mathcal{S}$ and $\mathcal{S}_T$ Is Defined by a Left Multiplier T

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Let $\mathcal{S}$ be a semigroup with a left multiplier T on $\mathcal{S}$. There exists a new semigroup $\mathcal{S}_T$, which depends on $\mathcal{S}$ and T , which has the same underlying space as $\mathcal{S}$. We study the question of involutions on $\mathcal{S}_T$ and a Banach algebra $\mathfrak{A}_T$. We find a condition of T under which $\mathcal{S}_T$ and the second dual $\mathfrak{A}_T^{**}$ admit an involution. We will show that $\mathfrak{A}_T$ is C^* -algebra if and only if $T : \mathfrak{A}_T \rightarrow \mathfrak{A}$ is an isometry, under mild conditions. Also, $\mathfrak{A}$ is C^* -algebra if and only if so is $\mathfrak{A}_T$, under other minor conditions.

1. Introduction and Definitions

1.1. Introduction. Birtel in his paper [1, Theorem 5] has introduced a multiplication on a commutative Banach algebra. In [2, Theorem 1.3.1], Larsen has defined a new Banach algebra $\mathfrak{A}_T$ which is related to a certain semisimple commutative Banach algebra $\mathfrak{A}$ and a multiplier T of $\mathfrak{A}$ which preserves the regular maximal ideal space and $\text{Mul}(\mathfrak{A}) = \text{Mul}(\mathfrak{A}_T)$, where $\text{Mul}(\mathfrak{A})$ is the algebra of all multipliers of $\mathfrak{A}$. The multiplicative linear functionals of $\mathfrak{A}$ and $\mathfrak{A}_T$ corresponding to given ideals in $\Delta(\mathfrak{A}) = \Delta(\mathfrak{A}_T)$ are not the same. They do, however, only differ by a multiplicative constant. Yoo in his paper [3] has shown that $\text{Mul}(\mathfrak{A})$ is a C^* -algebra if $\mathfrak{A}$ is C^* -algebra. The second author studied the implications for properties of $\mathfrak{A}$ and $\mathfrak{A}_T$. It is shown that, under certain conditions, the Banach algebras $\mathfrak{A}$ and $\mathfrak{A}_T$ have the same properties and, in the other case, they have different properties [4].

The purpose of this paper is to continue the subject; that is, we investigate some properties between $\mathfrak{A}_T$ and $\mathfrak{A}$, in the category of $*$ -semigroup. We establish necessary and sufficient conditions for which $\mathfrak{A}_T$ and $\mathfrak{A}$ are C^* -algebra.

1.2. Definition. The term *semigroup* will describe a nonempty set $\mathcal{S}$ endowed with a associative binary operation mapping

on $\mathcal{S} \times \mathcal{S}$ into $\mathcal{S}$. If $\mathcal{S}$ is also a Hausdorff topological space and the binary operation is continuous for the product topology of $\mathcal{S} \times \mathcal{S}$ then $\mathcal{S}$ is said to be *Hausdorff topological semigroup*. Let $\mathcal{S}$ be a semigroup. A map $T : \mathcal{S} \rightarrow \mathcal{S}$ is a left (resp., right) multiplier on $\mathcal{S}$ if

$$T(ab) = T(a)b \quad (T(ab) = aT(b)) \quad (1)$$

for all $a, b \in \mathcal{S}$. The class of left (resp., right) multiplier on $\mathcal{S}$ is denoted by $\text{Mul}_\ell(\mathcal{S})$ (resp., $\text{Mul}_r(\mathcal{S})$). It is easy to check that $\text{Mul}_\ell(\mathcal{S})$ and $\text{Mul}_r(\mathcal{S})$ are unital semigroup under the operation composition.

An operator T is a *multiplier* on $\mathcal{S}$ if $T \in \text{Mul}_\ell(\mathcal{S}) \cap \text{Mul}_r(\mathcal{S})$. The space of all multipliers on $\mathcal{S}$ is denoted by $\text{Mul}(\mathcal{S})$. A pair (T, R) is a *double centralizer* on $\mathcal{S}$ if

$$\begin{aligned} T &\in \text{Mul}_\ell(\mathcal{S}), \quad R \in \text{Mul}_r(\mathcal{S}), \\ aT(b) &= R(a)b \quad (a, b \in \mathcal{S}). \end{aligned} \quad (2)$$

For $s \in \mathcal{S}$, define $L_s : \mathcal{S} \rightarrow \mathcal{S}, R_s : \mathcal{S} \rightarrow \mathcal{S}$ by

$$L_s(t) = st, \quad R_s(t) = ts \quad (t \in \mathcal{S}) \quad (3)$$

Then $L_s \in \text{Mul}_\ell(\mathcal{S}), R_s \in \text{Mul}_r(\mathcal{S}), L_s L_t = L_{st}$, and $R_s R_t = R_{ts}$.

