Research Article

Riesz Basicity for General Systems of Functions

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In this paper we find the general conditions for a complete biorthogonal conjugate system to form a Riesz basis. We show that if a complete biorthogonal conjugate system is uniformly bounded and its coefficient space is solid, then the system forms a Riesz basis. We also construct affine Riesz bases as an application to the main result.

1. Main Result

The aim of this paper is to find the general conditions for a complete biorthogonal conjugate system to form a Riesz basis, following the results obtained by Bari [1], Christensen [2], Sarsenbi with coauthors [3–5], San Antolin and Zalik [6], and Guo [7].

Let $\{u_n(x)\}_{n=1}^{\infty}$ and $\{v_n(x)\}_{n=1}^{\infty}$ be a complete biorthogonal conjugate system of functions from $L^2(0, 1)$ space.

By system coefficient space $\{u_n(x)\}_{n=1}^{\infty}$ we denote the space $X(u)$ of all the numeric sequences $a = \{a_n\}_{n=1}^{\infty}$ such that the series $\sum_{n=1}^{\infty} a_n u_n(x)$ converges in $L^2(0, 1)$. It is evident that coefficient space $X(u)$ is complete under the norm $\|a\|_{X(u)} = \sup_{n \in \mathbb{N}} \|\sum_{k=1}^n a_k u_k(x)\|$, and a natural basis $\varepsilon_i = \{\delta_{ij}\}_{j=1}^{\infty}$, $i \in \mathbb{N}$, where δ_{ij} is a Kronecker delta, forms a $X(u)$ space basis.

A Banach coordinate space X of numeric sequences $a = \{a_n\}_{n=1}^{\infty}$ is said to be solid if $b \in X$ follows from $a \in X$ and $|b_n| \leq |a_n|$, $n \in \mathbb{N}$ (the inequality $\|b\|_X \leq \|a\|_X$, as it is put by the precise definition, is not required here).

It is clear that $X(u)$ space is solid if natural basis is an unconditional basis for $X(u)$. The latter follows from unconditional basicity for a system $\{u_n(x)\}_{n=1}^{\infty}$.

Theorem 1. Let $\{u_n(x)\}_{n=1}^{\infty}$ and $\{v_n(x)\}_{n=1}^{\infty}$ be a complete biorthogonal conjugate system of functions that is uniformly bounded:

$$\int_0^1 |u_n(x)|^2 dx \leq C, \quad \int_0^1 |v_n(x)|^2 dx \leq C, \quad n \in \mathbb{N}. \quad (1)$$

Let there be given coefficient spaces $X(u)$ and $X(v)$ which are both solid. Then $\{u_n(x)\}_{n=1}^{\infty}$ and $\{v_n(x)\}_{n=1}^{\infty}$ system form a Riesz basis.

Proof. We consider the series $\sum_{n=1}^{\infty} a_n u_n(x)$ for a numeric sequence $\{a_n\}_{n=1}^{\infty} \in \ell^2$ and show that series converges for almost all choices of signs, that is, series

$$\sum_{n=1}^{\infty} a_n r_n(t) u_n(x), \quad (2)$$

where $\{r_n(t)\}_{n=1}^{\infty}$ is the Rademacher system and converges for almost all $t \in [0, 1]$ in L^2 metrics by variable x (e.g., [8, Chapter 2]).

We use the results from [8] claiming that convergence of series $\sum_{n=1}^{\infty} f_n(x)$ for almost all choices of signs is equivalent to

$$\left(\sum_{n=1}^{\infty} |f_n(x)|^2 \right)^{1/2} \in L^2(0, 1). \quad (3)$$

For the series considered $\sum_{n=1}^{\infty} a_n u_n(x)$, by Levi's theorem we have

$$\begin{aligned} \int_0^1 \sum_{n=1}^{\infty} |a_n u_n(x)|^2 dx &= \sum_{n=1}^{\infty} |a_n|^2 \int_0^1 |u_n(x)|^2 dx \\ &\leq C \sum_{n=1}^{\infty} |a_n|^2 < \infty, \end{aligned} \quad (4)$$

meaning

$$\left(\sum_{n=1}^{\infty} |a_n u_n(x)|^2 \right)^{1/2} \in L^2(0, 1). \quad (5)$$

Convergence of the series $\sum_{n=1}^{\infty} a_n u_n(x)$ for almost all choices of signs is shown.

Now take a fixed $t_0 \in [0, 1]$ such that the series $\sum_{n=1}^{\infty} a_n r_n(t_0) u_n(x)$ converges in $L^2(0, 1)$ space. By the solidity condition for coefficient space $X(u)$ in $L^2(0, 1)$, the series $\sum_{n=1}^{\infty} a_n u_n(x)$ converges, too.

Thus for any numeric sequence $a = \{a_n\}_{n=1}^{\infty} \in \ell^2$ the series $\sum_{n=1}^{\infty} a_n u_n(x)$ converges in $L^2(0, 1)$. Then the following equivalent inequalities are satisfied:

$$\begin{aligned} \left\| \sum_{n=1}^{\infty} a_n u_n \right\|^2 &\leq B \sum_{n=1}^{\infty} |a_n|^2, \quad \{a_n\}_{n=1}^{\infty} \in \ell^2, \\ \sum_{n=1}^{\infty} |f, u_n|^2 &\leq B \|f\|^2, \quad f \in L^2(0, 1). \end{aligned} \quad (6)$$

This means that $\{u_n(x)\}_{n=1}^{\infty}$ is Bessel system.

