Research Article

Another Class of Distances and Continuous Quasi-Distances in Product Spaces

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We construct a class of continuous quasi-distances in a product of metric spaces and show that, generally, when the parameter λ (as shown in the paper) is positive, d is a distance and when $\lambda < 0$, d is only a continuous quasi-distance, but not a distance. It is remarkable that the same result in relation to the sign of λ was found for two other classes of continuous quasi-distances (see Peppo (2010a, 2010b) and Peppo (2011)). This conclusion is due to the fact that E is a product space. For the purposes of our main result, a notion of density in metric spaces is introduced.

1. Introduction

In this paper a quasi-distance d on a set E is defined as a function $d : E^2 \mapsto [0; +\infty[$ with the usual properties of a metric and a weaker version of the triangle inequality:

$$d(x, y) \leq k [d(x, z) + d(y, z)], \quad (1)$$

where $k \geq 1$.

This function is not always continuous with respect to the d -topology generated by itself in the same manner as by a distance. This is because the “open” balls $B(a, r) = \{x \in E : d(x, a) < r\}$, which form a base for a complete system of neighbourhoods of $a \in E$, are not always open sets in the d -topology. Examples of “open” balls that are not open sets can be found in [1].

It is known that for every quasi-distance d , there exists another quasi-distance d' , whose topology and uniformity are the same as those of d and the open d' -balls are open sets [2].

But if any condition or relation is satisfied with respect to a quasi-distance d , we do not know if, in general, the same condition or relation will be satisfied with respect to the “new” quasi-distance d' , even if d is equivalent to d' .

For this reason, the continuity of a quasi-distance cannot be omitted without “danger.”

In fixed-point theory, some authors require the continuity of the used quasi-distance as complementary condition (see, among others, [3]).

The most interesting fact about the quasi-metric spaces is that, in many applications, they constitute a more general setting than the metric spaces without losing the good properties of these last spaces (see, among others, [4]).

In [5–7] we proved that, generally, the functions defined in a product space for two points, $P(x_1, x_2, \dots, x_n)$ and $Q(y_1, y_2, \dots, y_n)$, respectively, by

$$d(P, Q) = \begin{cases} \sum_{i=1}^n a_i d_i(x_i, y_i) & \text{if } \sum_{i=1}^n \beta_i d_i(x_i, y_i) \geq 0, \\ \sum_{i=1}^n a'_i d_i(x_i, y_i) & \text{if } \sum_{i=1}^n \beta_i d_i(x_i, y_i) < 0, \end{cases}$$

$$d(P, Q) = \begin{cases} \frac{\sum_{i=1}^n a_i d_i(x_i, y_i)}{1 + \sum_{i=1}^n a_i d_i(x_i, y_i)} & \text{if } \sum_{i=1}^n \beta_i d_i(x_i, y_i) \geq 0, \\ \frac{\sum_{i=1}^n a'_i d_i(x_i, y_i)}{1 + \sum_{i=1}^n a'_i d_i(x_i, y_i)} & \text{if } \sum_{i=1}^n \beta_i d_i(x_i, y_i) < 0, \end{cases} \quad (2)$$

with $a_i - a'_i = \lambda \beta_i$ for all $i \in I$, are distances when $\lambda > 0$ but only continuous quasi-distances when $\lambda < 0$.

In this paper we revisit this problem for another family of functions in a product space defined by

$$d(P, Q) = \begin{cases} \sqrt{\sum_{i=1}^n a_i^2 d_i^2(x_i, y_i)} & \text{if } \sum_{i=1}^n \beta_i d_i^2(x_i, y_i) \geq 0, \\ \sqrt{\sum_{i=1}^n a_i'^2 d_i^2(x_i, y_i)} & \text{if } \sum_{i=1}^n \beta_i d_i^2(x_i, y_i) < 0, \end{cases} \quad (3)$$

with $a_i^2 - a_i'^2 = \lambda \beta_i$ and we find the same result.

This makes us think that the conclusions reached are not coincidental.

We have reason to believe that a similar result holds for other classes of quasi-distances in product spaces.

Additionally, we notice that the iterated quasi-triangle constants of our quasi-distances,

$$d(P_1, P_n) \leq k [d(P_1, P_2) + d(P_2, P_3) + \cdots + d(P_{n-1}, P_n)], \quad (4)$$

which we are calling *generalized improved quasi-triangle inequality*, improve the general one:

$$\begin{aligned} d(P_1, P_n) &\leq k d(P_1, P_2) + k^2 d(P_2, P_3) \\ &+ \cdots + k^{n-3} d(P_{n-3}, P_{n-2}) \\ &+ k^{n-2} d(P_{n-2}, P_{n-1}) + k^{n-2} d(P_{n-1}, P_n) \end{aligned} \quad (5)$$

(the two last coefficients are k^{n-2} ; it is not a mistake).

In Section 2 we introduce the notion of a density condition in metric spaces for the purposes of our main result.

In Section 4 we give a counterexample proving the importance of the $(\alpha, \sqrt{-\beta_i/\beta_j})$ -density condition in part III of our main result.

2. Density in Metric Spaces

In the third part of our main result (Section 3) we will need metric spaces satisfying an additional property of density that we are calling (α, k) -density condition.