If, in addition, $\mathcal{S}$ is a Banach algebra then a left (resp., right) multiplier must be a linear map. In this case, we say that a multiplier is on Banach algebra (see [5, Definition 1.4.25]).

A semigroup $\mathcal{S}$ is left (resp., right) faithful if $sx = tx$ (resp., $xs = xt$) for all $x \in \mathcal{S}$ then $s = t$. A semigroup $\mathcal{S}$ is faithful if it is left and right faithful. If $\mathcal{S}$ is faithful then $T : \mathcal{S} \rightarrow \mathcal{S}$ is multiplier if and only if $sTt = (Ts)t$ for all $s, t \in \mathcal{S}$ [6].

For a Banach algebra $\mathfrak{A}$, the left (resp., right) annihilator of $\mathfrak{A}$ is denoted by

$$\begin{aligned} \text{Ann}_\ell(\mathfrak{A}) &= \{a \in \mathfrak{A} : a\mathfrak{A} = 0\} \\ (\text{Ann}_r(\mathfrak{A}) &= \{a \in \mathfrak{A} : \mathfrak{A}a = 0\}). \end{aligned} \quad (4)$$

The set $\text{Ann}_\ell(\mathfrak{A}) \cap \text{Ann}_r(\mathfrak{A})$ is called the annihilator of $\mathfrak{A}$. The Banach algebra $\mathfrak{A}$ is left (resp., right) faithful if $\text{Ann}_\ell(\mathfrak{A}) = \{0\}$ (resp., $\text{Ann}_r(\mathfrak{A}) = \{0\}$).

A net $(e_\alpha)_{\alpha \in I}$ in a Hausdorff topological semigroup $\mathcal{S}$, where I is a directed set, is a left (resp., right) approximate identity, or, briefly l.a.i. (resp., r.a.i.) if $\lim_\alpha e_\alpha a = a$ (resp., $\lim_\alpha a e_\alpha = a$).

A net $(e_\alpha)_{\alpha \in I}$ is an approximate identity (briefly, a.i.) if it is a left and right approximate identity.

A net (e_α) in Banach algebra $\mathfrak{A}$ is a bounded approximate identity (briefly, b.a.i.) if, for all $a \in \mathfrak{A}$,

$$\lim_\alpha e_\alpha a = a = \lim_\alpha a e_\alpha, \quad \sup_{\alpha \in I} \|e_\alpha\| < \infty. \quad (5)$$

If a Hausdorff topological semigroup $\mathcal{S}$ has an a.i., then for all $T \in \text{Mul}_\ell(\mathcal{S})$ (resp., $R \in \text{Mul}_r(\mathcal{S})$),

$$\begin{aligned} T(a) &= \lim_\alpha L_{T(e_\alpha)}(a) \\ (\text{resp. } R(a) &= \lim_\alpha R_{T(e_\alpha)}(a)), \end{aligned} \quad (6)$$

$(a \in \mathcal{S}).$

Now, suppose that $\mathcal{S}$ and $\mathfrak{A}$ are topological semigroup and Banach algebra, respectively, and let T be a left multiplier. We define a new product on $\mathcal{S}$ and a new product and a norm on $\mathfrak{A}$ by

$$\begin{aligned} a \circ b &= aT(b) \quad (a, b \in \mathcal{S}, \mathfrak{A}) \\ \|c\|_T &= \|c\| \|T\| \quad (c \in \mathfrak{A}). \end{aligned} \quad (7)$$

The set $\mathcal{S}$ (resp., $\mathfrak{A}$) with the new product “ $\circ$ ” (resp., $(\mathfrak{A}, \circ)$) with the new norm $\|\cdot\|_T$ is denoted by $\mathcal{S}_T$ (resp., $\mathfrak{A}_T$). It is easy to see that $\mathcal{S}_T$ is semigroup and if T is continuous then $\mathcal{S}_T$ is topological semigroup. Also, the algebra $\mathfrak{A}_T$ is a Banach algebra (see [2, Theorem 1.3.1] and [4, Theorem 2.1]).

An involution on a semigroup $\mathcal{S}$ is a map $*$: $\mathcal{S} \rightarrow \mathcal{S}$ such that (i) $(s^*)^* = s$, (ii) $(st)^* = t^*s^*$ ($s, t \in \mathcal{S}$), (iii) if involution map is continuous, we say that $(\mathcal{S}, *)$ is topological $*$ -semigroup, and the involution map is an isomorphism and a homeomorphism of $\mathcal{S}$ onto $\mathcal{S}$.