Besselian property for a system $\{v_n(x)\}_{n=1}^{\infty}$ is proved in the same way. It is clear that Besselian property for both biorthogonal conjugate systems $\{u_n(x)\}_{n=1}^{\infty}$ and $\{v_n(x)\}_{n=1}^{\infty}$ implies the Riesz basicity for these systems. $\square$

Remark 2. Note that in Theorem 1 we can replace the coefficient space $X(u)$ with $X(u) \cap \ell^2$ and $X(v)$ with $X(v) \cap \ell^2$.

2. Affine Riesz bases

Let function $u : \mathbb{R} \rightarrow \mathbb{R}$ have a support $\text{supp } u \subset [0, 1]$. Using the representation $n = 2^k + j$, $k = 0, 1, \dots$, $j = 0, \dots, 2^k - 1$ for $n \in \mathbb{N}$, we assume

$$u_n(x) = u_{k,j}(x) = 2^{k/2} u(2^k x - j). \quad (7)$$

Besides, we suppose $u_0(x) = 1$, $x \in [0, 1]$. System of functions $\{u_n(x)\}_{n=0}^{\infty}$ is called an affine system generated by a function u . Here and elsewhere we assume

$$u \in L^2(0, 1), \quad \int_0^1 u(x) dx = 0. \quad (8)$$

Note that the classic example of an affine system of functions is the Haar wavelet $\{h_n(x)\}_{n=0}^{\infty}$ generated by the function

$$h(x) = \begin{cases} 1, & x \in \left[0, \frac{1}{2}\right), \\ -1, & x \in \left[\frac{1}{2}, 1\right), \\ 0, & x \notin [0, 1). \end{cases} \quad (9)$$

We enumerate the functions of Rademacher system $\{r_k\}_{k=0}^{\infty}$

$$r_k = 2^{-k/2} \sum_{j=0}^{2^k-1} h_{k,j}, \quad k = 0, 1, \dots \quad (10)$$

We suppose that an affine system $\{u_n(x)\}_{n=0}^{\infty}$ generator u can be represented by Rademacher system

$$u = \sum_{k=0}^{\infty} a_k r_k, \quad \sum_{k=0}^{\infty} |a_k|^2 < \infty. \quad (11)$$

In this case we have the following completeness criterion for a system $\{u_n(x)\}_{n=0}^{\infty}$. Let the function

$$U(z) = \sum_{k=0}^{\infty} a_k z^k, \quad |z| < 1, \quad (12)$$

be analytic in the unit disk with coefficients a_k from (11).

Theorem 3 (see [9]). *A necessary and sufficient condition for an affine system $\{u_n(x)\}_{n=0}^{\infty}$ to be complete in $L^2(0, 1)$ space is that analytic function $U(z)$ is outer function.*

The following results are true for function u in the form (11).

Theorem 4. *System $\{v_n(x)\}_{n=0}^{\infty}$ that is biorthogonal conjugate to the affine system $\{u_n(x)\}_{n=0}^{\infty}$ exists and is complete in $L^2(0, 1)$ space if $a_0 \neq 0$.*

Proof. Suppose

$$V(z) = \frac{1}{U(z)} = \sum_{k=0}^{\infty} b_k z^k, \quad (13)$$

that is,

$$a_0 b_0 = 1, \quad \sum_{\nu=0}^k a_{\nu} b_{k-\nu} = 0, \quad k \geq 1. \quad (14)$$

Then it follows from the results of [10] that

$$v_n = v(\alpha_1, \dots, \alpha_k) = \sum_{\nu=0}^k 2^{-(k-\nu)/2} b_{k-\nu} h(\alpha_1, \dots, \alpha_{\nu}), \quad (15)$$

where $n \in \mathbb{N}$ and $n = 2^k + \sum_{\nu=1}^k \alpha_{\nu} 2^{k-\nu}$ is binary expansion, $h(\alpha_1, \dots, \alpha_{\nu}) = h_m$ is the Haar function for $m = 2^{\nu} + \sum_{\mu=1}^{\nu} \alpha_{\mu} 2^{\nu-\mu}$, and $v_0(x) = 1$, $x \in [0, 1]$. The explicit representation (15) shows that v_n is a Haar polynomial of degree n . Hence it follows that the system $\{v_n(x)\}_{n=0}^{\infty}$ is complete. $\square$

Now we can formulate the Riesz basicity test for affine system $\{u_n(x)\}_{n=0}^{\infty}$ with form (11) generator, based on Theorem 1.

Theorem 5. *Let analytic function $U(z)$ have an absolutely convergent Taylor-series expansion*

$$\sum_{k=0}^{\infty} |a_k| < \infty \quad (16)$$

and $U(z)$ does not vanish in the closed unit disk ($|z| \leq 1$). Then an affine system of functions $\{u_n(x)\}_{n=0}^{\infty}$ forms a Riesz basis.

Proof. By the conditions of the theorem, $U(z)$ is outer function. By Theorem 3, an affine system $\{u_n(x)\}_{n=0}^{\infty}$ is complete in $L^2(0, 1)$ space. By Theorem 4, biorthogonal conjugate system $\{v_n(x)\}_{n=0}^{\infty}$ is complete, too.

Obviously, $\|u_n\| \leq \max\{1, \|u\|\}$. From representation (15) we get

$$\|v_n\|^2 \leq \sum_{k=0}^{\infty} 2^{-k} |b_k|^2 < \infty, \quad n \in \mathbb{N}. \quad (17)$$

We need to take into account that by Wiener theorem on absolutely convergent Taylor series we have

$$\sum_{k=0}^{\infty} |b_k| < \infty. \quad (18)$$

Finally, from results of [11] it follows that $X(u) \cap \ell^2 = \ell^2$ and $X(v) \cap \ell^2 = \ell^2$, so all the conditions from Theorem 1 including the Remark are satisfied. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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