Definition 1. For an $\alpha \in]0; 1[$ and a positive number k , the pair of metric spaces (E, d) and (E', d') satisfies the (α, k) -density condition if the space E contains at least three distinct points, x, y, z , satisfying the relations

$$d(x, z) = \alpha d(x, y), \quad d(y, z) = (1 - \alpha) d(x, y) \quad (6)$$

and for every $\beta \in]\alpha; 1[$ there exist at least three distinct points $x', y', z' \in E'$, satisfying the relations

$$\begin{aligned} d'(x', z') &= \beta d'(x', y'), \\ d'(y', z') &= (1 - \beta) d'(x', y'), \\ d'(x', y') &= k d(x, y). \end{aligned} \quad (7)$$

In terms of segments, Definition 1 may be expressed as follows.

For an $\alpha \in]0; 1[$ and a positive number k , the pair of metric spaces (E, d) and (E', d') satisfies the (α, k) -density condition if E contains at least a segment, $[x, y]$, containing a point z dividing it at the ratio $\alpha : 1 - \alpha$ and, for every $\beta \in]\alpha; 1[$, E' contains at least a segment $[x', y']$, k time “longer” in (E', d') than $[x, y]$ in (E, d) and containing a point z' dividing it at the ratio $\beta : 1 - \beta$.

By segment we mean the set defined by the following:

Definition 2. For two points x, y in a metric space (E, d) , the segment $[x; y]$ is the set of points $z \in E$ satisfying

$$d(x, y) = d(x, z) + d(z, y). \quad (8)$$

It is obvious that the extremities x and y belong to the segment $[x; y]$, but it may only be reduced at its two extremities.

We note also that, despite appearances, if $u \in [x; y]$, $v \in [x; y]$, and $d(x, u) = d(x, v)$, this does not mean, in general, that $u = v$, as in the example below (of course, in the particular case when $d(x, u) = d(x, v) = 0$, we have $u = v = x$).

Example 3. Let a, b be two strictly positive numbers satisfying $\max(a, b) \leq \min(2a, 2b)$, $c = a + b$, c_1 a number satisfying $\max(a, b) \leq c_1 \leq \min(2a, 2b)$, and E a set containing whatever four points $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$, and $D(x_4, y_4)$ of $\mathbb{R}^2$. Define the function d by

$$\begin{aligned} d(A, B) &= d(B, A) = c, & d(A, C) &= d(C, A) = b, \\ d(B, C) &= d(C, B) = a, \\ d(A, D) &= d(D, A) = b, & d(B, D) &= d(D, B) = a, \\ d(C, D) &= d(D, C) = c_1, \\ d(A, A) &= d(B, B) = d(C, C) = d(D, D) = 0. \end{aligned} \quad (9)$$

The two first properties of a distance are obviously satisfied.

So, to prove that d is a distance, it suffices to prove the triangle inequality in all possible “triangles” formed with the points of E , that is, ABC , ABD , ACD , and BCD (Figure 1).

In the “triangles” ABC and ABD , the “sides” are a, b, c and the greatest one is $c = a + b$, so the “triangle” inequality is satisfied.

In the “triangles” ACD and BCD , the “sides” are, respectively, b, b, c_1 and a, a, c_1 and the greatest one is, by our choice, c_1 ; or $c_1 \leq \min(2a, 2b) \leq 2a$, and $c_1 \leq \min(2a, 2b) \leq 2b$, so the “triangle” inequality is satisfied.

The points C and D belong to $[A, B]$, because $d(C, A) + d(C, B) = b + a = c = d(A, B)$ and $d(D, A) + d(D, B) = b + a = c = d(A, B)$. On the other hand $d(C, A) = d(D, A) = b$, but, generally, $C \neq D$.

We note that the (α, k) -density condition, for an $\alpha \in]0; 1[$ and a positive number k , as it is required in the third part of

FIGURE 1

FIGURE 2

our main result (Section 3), is not a very restrictive condition. For example, if d is the usual distance in R^2 , $\alpha = 1/3$, and $k = 2$, in order that the pair $((E, d); (E', d'))$ satisfies the $(1/3, 2)$ -density condition, it suffices that (E, d) contains the set $\{(1; 1), (2; 1), (4; 1)\}$, while (E', d') contains the set $\{(1; 2)\} \cup \{(x; 2) : 3 \leq x \leq 7\}$ as in Figure 2.

Clearly, every pair of normed vector spaces over R satisfies the (α, k) -density condition for each $\alpha \in]0; 1[$ and for each positive number k .

3. Main Result

Theorem 4. Let (E_i, d_i) for $i \in \{1, 2, \dots, n\}$ be metric spaces, λ and β_i real numbers, a_i and a_i' positive numbers satisfying $a_i^2 - a_i'^2 = \lambda \beta_i$, and $P(x_1, x_2, \dots, x_n)$ and $Q(y_1, y_2, \dots, y_n)$ two arbitrary points of $E = \prod_{i=1}^n E_i$. Then

(i) the function $d : E^2 \mapsto R^+$ defined by

$$d(P, Q) = \begin{cases} \sqrt{\sum_{i=1}^n a_i^2 d_i^2(x_i, y_i)} & \text{if } \sum_{i=1}^n \beta_i d_i^2(x_i, y_i) \geq 0 \\ \sqrt{\sum_{i=1}^n a_i'^2 d_i^2(x_i, y_i)} & \text{if } \sum_{i=1}^n \beta_i d_i^2(x_i, y_i) < 0 \end{cases} \quad (10)$$

is a continuous quasi-distance on E satisfying the generalized improved triangle inequality for every m points;

(ii) if ≥ 0 or β_i 's are all nonnegative or all nonpositive, d is a distance;

(iii) if $\lambda < 0$ and there exist two indexes $(i, j) \in \{1, 2, \dots, n\}$ with $\beta_i < 0$, $\beta_j > 0$ and an $\alpha \in]0; 1[$ such that the spaces E_i and E_j satisfy the $(\alpha, \sqrt{-\beta_i/\beta_j})$ -density condition, the continuous quasi-distance d is not a distance.