For an involution in Banach algebra $\mathfrak{A}$, we add the following definition:

$$(iv) (\alpha s + \beta t)^* = \bar{\alpha}s^* + \bar{\beta}t^* \quad (\alpha, \beta \in \mathbb{C}, s, t \in \mathfrak{A}).$$

A Banach algebra with an isometric involution is a Banach $*$ -algebra. A C^* -algebra is a Banach $*$ -algebra $\mathfrak{A}$ such that

$$\|a^*a\| = \|a\|^2 \quad (a \in \mathfrak{A}). \quad (8)$$

Now, suppose that $\mathcal{S}$ is a $*$ -semigroup; then a map $T : \mathcal{S} \rightarrow \mathcal{S}$ is said to be a $*$ -mapping if $T(a^*) = T(a)^*$.

Define $T^\Delta(a) = T(a^*)^*$. Then T is $*$ -mapping if and only if $T^\Delta = T$. We say an element $s \in \mathcal{S}$ is positive if $s = x^*x$ for some $x \in \mathcal{S}$. We denote the set of all positive elements of $\mathcal{S}$ by $\mathcal{S}^+$; that is, $\mathcal{S}^+ = \{x^*x : x \in \mathcal{S}\}$.

Now, let $\mathcal{S}_1$ be also $*$ -semigroup. Then, the map $T : \mathcal{S} \rightarrow \mathcal{S}_1$ is a positive map if $T(\mathcal{S}^+) \subseteq \mathcal{S}_1^+$.

A complex value function φ on $\mathcal{S}$ is said to be positive definite if $\sum_{i,j=1}^n c_i \bar{c}_j \varphi(x_j^* x_i) \geq 0$ for all $c_1, c_2, \dots, c_n \in \mathbb{C}$ and $x_1, x_2, \dots, x_n \in \mathcal{S}$, $n \in \mathbb{N}$. We write $P(\mathcal{S})$ (resp., $P(\mathcal{S}_T)$) for all positive definite functions on $\mathcal{S}$ (resp., $\mathcal{S}_T$).

In this paper, the nonzero character space on a Banach algebra of $\mathfrak{A}$ is denoted by $\Phi_{\mathfrak{A}}$ and it is the set of all multiplicative linear functionals on $\mathfrak{A}$.

Define the set of positive elements of $\mathfrak{A}$ by

$$\mathfrak{A}^+ = \left\{ \sum_{i=1}^n a_i^* a_i : a_i \in \mathfrak{A}, n \in \mathbb{N} \right\}. \quad (9)$$

A linear functional on $\mathfrak{A}$ is positive if $f(\mathfrak{A}^+) \subseteq \mathbb{R}^+$. The set of all positive linear functionals on $\mathfrak{A}$ is denoted by $P_{\mathfrak{A}}$. Let $f \in P_{\mathfrak{A}}$. Then f is a positive trace if

$$f(aa^*) = f(a^*a) \quad (a \in \mathfrak{A}). \quad (10)$$

The $*$ -algebra $\mathfrak{A}$ is hermitian if $\sigma(a) \subseteq \mathbb{R}$ for all $a \in \mathfrak{A}$, $a = a^*$, where $\sigma(a)$ is the spectrum of a .

Let $\mathfrak{A}$ be a Banach algebra and $\mathfrak{A}^*$, $\mathfrak{A}^{**}$ be its first and second dual space of $\mathfrak{A}$. For any element $a \in \mathfrak{A}$, the image of a in $\mathfrak{A}^{**}$, under the canonical mapping, will be denoted by $\hat{a}$. Let $F, G \in \mathfrak{A}^{**}$. Then, we can follow the two Arens products and equip $\mathfrak{A}^{**}$ with the first Arens product $\square$ and second Arens product $\diamond$, which is defined by using iterated limits as follows:

$$\begin{aligned} F \square G &= w^* - \lim_{a^\wedge \rightarrow F} w^* - \lim_{b^\wedge \rightarrow G} (ab)^\wedge, \\ F \diamond G &= w^* - \lim_{b^\wedge \rightarrow G} w^* - \lim_{a^\wedge \rightarrow F} (ab)^\wedge. \end{aligned} \quad (11)$$

The Banach algebra $\mathfrak{A}$ is said to be Arens regular if $F \square G = F \diamond G$, for all $F, G \in \mathfrak{A}^{**}$.

The details of these constructions may be found in many places, including the book [5] and the articles [7–9].

2. Topological Semigroup $\mathcal{S}_T$ with an Involution

We commence our study of $*$ -semigroup $\mathcal{S}_T$ by considering the same underlying $*$ -semigroup $\mathcal{S}$. In the first section, we suppose that $\mathcal{S}$ is $*$ -semigroup. We can find some conditions under which $\mathcal{S}_T$ is a $*$ -semigroup. Then, under those conditions, we begin to consider the position of T and obtain the best situation possible in general.

Lemma 1. Let $\mathcal{S}$ be $*$ -topological semigroup. If T is a $*$ -mapping then T is a multiplier. The converse holds, if T is continuous and there exists an approximate identity (e_α) such that $T(e_\alpha)^* = T(e_\alpha)$ for all α .

