Research Article

A Remark on the Homogeneity of Isosceles Orthogonality

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Inspired by the definition of homogeneous direction of isosceles orthogonality, we introduce the notion of almost homogeneous direction of isosceles orthogonality and show that, surprisingly, these two notions coincide. Several known characterizations of inner products are improved.

1. Introduction

Throughout this paper, $X = (X, \|\cdot\|)$ always denotes a real normed linear space with origin o , unit ball B_X , and unit sphere S_X . The dimension of such a space is always assumed to be at least 2.

Isosceles orthogonality, an extension of the orthogonality in inner product spaces to normed linear spaces, was introduced by James in [1]. Two vectors (or points) x and y in X are said to be *isosceles orthogonal* (denoted by $x \perp_I y$) if and only if

$$\|x + y\| = \|x - y\|. \quad (1)$$

In the same paper James proved that the implication

$$x, y \in X, \quad \alpha \in \mathbb{R}, \quad x \perp_I y \implies x \perp_I \alpha y \quad (2)$$

characterizes inner product spaces. In other words, isosceles orthogonality is *homogeneous* if and only if the underlying space is an inner product space. Although implication (2) does not hold in a normed linear space which is not an inner product space, it might hold “locally.” For example, let X be the normed linear space on $\mathbb{R}^2$ with the maximum norm and $x = (1, 1)$. Then

$$y \in X, \quad \alpha \in \mathbb{R}, \quad x \perp_I y \implies x \perp_I \alpha y. \quad (3)$$

In fact, in this example one can easily verify that $x \perp_I y$ if and only if y is a multiple of $(-1, 1)$. The existence of vectors satisfying implication (3) motivated the authors of [2] to introduce the following definition.

Definition 1. A unit vector x in X such that implication (3) holds is called a homogeneous direction of isosceles orthogonality.

We denote by H_X the set of all homogeneous directions of isosceles orthogonality. Note that if X is an inner product space then $H_X = S_X$ and that H_X might be an empty set for some normed linear space X (cf. Example 2.1 in [1]).

In the book [3], Amir mentioned the following implication similar to (2) which also characterizes inner product spaces (cf. [3, (4.12)] and [4, I₁]):

$$\text{there exists a number } \alpha \in (0, 1) \text{ such that } x, y \in X, \quad (4) \\ x \perp_I y \implies x \perp_I \alpha y.$$

A related characterization can be found in [3, (10.13)] (or [5]). By modifying (4) a little bit, we can introduce the notion of almost homogeneous direction of isosceles orthogonality. Note that, in the following definition, we do not need the number α to be chosen uniformly.

Definition 2. Let x be a unit vector in a normed linear space X . If for each $y \in X$, which is isosceles orthogonal to x , there exists a number $\alpha := \alpha(x, y) \in (0, 1)$ such that $x \perp_I \alpha y$, then x is called an *almost homogeneous direction of isosceles orthogonality*.

We denote by H'_X the set of all almost homogeneous directions of isosceles orthogonality. It is clear that

$$x \in H_X \implies x \in H'_X. \quad (5)$$

We will show that, surprisingly, the converse of (5) is also true.

2. Main Result

We will use also the notion of Birkhoff orthogonality. A vector x is said to be *Birkhoff orthogonal* (see [6, 7]) to another vector y if the inequality $\|x + \alpha y\| \geq \|x\|$ holds for each real number α . For relations between isosceles orthogonality and Birkhoff orthogonality and related results, we refer to the survey [8].

Lemma 3 (cf. Section 3 in [7] or Theorems 4.8 and 4.9 in [8]). *Let $x \neq 0$ and y be two vectors in a normed linear space. Then $x \perp_B y$ if and only if $\gamma \in [\alpha(x, y), \beta(x, y)]$, where*

$$\begin{aligned}\alpha(x, y) &:= -\frac{1}{\|x\|} \lim_{\eta \rightarrow 0^+} \frac{\|x + \eta y\| - \|x\|}{\eta}, \\ \beta(x, y) &:= -\frac{1}{\|x\|} \lim_{\eta \rightarrow 0^-} \frac{\|x + \eta y\| - \|x\|}{\eta}.\end{aligned}\quad (6)$$

Therefore, a point $z \in y + \langle -x, x \rangle$ satisfies $x \perp_B z$ if and only if

$$z \in [\alpha(x, y)x + y, \beta(x, y)x + y]. \quad (7)$$

Clearly, the midpoint of the segment $[\alpha(x, y)x + y, \beta(x, y)x + y]$ is the unique point $z \in y + \langle -x, x \rangle$ satisfying $\alpha(x, z) + \beta(x, z) = 0$.