Proof. (i) First, we will prove that d is a quasi-distance and then that d is a continuous function of its two variables with respect to the d -topology.

It is clear that $d(P, Q) \geq 0$, that d is symmetric, and that $d(P, Q) = 0 \Leftrightarrow P = Q$.

Let $K_1 = \min \{a_i, a_i'\}_{1 \leq i \leq n}$, $K_2 = \max \{a_i, a_i'\}_{1 \leq i \leq n}$, and $D(P, Q) = \sqrt{\sum_{i=1}^n d_i^2(x_i, y_i)}$.

Clearly, for whatever points P and Q , we have

$$K_1 D(P, Q) \leq d(P, Q) \leq K_2 D(P, Q). \quad (11)$$

As D is a distance, we can write, for whatever points $P_1(x_1^{(1)}, x_2^{(1)}, \dots, x_n^{(1)})$, $P_2(x_1^{(2)}, x_2^{(2)}, \dots, x_n^{(2)})$, ..., $P_i(x_1^{(i)}, x_2^{(i)}, \dots, x_n^{(i)})$,

$$\begin{aligned} d(P_1, P_i) &\leq K_2 D(P_1, P_i) \\ &\leq K_2 [D(P_1, P_2) + D(P_2, P_3) + \dots + D(P_{i-1}, P_i)] \\ &\leq \frac{K_2}{K_1} [d(P_1, P_2) + d(P_2, P_3) + \dots + d(P_{i-1}, P_i)]. \end{aligned} \quad (12)$$

This means that d is a quasi-distance with constant $K = K_2/K_1$, and the same constant holds for whatever m points. Let us show now that d is continuous with respect to the d -topology.

Inequalities (11) show that d and D are topologically equivalent. Hence to prove that d is continuous, we will show that for every two sequences, $P_k(x_1^{(k)}, x_2^{(k)}, \dots, x_n^{(k)})$ and $Q_k(y_1^{(k)}, y_2^{(k)}, \dots, y_n^{(k)})$, converging, respectively, to $P(x_1, x_2, \dots, x_n)$ and $Q(y_1, y_2, \dots, y_n)$ for $k \rightarrow +\infty$, $d(P_k, Q_k)$ converges to $d(P, Q)$.

For the points $P(x_1, x_2, \dots, x_n)$ and $Q(y_1, y_2, \dots, y_n)$, in the following, we will denote $g(P, Q) = \sqrt{\sum_{i=1}^n a_i^2 d_i^2(x_i, y_i)}$,

$h(P, Q) = \sqrt{\sum_{i=1}^n a_i'^2 d_i^2(x_i, y_i)}$, and $A = \sum_{i=1}^n \beta_i d_i^2(x_i, y_i)$.

If $A > 0$ ($A < 0$), an integer p exists such that for $k \in N$, $k \geq p$ we have $\sum_{i=1}^n \beta_i d_i^2(x_i^{(k)}, y_i^{(k)}) > 0$ ($\sum_{i=1}^n \beta_i d_i^2(x_i^{(k)}, y_i^{(k)}) < 0$) (because the sequence $\sum_{i=1}^n \beta_i d_i^2(x_i^{(k)}, y_i^{(k)})$ converges to A).

For $k \in N$, $k \geq p$, we have $d(P_k, Q_k) = g(P_k, Q_k)$ ($d(P_k, Q_k) = h(P_k, Q_k)$) and, consequently, $d(P_k, Q_k)$ converges to $g(P, Q) = d(P, Q)$ and ($h(P, Q) = d(P, Q)$).

If $A = 0$, from $a_i^2 - a_i'^2 = \lambda\beta_i$ we deduce

$$\begin{aligned} \sqrt{\sum_{i=1}^n a_i^2 d_i^2(x_i, y_i)} &= \sqrt{\sum_{i=1}^n (a_i'^2 + \lambda\beta_i) d_i^2(x_i, y_i)} \\ &= \sqrt{\sum_{i=1}^n a_i'^2 d_i^2(x_i, y_i) + \lambda A} \\ &= \sqrt{\sum_{i=1}^n a_i'^2 d_i^2(x_i, y_i)}. \end{aligned} \quad (13)$$

That is, $g(P, Q) = h(P, Q)$.

It follows that $g(P_k, Q_k)$ and $h(P_k, Q_k)$ converge to the same limit $g(P, Q) = h(P, Q) = d(P, Q)$. We can conclude that $d(P_k, Q_k)$, formed by the terms of the two sequences above, also converges to $d(P, Q)$. Thus, d is continuous and the “open” balls $B(P_0, r) = \{P \in E : d(P, P_0) < r\}$ really are, in fact, open sets for the d -topology.