Proof. Let T be a left multiplier. Then

$$T^\Delta(ab) = T((ab)^*)^* = (T(b^*)a^*)^* = aT^\Delta(b) \quad (12)$$

for all $a, b \in \mathcal{S}$. This yields that T^Δ is a right multiplier. On the other hand, T is a $*$ -mapping. Then

$$T(a) = T(a^{**}) = T(a^*)^* = T^\Delta(a), \quad (13)$$

for all $a \in \mathcal{S}$. It follows that T is a multiplier.

Conversely, let (e_α) be an approximate identity with $T(e_\alpha)^* = T(e_\alpha)$ for all α . Then

$$\begin{aligned} T(a^*) &= \lim_\alpha T(e_\alpha a^*) = \lim_\alpha T(e_\alpha) a^* \\ &= \lim_\alpha (aT(e_\alpha))^* = \lim_\alpha (T(ae_\alpha))^* \\ &= T(a)^*, \end{aligned} \quad (14)$$

for all $a \in \mathcal{S}$. Then T is a $*$ -mapping. $\square$

Theorem 2. Let $\mathcal{S}$ be a $*$ -semigroup. Then

- (i) $\mathcal{S}_T$ is $*$ -semigroup if and only if (T, T^Δ) is a double centralizer;
- (ii) if T is a $*$ -mapping then $\mathcal{S}_T$ is $*$ -semigroup. The converse holds if T is a multiplier and $\mathcal{S}$ is faithful.

Proof. (i) Suppose that $\mathcal{S}_T$ is a $*$ -mapping and T is a left multiplier. Then T^Δ is a right multiplier and

$$\begin{aligned} aT(b) &= a \circ b = (b^* \circ a^*)^* \\ &= T(a^*)^* b = T^\Delta(a) b \quad (a, b \in \mathcal{S}). \end{aligned} \quad (15)$$

It follows that (T, T^Δ) is double centralizer.

For the converse, suppose that $aT(b) = T^\Delta(a)b$, for all a, b in $\mathcal{S}$. Then

$$\begin{aligned} (a \circ b)^* &= (aT(b))^* = (T^\Delta(a)b)^* \\ &= b^* T(a^*) = b^* \circ a^*. \end{aligned} \quad (16)$$

(ii) Suppose that T is a $*$ -mapping then $T = T^\Delta$ and then T is a multiplier. This follows that (T, T^Δ) is a double centralizer. By (i), $\mathcal{S}_T$ is a $*$ -semigroup.

For the converse, let T be a multiplier and $\mathcal{S}_T$ be a $*$ -semigroup. Then, by (i), $aT(b) = T^\Delta(a)b$, for all a, b in $\mathcal{S}$. Therefore,

$$T(a^*)^* b = T^\Delta(a) b = aT(b) = T(a) b \quad (17)$$

for all a, b in $\mathcal{S}$. This yields that $T(a^*) = T(a)^*$, since $\mathcal{S}$ is left faithful. $\square$

Definition 3. The center of a semigroup $\mathcal{S}$ is the subset $\mathfrak{Z}(\mathcal{S})$ of $\mathcal{S}$ which is defined by

$$\mathfrak{Z}(\mathcal{S}) = \{a \in \mathcal{S} : ab = ba \ \forall b \in \mathcal{S}\}. \quad (18)$$

It is easy to see that $\mathfrak{Z}(\mathcal{S})$ is a commutative subsemigroup of $\mathcal{S}$. Now, we have the following result.

Corollary 4. Let $\mathcal{S}$ be a left (resp., right) faithful $*$ -semigroup. Then L_a (resp., R_a) is a $*$ -mapping if and only if $a \in \mathfrak{Z}(\mathcal{S})$ and $a^* = a$.

Proof. By the symmetry of the situation L_a and R_a , we establish the case of L_a . Suppose that L_a is a $*$ -mapping. Then, by Lemma 1, L_a is a multiplier. Hence, $L_a(x)y = xL_a(y)$.

So, for all x, y in $\mathcal{S}$,

$$axy = xay. \quad (19)$$

On the other hand, $\mathcal{S}$ is left faithful; then $ax = xa$, for all x and then $a \in \mathfrak{Z}(\mathcal{S})$. Now, by using the equality $ax = xa$, for all $x \in \mathcal{S}$, we also see that L_a is a $*$ -mapping if and only if so is L_{a^*} ; that is, if L_{a^*} is a $*$ -mapping, then, for all x in $\mathcal{S}$,

$$\begin{aligned} L_{a^*}(x) &= a^* x = L_{a^*}(x^{**}) = L_{a^*}(x^*)^* = (a^* x^*)^* \\ &= xa = ax = L_a(x). \end{aligned} \quad (20)$$

So, we have $a^* x = ax$, for all x in $\mathcal{S}$. Then $a = a^*$.