One can easily verify the following lemma.

Lemma 4. *Let x and y be two linearly independent vectors in X and λ a positive number. Then*

$$\alpha(x, \lambda y) = \lambda \alpha(x, y), \quad \beta(x, \lambda y) = \lambda \beta(x, y). \quad (8)$$

Lemma 5. *If $u \in H'_X$, then, for each point z that is isosceles orthogonal to u , $\alpha(u, z) + \beta(u, z) = 0$.*

Proof. We only consider the nontrivial case when $z \neq 0$. Set

$$y = \frac{1}{2} (\alpha(u, z) + \beta(u, z)) u + z. \quad (9)$$

Then one can verify that

$$\alpha(u, y) + \beta(u, y) = 0. \quad (10)$$

In the following we show that $y = z$, which will complete the proof.

By the construction of y there exists a real number α_0 such that $z = y + \alpha_0 u$. Since $u \in H'_X$, there exists a sequence $\{\gamma_n\}_{n=1}^\infty$ of positive numbers, which is strictly decreasing and converges to 0, such that $u \perp_I \gamma_n z$ for each integer $n \geq 1$. Thus we have the equality

$$\|u + \gamma_n(\alpha_0 u + y)\| = \|u - \gamma_n(\alpha_0 u + y)\| \quad n = 1, 2, \dots \quad (11)$$

or, equivalently,

$$\left\| \frac{1}{\gamma_n} u + (\alpha_0 u + y) \right\| = \left\| \frac{1}{\gamma_n} u - (\alpha_0 u + y) \right\| \quad n = 1, 2, \dots \quad (12)$$

When γ_n is sufficiently close to 0, one can verify the following:

$$\begin{aligned}& \left\| \frac{1}{\gamma_n} u + (\alpha_0 u + y) \right\| - \left\| \frac{1}{\gamma_n} u + y \right\| \\&= \left\| \frac{1}{\gamma_n} u + \frac{1/\gamma_n}{1/\gamma_n + \alpha_0} y + \alpha_0 u + \frac{\alpha_0}{1/\gamma_n + \alpha_0} y \right\| - \left\| \frac{1}{\gamma_n} u + y \right\| \\&= \left\| \frac{1}{\gamma_n} u + \frac{1/\gamma_n}{1/\gamma_n + \alpha_0} y \right\| + \alpha_0 \left\| u + \frac{1}{1/\gamma_n + \alpha_0} y \right\| \\&\quad - \left\| \frac{1}{\gamma_n} u + y \right\|.\end{aligned}\quad (13)$$

By combining (13) with the triangle inequalities we obtain

$$\begin{aligned}& \left\| \frac{1}{\gamma_n} u + y \right\| - \left| \frac{\alpha_0}{1/\gamma_n + \alpha_0} \right| \|y\| \\&\quad + \alpha_0 \left\| u + \frac{1}{1/\gamma_n + \alpha_0} y \right\| - \left\| \frac{1}{\gamma_n} u + y \right\| \\&\leq \left\| \frac{1}{\gamma_n} u + (\alpha_0 u + y) \right\| - \left\| \frac{1}{\gamma_n} u + y \right\| \\&\leq \left| \frac{1/\gamma_n}{1/\gamma_n + \alpha_0} - 1 \right| \|y\| + \alpha_0 \left\| u + \frac{1}{1/\gamma_n + \alpha_0} y \right\|,\end{aligned}\quad (14)$$

from which it follows that

$$\lim_{n \rightarrow \infty} \left(\left\| \frac{1}{\gamma_n} u + (\alpha_0 u + y) \right\| - \left\| \frac{1}{\gamma_n} u + y \right\| \right) = \alpha_0 \|u\|. \quad (15)$$