(ii) If β_i s are all nonnegative, $d(P, Q) = g(P, Q)$ for every P and Q ; if β_i s are all nonpositive, $d(P, Q) = h(P, Q)$ for every P and Q (recall that if $\sum_{i=1}^n \beta_i d_i^2(x_i, y_i) = 0$, $d(P, Q) = g(P, Q) = h(P, Q)$). As g and h are distances, d is a distance too.

If $\lambda = 0$, $a_i = a_i'$ for every $i \in \{1, 2, \dots, n\}$, so $d(P, Q) = g(P, Q) = h(P, Q)$ for every P and Q ; that is, d is a distance.

Suppose now that $\lambda > 0$; for two arbitrary points $P(x_1, x_2, \dots, x_n)$ and $Q(y_1, y_2, \dots, y_n)$, denoting again $A = \sum_{i=1}^n \beta_i d_i^2(x_i, y_i)$ we have

$$\begin{aligned} g(P, Q) &= \sqrt{\sum_{i=1}^n a_i^2 d_i^2(x_i, y_i)} = \sqrt{\sum_{i=1}^n (a_i'^2 + \lambda\beta_i) d_i^2(x_i, y_i)} \\ &= \sqrt{\sum_{i=1}^n a_i'^2 d_i^2(x_i, y_i) + \lambda A}. \end{aligned} \quad (14)$$

So, for $\lambda > 0$ it follows that

$$\begin{aligned} A \geq 0 &\implies g(P, Q) \geq h(P, Q), \\ A < 0 &\implies g(P, Q) < h(P, Q). \end{aligned} \quad (15)$$

□

This means that if $\lambda > 0$, $d = \max(g, h)$, and since the maximum of two distances is a distance, d is a distance.

Notice that in this part of the theorem we did not need any supplementary density condition on the spaces (E_i, d_i) .

(iii) Suppose now that $\lambda < 0$ and that there exist an $\alpha \in]0; 1[$ and two indexes $i, j \in \{1, 2, \dots, n\}$ such that $\beta_i < 0$, $\beta_j > 0$, and the pair (E_i, E_j) satisfies the $(\alpha, \sqrt{-\beta_i/\beta_j})$ -density condition.

We choose a $\beta \in]\alpha; 1[$ satisfying the relation

$$\begin{aligned} &-\beta_i \beta_j a_j^2 a_i^2 (\beta - \alpha) \\ &< \lambda^2 \beta_i^2 \beta_j^2 (\beta - \alpha) \left(1 - \frac{\alpha + \beta}{2}\right)^2 \\ &+ 2(-\beta_i a_j^2 \beta + \beta_j a_i^2 \alpha) \lambda \beta_i \beta_j \left(1 - \frac{\alpha + \beta}{2}\right). \end{aligned} \quad (16)$$

Such a $\beta > \alpha$ exists because, passing into limit in (16) as $\beta \rightarrow \alpha^+$, we obtain

$$0 < 2\lambda \beta_i \beta_j (1 - \alpha) \alpha (\beta_j a_i^2 - \beta_i a_j^2) \quad (17)$$

that is clearly true.

As (E_i, E_j) satisfies the $(\alpha, \sqrt{-\beta_i/\beta_j})$ -density condition, there exist three points $x_i, y_i, z_i \in E_i$, satisfying the relations $d_i(x_i, z_i) = \alpha d_i(x_i, y_i)$ and $d_i(y_i, z_i) = (1 - \alpha) d_i(x_i, y_i)$, and three points $x_j, y_j, z_j \in E_j$, satisfying the relations

$$\begin{aligned} d_j(x_j, z_j) &= \beta d_j(x_j, y_j), \\ d_j(y_j, z_j) &= (1 - \beta) d_j(x_j, y_j), \\ d_j(x_j, y_j) &= \sqrt{-\frac{\beta_i}{\beta_j}} d_i(x_i, y_i). \end{aligned} \quad (18)$$

With the points $x_i, y_i, z_i \in E_i$ and $x_j, y_j, z_j \in E_j$ we construct the points $P(\dots, x_i, \dots, x_j, \dots)$, $Q(\dots, y_i, \dots, y_j, \dots)$, and $R(\dots, z_i, \dots, z_j, \dots)$ of E (except the i th and the j th coordinates, all the other coordinates are equal in P, Q , and R) and will prove that

$$d(P, Q) > d(P, R) + d(Q, R). \quad (19)$$

For the points P and Q we have from (18) that

$$\sum_{k=1}^n \beta_k d_k^2(x_k, y_k) = \beta_i d_i^2(x_i, y_i) + \beta_j d_j^2(x_j, y_j) = 0, \quad (20)$$

$$\text{so } d(P, Q) = \sqrt{a_i^2 d_i^2(x_i, y_i) + a_j^2 d_j^2(x_j, y_j)} = \sqrt{(a_i^2 - (\beta_i/\beta_j) a_j^2) d_i^2(x_i, y_i)}.$$

For the points P and R ,

$$\begin{aligned} \sum_{k=1}^n \beta_k d_k^2(x_k, z_k) &= \beta_i d_i^2(x_i, z_i) + \beta_j d_j^2(x_j, z_j) \\ &= \beta_i \alpha^2 d_i^2(x_i, y_i) + \beta_j \beta^2 \left(-\frac{\beta_i}{\beta_j}\right) d_i^2(x_i, y_i) \\ &= \beta_i d_i^2(x_i, y_i) (\alpha^2 - \beta^2) > 0, \end{aligned} \quad (21)$$

because $\beta_i < 0$ and $0 < \alpha < \beta$, so $d(P, R) = \sqrt{a_i^2 d_i^2(x_i, z_i) + a_j^2 d_j^2(x_j, z_j)} = \sqrt{(a_i^2 \alpha^2 - a_j^2 \beta^2 (\beta_i/\beta_j)) d_i^2(x_i, y_i)}$.