For the converse, let $a = a^*$ and $a \in \mathfrak{Z}(\mathcal{S})$. Then

$$L_a(x^*) = L_{a^*}(x^*) = (xa)^* = (ax)^* = L_a(x)^*. \quad (21)$$

Hence, L_a is a $*$ -mapping. $\square$

Corollary 5. Let $\mathcal{S}$ and $\mathcal{S}_T$ be a $*$ -semigroup and let $\mathcal{S}$ be left faithful. Then, T is a $*$ -mapping if and only if T is a multiplier.

Proof. Let $\mathcal{S}$ and $\mathcal{S}_T$ be two $*$ -semigroups. Then

$$aT(b) = a \circ b = (b^* \circ a^*)^* = T(a^*)^* b \quad (a, b \in \mathcal{S}). \quad (22)$$

So, by using this equality, T is a $*$ -mapping if and only if T is a multiplier. $\square$

Theorem 6. Let $\mathcal{S}$ be a $*$ -semigroup, $T \in \text{Mul}_\ell(\mathcal{S})$ with a $*$ -mapping if $T: \mathcal{S}_T \rightarrow \mathcal{S}$ is a map, then, for each $f \in \mathcal{P}(\mathcal{S})$ and $\{s_1, \dots, s_n\} \subseteq \mathcal{S}$, we have

$$0 \leq \sum_{i,j=1}^n (fT(s_i^* \circ s_j) fT(s_j^* \circ s_i)) \leq \left[\sum_{i=1}^n fT(s_i^* \circ s_i) \right]^2. \quad (23)$$

Proof. Suppose that T is a $*$ -mapping. Then, by Theorem 2(ii), $\mathcal{S}_T$ is $*$ -semigroup. If $f \in \mathcal{P}(\mathcal{S})$ and $\{t_1, \dots, t_m\} \subseteq \mathcal{S}$, $\{c_1, \dots, c_m\} \subseteq \mathbb{C}$ then we have

$$\begin{aligned} \sum_{i,j=1}^m c_i \bar{c}_j fT(t_i^* \circ t_j) &= \sum_{i,j=1}^m c_i \bar{c}_j f(T(t_i^*) T(t_j)) \\ &= \sum_{i,j=1}^m c_i \bar{c}_j f(T(t_i)^* T(t_j)) \geq 0. \end{aligned} \quad (24)$$

Hence, $fT \in \mathcal{P}(\mathcal{S}_T)$. Now, suppose that $\{s_1, \dots, s_n\} \subseteq \mathcal{S}$. Then, we have the following assertions:

- (i) $fT(s_i^* \circ s_j) = \overline{fT(s_j^* \circ s_i)}$;
- (ii) $fT(s_i^* \circ s_i) \geq 0$;
- (iii) $fT(s_i^* \circ s_j)fT(s_j^* \circ s_i) = |fT(s_i^* \circ s_j)|^2 \leq fT(s_i^* \circ s_i)fT(s_j^* \circ s_j)$ (see [10, Lemma II.3] or [11, Page 68, 69]).

It follows that

$$\begin{aligned} 0 &\leq \sum_{i,j=1}^n |fT(s_i^* \circ s_j)|^2 = \sum_{i,j=1}^n fT(s_i^* \circ s_j) fT(s_j^* \circ s_i) \\ &\leq \sum_{i,j=1}^n fT(s_i^* \circ s_i) fT(s_j^* \circ s_j) \quad (25) \\ &= \left[\sum_{i=1}^n fT(s_i^* \circ s_i) \right]^2. \end{aligned}$$

□

3. $\mathfrak{A}_T$ as a C^* -Algebra

We know that a Banach C^* -algebra is a Banach $*$ -algebra which has an isometric involution and the norm on algebra is C^* -norm; that is, $\|a^*a\| = \|a\|^2$ for all $a \in \mathfrak{A}$.

Now, we begin by considering the relation between the C^* -algebra $\mathfrak{A}$ and $\mathfrak{A}_T$. In general, there are two subjects; then, we have to consider them. First, we find out some conditions of $T \in \text{Mul}(\mathfrak{A})$ if $\mathfrak{A}_T$ is C^* -algebra; second, under such conditions, $\mathfrak{A}_T$ is C^* -algebra if and only if so is $\mathfrak{A}$. Now, we have established the following theorem.

Throughout this paper assume that T is a continuous left multiplier on $\mathcal{S}$.

Theorem 7. *Let $\mathfrak{A}$ be a Banach algebra. Then consider the following.*

- (i) *The operator $T : \mathfrak{A}_T \rightarrow \mathfrak{A}$ is homomorphism and norm decreasing.*
- (ii) *If $\mathfrak{A}$ is a $*$ -algebra and T is a $*$ -mapping then T is a positive and $P_{\mathfrak{A}}T \subseteq P_{\mathfrak{A}_T}$.*
- (iii) *Let $\mathfrak{A}$ be hermitian. Then every element of $\Phi_{\mathfrak{A}}T$ is a positive trace if T is a $*$ -mapping.*

Proof. (i) Let $a, b \in \mathfrak{A}$. Then

$$T(a \circ b) = T(aT(b)) = T(a)T(b), \quad \|T(a)\| \leq \|a\|_T. \quad (26)$$

So, the result follows.