Similarly, when γ_n is sufficiently close to 0, we have the equality

$$\begin{aligned}& \left\| \frac{1}{\gamma_n} u - (\alpha_0 u + y) \right\| - \left\| \frac{1}{\gamma_n} u - y \right\| \\&= \left\| \frac{1}{\gamma_n} u - \frac{1}{1 - \alpha_0 \gamma_n} y - \alpha_0 u + \frac{\alpha_0 \gamma_n}{1 - \alpha_0 \gamma_n} y \right\| - \left\| \frac{1}{\gamma_n} u - y \right\| \\&= \left\| \frac{1}{\gamma_n} u - \frac{1}{1 - \alpha_0 \gamma_n} y \right\| - \alpha_0 \left\| u - \frac{\gamma_n}{1 - \alpha_0 \gamma_n} y \right\| - \left\| \frac{1}{\gamma_n} u - y \right\|.\end{aligned}\quad (16)$$

From (16) and the triangle inequalities it follows that

$$\begin{aligned}& \left\| \frac{1}{\gamma_n} u - y \right\| - \left\| \frac{\alpha_0 \gamma_n}{1 - \alpha_0 \gamma_n} y \right\| \\&\quad - \alpha_0 \left\| u - \frac{\gamma_n}{1 - \alpha_0 \gamma_n} y \right\| - \left\| \frac{1}{\gamma_n} u - y \right\| \\&\leq \left\| \frac{1}{\gamma_n} u - (\alpha_0 u + y) \right\| - \left\| \frac{1}{\gamma_n} u - y \right\| \\&\leq \left\| \frac{\alpha_0 \gamma_n}{1 - \alpha_0 \gamma_n} y \right\| - \alpha_0 \left\| u - \frac{\gamma_n}{1 - \alpha_0 \gamma_n} y \right\|.\end{aligned}\quad (17)$$

From (17) it follows that

$$-\lim_{n \rightarrow \infty} \left(\left\| \frac{1}{\gamma_n} u - (\alpha_0 u + y) \right\| - \left\| \frac{1}{\gamma_n} u - y \right\| \right) = \alpha_0 \|u\|. \quad (18)$$

By adding (15) and (18), we obtain

$$\begin{aligned}
 2\alpha_0 \|u\| &= \lim_{n \rightarrow \infty} \left(\left\| \frac{1}{\gamma_n} u + (\alpha_0 u + y) \right\| - \left\| \frac{1}{\gamma_n} u + y \right\| \right. \\
 &\quad \left. - \left\| \frac{1}{\gamma_n} u - (\alpha_0 u + y) \right\| + \left\| \frac{1}{\gamma_n} u - y \right\| \right) \\
 &= \lim_{n \rightarrow \infty} \left(\left\| \frac{1}{\gamma_n} u - y \right\| - \left\| \frac{1}{\gamma_n} u + y \right\| \right) \\
 &= - \lim_{n \rightarrow \infty} \left(\frac{\|u - \gamma_n y\| - \|u\|}{-\gamma_n} + \frac{\|u + \gamma_n y\| - \|u\|}{\gamma_n} \right). \quad (19)
 \end{aligned}$$

Then it follows from (10) that $\alpha_0 = 0$. Therefore $z = y$, as claimed. $\square$

Our main result is the following theorem.

Theorem 6. *If $u \in H'_X$, then $u \in H_X$.*

Proof. Let z be an arbitrary point that is isosceles orthogonal to u and λ an arbitrary number. We only need to consider the case $\lambda > 0$. By Lemma 5,

$$\alpha(u, z) + \beta(u, z) = 0. \quad (20)$$

Thus

$$\alpha(u, \lambda z) + \beta(u, \lambda z) = 0. \quad (21)$$

Moreover, it is clear that λz is the unique point p in $\lambda z + \langle -u, u \rangle$ such that $\alpha(u, p) + \beta(u, p) = 0$. Lemma 5 and the existence of a point in $\lambda z + \langle -u, u \rangle$ that is isosceles orthogonal to u show that $u \perp_I \lambda z$, and the proof is complete. $\square$

By Theorems 6 and 2 in [2] we obtain the following corollary, which improves (4.12) and (10.13) in [3]. Corollary 7 is formally stronger than Theorem 2 in [2].

Corollary 7. *Let X be a Banach space whose dimension is at least two. If the relative interior of H'_X in S_X is not empty, then X is a Hilbert space.*

Conflict of Interests

The authors declare that they have no conflict of interests to this work.

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