For the points Q and R,

$$\begin{aligned}
 \sum_{k=1}^n \beta_k d_k^2(x_k, y_k) &= \beta_i d_i^2(y_i, z_i) + \beta_j d_j^2(y_j, z_j) \\
 &= \beta_i(1-\alpha)^2 d_i^2(x_i, y_i) \\
 &\quad - \beta_j(1-\beta)^2 \frac{\beta_i}{\beta_j} d_i^2(x_i, y_i) \\
 &= \beta_i d_i^2(x_i, y_i) [(1-\alpha)^2 - (1-\beta)^2] < 0,
 \end{aligned} \tag{22}$$

because $\beta_i < 0$ and $(1-\alpha)^2 > (1-\beta)^2$,
 so $d(Q, R) = \sqrt{a_i'^2 d_i^2(y_i, z_i) + a_j'^2 d_j^2(y_j, z_j)} = \sqrt{a_i'^2(1-\alpha)^2 - a_j'^2(1-\beta)^2} (\beta_i/\beta_j) d_i(x_i, y_i)$.

We are ready to prove now the inequality $d(P, Q) > d(P, R) + d(Q, R)$, transforming it equivalently to

$$\begin{aligned}
 \sqrt{a_i^2 - \frac{\beta_i}{\beta_j} a_j^2} d_i(x_i, y_i) &> \sqrt{a_i^2 \alpha^2 - a_j^2 \beta^2 \frac{\beta_i}{\beta_j}} d_i(x_i, y_i) \\
 &\quad + \sqrt{a_i'^2(1-\alpha)^2 - a_j'^2(1-\beta)^2} \frac{\beta_i}{\beta_j} \\
 &\quad \times d_i(x_i, y_i) \\
 \iff \sqrt{a_i^2 - \frac{\beta_i}{\beta_j} a_j^2} &> \sqrt{a_i^2 \alpha^2 - a_j^2 \beta^2 \frac{\beta_i}{\beta_j}} \\
 &\quad + \sqrt{a_i'^2(1-\alpha)^2 - a_j'^2(1-\beta)^2} \frac{\beta_i}{\beta_j} \\
 \iff \sqrt{\beta_j a_i^2 - \beta_i a_j^2} &> \sqrt{\beta_j a_i^2 \alpha^2 - \beta_i a_j^2 \beta^2} \\
 &\quad + \sqrt{\beta_j a_i'^2(1-\alpha)^2 - \beta_i a_j'^2(1-\beta)^2}.
 \end{aligned} \tag{23}$$

From $\beta_j a_i^2 > \beta_j a_i^2 \alpha^2$ and $-\beta_i a_j^2 > -\beta_i a_j^2 \beta^2$, we have $\sqrt{\beta_j a_i^2 - \beta_i a_j^2} > \sqrt{\beta_j a_i^2 \alpha^2 - \beta_i a_j^2 \beta^2}$, so $\sqrt{\beta_j a_i^2 - \beta_i a_j^2} - \sqrt{\beta_j a_i^2 \alpha^2 - \beta_i a_j^2 \beta^2}$ is positive and we can equivalently square

$$\begin{aligned}
 \iff \left(\sqrt{\beta_j a_i^2 - \beta_i a_j^2} - \sqrt{\beta_j a_i^2 \alpha^2 - \beta_i a_j^2 \beta^2} \right)^2 \\
 > \left(\sqrt{\beta_j a_i'^2(1-\alpha)^2 - \beta_i a_j'^2(1-\beta)^2} \right)^2
 \end{aligned} \tag{24}$$

$$\begin{aligned}
 \iff \beta_j a_i^2 - \beta_i a_j^2 + \beta_j a_i^2 \alpha^2 - \beta_i a_j^2 \beta^2 \\
 - 2\sqrt{\beta_j a_i^2 - \beta_i a_j^2} \sqrt{\beta_j a_i^2 \alpha^2 - \beta_i a_j^2 \beta^2} \\
 > \beta_j a_i'^2(1-\alpha)^2 - \beta_i a_j'^2(1-\beta)^2
 \end{aligned} \tag{25}$$

(because $a_i'^2 = a_i^2 - \lambda \beta_i$ and $a_j'^2 = a_j^2 - \lambda \beta_j$),