(ii) Suppose that T is a $*$ -mapping. Then, by Theorem 2, (ii), $\mathfrak{A}_T$ is a $*$ -algebra. Let $a \in \mathfrak{A}_T^+$. Then $a = \sum_{i=1}^n a_i^* \circ a_i$ for some $a_i, a_i^* \in \mathfrak{A}$ ($1 \leq i \leq n$). Hence, $T(a_i^*) = T(a_i)^*$ for each i , and

$$T(a) = \sum_{i=1}^n T(a_i^* T(a_i)) = \sum_{i=1}^n T(a_i)^* T(a_i). \quad (27)$$

It follows that $T(\mathfrak{A}_T^+) \subseteq \mathfrak{A}^+$. If f is a positive functional on $\mathfrak{A}$ then

$$fT(\mathfrak{A}_T^+) \subseteq f(\mathfrak{A}^+) \subseteq \mathbb{R}^+. \quad (28)$$

It follows that $fT \in P_{\mathfrak{A}}$.

(iii) Suppose that $\mathfrak{A}$ is hermitian and $\varphi \in \Phi_{\mathfrak{A}}$. Then $\varphi(a^*) = \overline{\varphi(a)}$ for all $a \in \mathfrak{A}$ and φ is a positive trace [5, Proposition 1.10.22]. So, we have $\varphi \in P_{\mathfrak{A}}$. On the other hand,

$$\varphi T(a^* \circ a) = \varphi(T(a^*)T(a)) = \overline{\varphi(T(a))}\varphi(T(a)) \geq 0. \quad (29)$$

Then $\varphi T \in P_{\mathfrak{A}_T}$ and

$$\begin{aligned} \varphi(T(a^* \circ a)) &= \varphi(T(a))\varphi(T(a)^*) = \varphi(T(a)T(a^*)) \\ &= \varphi T(aT(a^*)) = \varphi T(a \circ a^*). \end{aligned} \quad (30)$$

It follows that $\varphi T(a^* \circ a) = \varphi T(a \circ a^*)$. So φT is a positive trace. □

Theorem 8. *Let $\mathfrak{A}_T$ be a C^* -algebra. Then consider the following.*

- (i) *The left multiplier T is bounded below.*
- (ii) *The map $T : \mathfrak{A}_T \rightarrow \mathfrak{A}$ is an isometry.*
- (iii) *$T(\mathfrak{A})$ is a closed set.*

Proof. (i) Suppose that $\mathfrak{A}_T$ is C^* -algebra. Take $a \in \mathfrak{A}$, we have

$$\begin{aligned} \|a\|_T^2 &= \|a^* \circ a\|_T = \|a^* T(a)\| \|T\| \\ &\leq \|T(a)\| \|a\| \|T\| = \|T(a)\| \|a\|_T. \end{aligned} \quad (31)$$

It follows that $\|a\|_T \leq \|T(a)\|$. So, T is bounded below and an injective.

(ii) By Theorem 7, (i), T is homomorphism and norm decreasing. Then $\|T(a)\| \leq \|a\|_T$. By (i), we then have $\|T(a)\| = \|a\|_T$.

(iii) By (ii), the left multiplier T satisfies

$$\|T(a)\| = \|T\| \|a\|, \quad (32)$$

for all $a \in \mathfrak{A}$. Now, suppose that $y \in \overline{T(\mathfrak{A})}$. Then there exists a sequence (x_n) in $\mathfrak{A}$ such that $(T(x_n))$ converges to $y \in \mathfrak{A}$. On the other hand,

$$\|T(x_n) - T(x_m)\| = \|T\| \|x_n - x_m\| \quad (33)$$

for all n, m . So, (x_n) is a Cauchy sequence in Banach algebra of $\mathfrak{A}$ and converges to some $x \in \mathfrak{A}$. Since T is continuous then $y = T(x) \in T(\mathfrak{A})$ and then $T(\mathfrak{A})$ is closed. □

In the next theorem, we find some conditions on T under which $\mathfrak{A}_T$ is C^* -algebra.

Theorem 9. *Let $\mathfrak{A}$ be a C^* -algebra and $T \in \text{Mul}(\mathfrak{A})$ with double centralizer (T, T^Δ) . Then, $\mathfrak{A}_T$ is a C^* -algebra if and only if $\|T(a)\| = \|a\|_T$, for all $a \in \mathfrak{A}^+$ (or $T : \mathfrak{A}_T^+ \rightarrow \mathfrak{A}$ is an isometry).*

Proof. Let $\mathfrak{A}_T$ be C^* -algebra. Then, by Theorem 8, (ii), $T : \mathfrak{A}_T \rightarrow \mathfrak{A}$ is an isometry and for all $a \in \mathfrak{A}^+$

$$\|T(a)\| = \|a\|_T. \quad (34)$$

For the converse, by Theorem 2, (i), $\mathfrak{A}_T$ is $*$ -algebra. It suffices to show that $\|a^* \circ a\|_T = \|a\|_T^2$, for all $a \in \mathfrak{A}$. Let $a \in \mathfrak{A}$. Then, $a^*a \in \mathfrak{A}^+$ and then

$$\begin{aligned} \|a^* \circ a\|_T &= \|a^*T(a)\| \|T\| = \|T(a^*a)\| \|T\| \\ &= \|a^*a\|_T \|T\| = \|a\|_T^2 \|T\|^2 \\ &= \|a\|_T^2. \end{aligned} \quad (35)$$

□

Theorem 10. Let $\mathfrak{A}$ be a C^* -algebra and T be a $*$ -mapping. Then $\mathfrak{A}_T$ is a C^* -algebra if and only if $T : \mathfrak{A}_T \rightarrow \mathfrak{A}$ is an isometry.

Proof. Let $\mathfrak{A}_T$ be a C^* -algebra. Then, by Theorem 8, (ii), $T : \mathfrak{A}_T \rightarrow \mathfrak{A}$ is an isometry.

For the converse, by Lemma 1, T is a multiplier and, (T, T^Δ) is a double centralizer. So, by Theorem 9, $\mathfrak{A}_T$ is a C^* -algebra. □

We pay attention to the extension of involution of $\mathfrak{A}$ to second dual $\mathfrak{A}^{**}$ that was studied by many authors in [7–9]. Now, we consider it for $\mathfrak{A}_T$.

Theorem 11. Let $\mathfrak{A}$ be a Banach algebra. Consider the following.

- (i) If (T, R) is a double centralizer then so is (T^{**}, R^{**}) , with respect the first (or second) Arens product on $\mathfrak{A}^{**}$.
- (ii) Let $\mathfrak{A}$ be $*$ -algebra and Arens regular, $T \in \text{Mul}_\ell(\mathfrak{A})$; then $\mathfrak{A}_{T^{**}}$ admits an involution if and only if (T, T^Δ) is a double centralizer.
- (iii) If T is $*$ -mapping then $\mathfrak{A}_{T^{**}}$ admits an involution.

Proof. Suppose that $T \in \text{Mul}_\ell(\mathfrak{A})$, $R \in \text{Mul}_r(\mathfrak{A})$ with $aT(b) = R(a)b$. Take $F \in \mathfrak{A}^{**}$, $G \in \mathfrak{A}^{**}$. Then, with the first Arens regular, we have

$$\begin{aligned} T^{**}(F \square G) &= w^* - \lim_{a^\wedge \rightarrow F} w^* - \lim_{b^\wedge \rightarrow G} T(ab)^\wedge \\ &= w^* - \lim_{a^\wedge \rightarrow F} w^* - \lim_{b^\wedge \rightarrow G} T(a)^\wedge \square b^\wedge \\ &= w^* - \lim_{a^\wedge \rightarrow F} T(a)^\wedge \square G \\ &= T^{**}(a) \square G. \end{aligned} \quad (36)$$

Now, with second Arens regular, we have

$$\begin{aligned} T^{**}(F \diamond G) &= w^* - \lim_{b^\wedge \rightarrow G} w^* - \lim_{a^\wedge \rightarrow F} T(ab)^\wedge \\ &= w^* - \lim_{b^\wedge \rightarrow G} w^* - \lim_{a^\wedge \rightarrow F} T(a)^\wedge \diamond b^\wedge \\ &= w^* - \lim_{b^\wedge \rightarrow G} T^{**}(F) \diamond b^\wedge \\ &= T^{**}(F) \diamond G. \end{aligned} \quad (37)$$

Hence $T^{**} \in \text{Mul}_\ell(\mathfrak{A}^{**})$. Similarly, $R^{**} \in \text{Mul}_r(\mathfrak{A}^{**})$ and $F \square T^{**}(G) = R^{**}(F) \square G$, $F \diamond T^{**}(G) = R^{**}(F) \diamond G$. So, (T^{**}, R^{**}) is a double centralizer with first and second Arens product.

(ii) Suppose that $\mathfrak{A}$ is $*$ -algebra. The involution, “ $*$,” has an extension to a continuous anti-isomorphism on $\mathfrak{A}^{**}$, denoted by the symbol “ $\sim$,” such that

$$\begin{aligned} (F \square G)^\sim &= w^* - \lim_{a^\wedge \rightarrow F} w^* - \lim_{b^\wedge \rightarrow G} [(ab)^\wedge]^* \\ &= w^* - \lim_{a^\wedge \rightarrow F} w^* - \lim_{b^\wedge \rightarrow G} [(b^*)^\wedge \diamond (a^*)^\wedge] \\ &= w^* - \lim_{a^\wedge \rightarrow F} w^* - \lim_{b^\wedge \rightarrow G} [(b^\wedge)^* \diamond (a^\wedge)^*] \\ &= G^\sim \diamond F^\sim, \end{aligned} \quad (38)$$

for all F and G in $\mathfrak{A}^{**}$. Similarly, $(F \diamond G)^\sim = G^\sim \square F^\sim$, for all F and G in $\mathfrak{A}^{**}$. So $\mathfrak{A}^{**}$ admits an involution if and only if $\mathfrak{A}$ is Arens regular. From (i) and Theorem 2, the proof is complete.