$$\begin{aligned}
 \iff -2\sqrt{\beta_j a_i^2 - \beta_i a_j^2} \sqrt{\beta_j a_i^2 \alpha^2 - \beta_i a_j^2 \beta^2} \\
 > \beta_j a_i^2(1-\alpha)^2 + \lambda \beta_i \beta_j (1-\beta)^2 - \beta_j a_i^2 \\
 &\quad - \beta_j a_i^2 \alpha^2 - \lambda \beta_i \beta_j (1-\alpha)^2 - \beta_i a_j^2(1-\beta)^2 \\
 &\quad + \beta_i a_j^2 + \beta_i a_j^2 \beta^2 \\
 \iff -2\sqrt{\beta_j a_i^2 - \beta_i a_j^2} \sqrt{\beta_j a_i^2 \alpha^2 - \beta_i a_j^2 \beta^2} \\
 > \beta_j a_i^2 [(1-\alpha)^2 - 1 - \alpha^2] \\
 &\quad + \lambda \beta_i \beta_j [(1-\beta)^2 - (1-\alpha)^2] \\
 &\quad - \beta_i a_j^2 [(1-\beta)^2 - 1 - \beta^2] \\
 \iff -2\sqrt{\beta_j a_i^2 - \beta_i a_j^2} \sqrt{\beta_j a_i^2 \alpha^2 - \beta_i a_j^2 \beta^2} \\
 > \beta_i a_j^2 2\beta - \beta_j a_i^2 2\alpha + \lambda \beta_i \beta_j (\alpha - \beta) (2 - \alpha - \beta) \\
 \iff \sqrt{\beta_j a_i^2 - \beta_i a_j^2} \sqrt{\beta_j a_i^2 \alpha^2 - \beta_i a_j^2 \beta^2} \\
 < -\beta_i a_j^2 \beta + \beta_j a_i^2 \alpha + \lambda \beta_i \beta_j (\beta - \alpha) \left(1 - \frac{\alpha + \beta}{2} \right)
 \end{aligned} \tag{26}$$

(we square)

$$\begin{aligned}
 \iff (\beta_j a_i^2 - \beta_i a_j^2) (\beta_j a_i^2 \alpha^2 - \beta_i a_j^2 \beta^2) \\
 < (-\beta_i a_j^2 \beta + \beta_j a_i^2 \alpha)^2 \\
 &\quad + \left[\lambda \beta_i \beta_j (\beta - \alpha) \left(1 - \frac{\alpha + \beta}{2} \right) \right]^2 \\
 &\quad + 2(-\beta_i a_j^2 \beta + \beta_j a_i^2 \alpha) \lambda \beta_i \beta_j (\beta - \alpha) \left(1 - \frac{\alpha + \beta}{2} \right) \\
 \iff \beta_j^2 a_i^4 \alpha^2 - \beta_i \beta_j a_j^2 a_i^2 \alpha^2 - \beta_i \beta_j a_j^2 a_i^2 \beta^2 + \beta_i^2 a_j^4 \beta^2 \\
 < \beta_i^2 a_j^4 \beta^2 + \beta_j^2 a_i^4 \alpha^2 - 2\beta_i \beta_j a_i^2 a_j^2 \alpha \beta \\
 &\quad + \lambda^2 \beta_i^2 \beta_j^2 (\beta - \alpha)^2 \left(1 - \frac{\alpha + \beta}{2} \right)^2 \\
 &\quad + 2(-\beta_i a_j^2 \beta + \beta_j a_i^2 \alpha) \lambda \beta_i \beta_j (\beta - \alpha) \left(1 - \frac{\alpha + \beta}{2} \right) \\
 \iff -\beta_i \beta_j a_j^2 a_i^2 \alpha^2 - \beta_i \beta_j a_j^2 a_i^2 \beta^2 \\
 < -2\beta_i \beta_j a_i^2 a_j^2 \alpha \beta + \lambda^2 \beta_i^2 \beta_j^2 (\beta - \alpha)^2 \left(1 - \frac{\alpha + \beta}{2} \right)^2 \\
 &\quad + 2(-\beta_i a_j^2 \beta + \beta_j a_i^2 \alpha) \lambda \beta_i \beta_j (\beta - \alpha) \left(1 - \frac{\alpha + \beta}{2} \right)
 \end{aligned}$$

$$\begin{aligned}
&\iff \beta_i \beta_j a_j^2 a_i^2 (2\alpha\beta - \alpha^2 - \beta^2) \\
&< \lambda^2 \beta_i^2 \beta_j^2 (\beta - \alpha)^2 \left(1 - \frac{\alpha + \beta}{2}\right)^2 \\
&\quad + 2(-\beta_i a_j^2 \beta + \beta_j a_i^2 \alpha) \lambda \beta_i \beta_j \\
&\quad \times (\beta - \alpha) \left(1 - \frac{\alpha + \beta}{2}\right) \\
&\iff -\beta_i \beta_j a_j^2 a_i^2 (\beta - \alpha)^2 \\
&< \lambda^2 \beta_i^2 \beta_j^2 (\beta - \alpha)^2 \left(1 - \frac{\alpha + \beta}{2}\right)^2 \\
&\quad + 2(-\beta_i a_j^2 \beta + \beta_j a_i^2 \alpha) \lambda \beta_i \beta_j \\
&\quad \times (\beta - \alpha) \left(1 - \frac{\alpha + \beta}{2}\right). \tag{27}
\end{aligned}$$

Dividing by $(\beta - \alpha)$ we have

$$\begin{aligned}
&\iff -\beta_i \beta_j a_j^2 a_i^2 (\beta - \alpha) \\
&< \lambda^2 \beta_i^2 \beta_j^2 (\beta - \alpha) \left(1 - \frac{\alpha + \beta}{2}\right)^2 \\
&\quad + 2(-\beta_i a_j^2 \beta + \beta_j a_i^2 \alpha) \lambda \beta_i \beta_j \left(1 - \frac{\alpha + \beta}{2}\right). \tag{28}
\end{aligned}$$

The last inequality is true from (16). The proof is over. So, d is not a distance.

Example 5. For $P(x_1, x_2)$ and $Q(y_1, y_2) \in R^2$, the function defined by

$$\begin{aligned}
&d(P, Q) \\
&= \begin{cases} \sqrt{2(x_1 - y_1)^2 + (x_2 - y_2)^2} & \text{if } 2(x_2 - y_2)^2 \geq (x_1 - y_1)^2, \\ \sqrt{(x_1 - y_1)^2 + 3(x_2 - y_2)^2} & \text{if } 2(x_2 - y_2)^2 < (x_1 - y_1)^2, \end{cases} \tag{29}
\end{aligned}$$

is a continuous quasi-distance, but not a distance. Here, $\lambda = -1$, $\beta_1 = -1$, $\beta_2 = 2$, $a_1^2 = 2$, $a_1'^2 = 1$, $a_2^2 = 1$, and $a_2'^2 = 3$, while the function defined by

$$\begin{aligned}
&d(P, Q) \\
&= \begin{cases} \sqrt{(x_1 - y_1)^2 + 3(x_2 - y_2)^2} & \text{if } 2(x_2 - y_2)^2 \geq (x_1 - y_1)^2, \\ \sqrt{2(x_1 - y_1)^2 + (x_2 - y_2)^2} & \text{if } 2(x_2 - y_2)^2 < (x_1 - y_1)^2, \end{cases} \tag{30}
\end{aligned}$$

is a distance.

Here, $\lambda = 1$, $\beta_1 = -1$, $\beta_2 = 2$, $a_1^2 = 1$, $a_1'^2 = 2$, $a_2^2 = 3$, and $a_2'^2 = 1$.

4. A Counterexample

Showing the importance of the $(\alpha, \sqrt{-\beta_i/\beta_j})$ -density condition in part III of the theorem.

Let $E_1 = \{x_1, y_1\}$, $E_2 = \{x_2, y_2\}$ be two metric spaces with distances, respectively, d_1, d_2 , $E = E_1 \times E_2 = \{P(x_1, x_2), Q(x_1, y_2), R(y_1, x_2), T(y_1, y_2)\}$, $x_i \neq y_i$, β_i , real numbers, and a_i, a_i' positive numbers satisfying, for $i = 1, 2$, $a_i^2 - a_i'^2 = \lambda\beta_i$, $\lambda < 0$, $\beta_1 < 0$, $\beta_2 > 0$, and $(a_1^2 - a_1'^2)(a_2^2 - a_2'^2) + 4a_1'^2 a_2^2 \geq 0 \iff \lambda^2 \beta_1 \beta_2 + 4a_1'^2 a_2^2 \geq 0$.

For whatever $\alpha \in]0; 1[$, it is obvious that the pair $((E_1, d_1), (E_2, d_2))$ does not satisfy the $(\alpha, \sqrt{-\beta_i/\beta_j})$ -density condition.

We will show that the function $d : E \times E \mapsto R^+$ defined for every two points $M(z_1, z_2), N(t_1, t_2) \in E$ by

$$d(M, N) = \begin{cases} \sqrt{a_1^2 d_1^2(z_1, z_2) + a_2^2 d_2^2(t_1, t_2)} & \text{if } \beta_1 d_1^2(z_1, z_2) + \beta_2 d_2^2(t_1, t_2) \geq 0 \\ \sqrt{a_1'^2 d_1^2(z_1, z_2) + a_2'^2 d_2^2(t_1, t_2)} & \text{if } \beta_1 d_1^2(z_1, z_2) + \beta_2 d_2^2(t_1, t_2) < 0 \end{cases} \tag{31}$$

is a distance.

For every three points of E , we will show that the greatest "distance" (we put quotes because we do not yet know if d is really a distance) between these points is inferior or equal to the sum of the two others.

4.1. First Case ($\beta_1 d_1^2(x_1, y_1) + \beta_2 d_2^2(x_2, y_2) \geq 0$). The six "distances" are

$$\begin{aligned}
&d(P, Q) = d(R, T) = a_2 d_2(x_2, y_2), \\
&d(P, R) = d(Q, T) = a_1' d_1(x_1, y_1), \\
&d(Q, R) = d(P, T) = \sqrt{a_1^2 d_1^2(x_1, y_1) + a_2^2 d_2^2(x_2, y_2)}. \tag{32}
\end{aligned}$$

As $a_1^2 - a_1'^2 = \lambda\beta_1 > 0$, we have $a_1 > a_1'$, so

$$\begin{aligned}
&\sqrt{a_1^2 d_1^2(x_1, y_1) + a_2^2 d_2^2(x_2, y_2)} \geq a_1 d_1(x_1, y_1) \geq a_1' d_1(x_1, y_1), \\
&\sqrt{a_1^2 d_1^2(x_1, y_1) + a_2^2 d_2^2(x_2, y_2)} \geq a_2 d_2(x_2, y_2). \tag{33}
\end{aligned}$$

The greatest of the six "distances" is $d(Q, R) = d(P, T) = \sqrt{a_1^2 d_1^2(x_1, y_1) + a_2^2 d_2^2(x_2, y_2)}$.

In each of the four possible "triangles," PQR , PQT , PRT , and QRT , there is a side equal to $\sqrt{a_1^2 d_1^2(x_1, y_1) + a_2^2 d_2^2(x_2, y_2)}$, another is equal to $a_1' d_1(x_1, y_1)$, and the third is $a_2 d_2(x_2, y_2)$.