(iii) Suppose that T is $*$ -mapping. Then (T, T^Δ) is double centralizer so, by (ii), $\mathfrak{A}_{T^{**}}$ admits an involution. □

Theorem 12. Let $\mathcal{S}$ be a Hausdorff topological semigroup with an involution and let $\mathfrak{A}$ be $*$ -algebra. Consider the following.

- (i) Let (T, T^Δ) be a double centralizer. Then $\mathfrak{A}$ is a $*$ -algebra if and only if so is $\mathfrak{A}_T$.
- (ii) Let T be a $*$ -mapping. Then, if $\mathfrak{A}_T$ is a C^* -algebra then so is $\mathfrak{A}$. The converse holds if $T : \mathfrak{A}_T \rightarrow \mathfrak{A}$ is an isometry.
- (iii) If T is $*$ -mapping then $\mathfrak{A}_{T^{**}}$ admits an involution.

Proof. (i) Suppose that $\mathcal{S}$ is $*$ -algebra. Then by Theorem 2, $\mathcal{S}_T$ is also $*$ -algebra.

Now, we consider the converse implication. Let $\mathcal{S}_T$ be $*$ -algebra and $a, b \in \mathcal{S}$. First, suppose that $a = T(a_1)$, $b = T(b_1)$, for some $a_1, b_1 \in \mathcal{S}$. So, we have

$$\begin{aligned} (ab)^* &= (T(a_1)T(b_1))^* = (T(a_1 \circ b_1))^* = T^\Delta((a_1 \circ b_1)^*) \\ &= T^\Delta(b_1^* \circ a_1^*) = T^\Delta(b_1^*T(a_1^*)) = T^\Delta(T^\Delta(b_1^*)a_1^*) \\ &= T^\Delta(b_1^*)T^\Delta(a_1^*) = (T(b_1))^*(T(a_1))^* = b^*a^*. \end{aligned} \quad (39)$$

Now, for a, b in $\mathcal{S} = \overline{T(\mathcal{S})}$, there are two nets (a_α) and (b_β) in $\mathcal{S}$ such that $a = \lim_\alpha T(a_\alpha)$, $b = \lim_\beta T(b_\beta)$. Since $(T(a_\alpha)T(b_\beta))^* = T(b_\beta)^*T(a_\alpha)^*$, for all α, β , then

$$\begin{aligned} (ab)^* &= \lim_\alpha \lim_\beta (T(a_\alpha)T(b_\beta))^* \\ &= \lim_\alpha \lim_\beta T(b_\beta)^*T(a_\alpha)^* \\ &= \left(\lim_\beta T(b_\beta) \right)^* \left(\lim_\alpha T(a_\alpha) \right)^* \\ &= b^*a^*. \end{aligned} \quad (40)$$

(ii) If T is a $*$ -mapping then $T = T^\Delta$. Therefore T is a multiplier and (T, T^Δ) is a double centralizer. If $\mathfrak{A}_T$ is a C^* -algebra then $\mathfrak{A}_T$ is a $*$ -algebra and T is an isometry and $T(\mathfrak{A})$ is closed set (by (i) and Theorem 8). So, by hypothesis, $T(\mathfrak{A}) = \mathfrak{A}$. Now, we consider C^* -norm on $\mathfrak{A}$. For $a \in \mathfrak{A}$, there is $a_1 \in \mathfrak{A}$, such that $a = T(a_1)$. So, we have

$$\begin{aligned} \|aa^*\| &= \|T(a_1)T(a_1)^*\| = \|T(a_1)T(a_1^*)\| \\ &= \|T(a_1 \circ a_1^*)\| = \|a_1 \circ a_1^*\|_T \\ &= \|a_1\|_T^2 = \|T(a_1)\|^2 \\ &= \|a\|^2. \end{aligned} \quad (41)$$

For the converse, if $\mathfrak{A}$ is C^* -algebra then by Theorem 2, (ii), $\mathfrak{A}_T$ is a $*$ -algebra and for $a \in \mathfrak{A}$

$$\begin{aligned} \|a \circ a^*\|_T &= \|aT(a^*)\|_T = \|T(aa^*)\| \|T\| \\ &= \|aa^*\|_T \|T\| = \|a\|_T^2. \end{aligned} \quad (42)$$

Then C^* -norm holds on $\mathfrak{A}_T$. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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