Therefore we have to prove only one “triangle inequality,” valid for the four triangles:

$$\sqrt{a_1'^2 d_1^2(x_1, y_1) + a_2'^2 d_2^2(x_2, y_2)} \leq a_1' d_1(x_1, y_1) + a_2' d_2(x_2, y_2), \quad (34)$$

or equivalently

$$\begin{aligned} & a_1^2 d_1^2(x_1, y_1) + a_2^2 d_2^2(x_2, y_2) \\ & \leq a_1'^2 d_1^2(x_1, y_1) + a_2'^2 d_2^2(x_2, y_2) \\ & \quad + 2a_1' a_2' d_1(x_1, y_1) d_2(x_2, y_2) \\ & \iff (a_1^2 - a_1'^2) d_1(x_1, y_1) \\ & \leq 2a_1' a_2' d_2(x_2, y_2) \iff \frac{d_1^2(x_1, y_1)}{d_2^2(x_2, y_2)} \\ & \leq \frac{4a_1'^2 a_2^2}{\lambda^2 \beta_1^2}. \end{aligned} \quad (35)$$

But from $\beta_1 d_1^2(x_1, y_1) + \beta_2 d_2^2(x_2, y_2) \geq 0$ we have $d_1^2(x_1, y_1)/d_2^2(x_2, y_2) \leq \beta_2/(-\beta_1)$ and from $\lambda^2 \beta_1 \beta_2 + 4a_1'^2 a_2^2 \geq 0$ we have $\beta_2/(-\beta_1) \leq 4a_1'^2 a_2^2/\lambda^2 \beta_1^2$, so $d_1^2(x_1, y_1)/d_2^2(x_2, y_2) \leq 4a_1'^2 a_2^2/\lambda^2 \beta_1^2$ and the triangle inequality is satisfied.

4.2. *Second Case* ($\beta_1 d_1^2(x_1, y_1) + \beta_2 d_2^2(x_2, y_2) < 0$). The six “distances” are

$$\begin{aligned} d(P, Q) &= d(R, T) = a_2 d_2(x_2, y_2), \\ d(P, R) &= d(Q, T) = a_1' d_1(x_1, y_1), \\ d(Q, R) &= d(P, T) = \sqrt{a_1'^2 d_1^2(x_1, y_1) + a_2'^2 d_2^2(x_2, y_2)}. \end{aligned} \quad (36)$$

As $a_2^2 - a_2'^2 = \lambda \beta_2 < 0$, we have $a_2 < a_2'$, so $\sqrt{a_1'^2 d_1^2(x_1, y_1) + a_2'^2 d_2^2(x_2, y_2)} \geq a_2' d_2(x_2, y_2) > a_2 d_2(x_2, y_2)$, and $\sqrt{a_1'^2 d_1^2(x_1, y_1) + a_2'^2 d_2^2(x_2, y_2)} \geq a_1' d_1(x_1, y_1)$.

The greatest of the six “distances” is $d(Q, R) = d(P, T) = \sqrt{a_1'^2 d_1^2(x_1, y_1) + a_2'^2 d_2^2(x_2, y_2)}$. In each of the four possible “triangles,” PQR , PQT , PRT , and QRT , there is a side equal to $\sqrt{a_1'^2 d_1^2(x_1, y_1) + a_2'^2 d_2^2(x_2, y_2)}$, another is equal to $a_1' d_1(x_1, y_1)$, and the third is $a_2 d_2(x_2, y_2)$. Therefore we have to prove only one “triangle inequality,” valid for the four triangles:

$$\begin{aligned} & \sqrt{a_1'^2 d_1^2(x_1, y_1) + a_2'^2 d_2^2(x_2, y_2)} \\ & \leq a_1' d_1(x_1, y_1) + a_2 d_2(x_2, y_2), \end{aligned} \quad (37)$$

that we square equivalently

$$\begin{aligned} & a_1'^2 d_1^2(x_1, y_1) + a_2'^2 d_2^2(x_2, y_2) \\ & \leq a_1'^2 d_1^2(x_1, y_1) + a_2^2 d_2^2(x_2, y_2) \\ & \quad + 2a_1' a_2 d_1(x_1, y_1) d_2(x_2, y_2) \\ & \iff (a_2'^2 - a_2^2) d_2(x_2, y_2) \\ & \leq 2a_1' a_2 d_1(x_1, y_1) \iff \frac{d_1^2(x_1, y_1)}{d_2^2(x_2, y_2)} \geq \frac{\lambda^2 \beta_2^2}{4a_1'^2 a_2^2}. \end{aligned} \quad (38)$$

But from $\beta_1 d_1^2(x_1, y_1) + \beta_2 d_2^2(x_2, y_2) < 0$ we have $d_1^2(x_1, y_1)/d_2^2(x_2, y_2) > \beta_2/(-\beta_1)$ and from $\lambda^2 \beta_1 \beta_2 + 4a_1'^2 a_2^2 \geq 0$ we have $\beta_2/(-\beta_1) \geq 4a_1'^2 a_2^2/\lambda^2 \beta_1^2$, so $d_1^2(x_1, y_1)/d_2^2(x_2, y_2) \geq \lambda^2 \beta_2^2/4a_1'^2 a_2^2$ and the triangle inequality is satisfied. So, d is a distance.

We note that if (E_i, d_i) are normed vector spaces, $(E_i, \|\cdot\|_i)$, very slight modifications transform our main result in its normed version and that, in this case, no density condition is needed.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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