Research Article

Stability for Functional Equation of Mixed Type in Non-Archimedean Normed Spaces

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We take into account the stability of the following functional equation
\[ f(x + ay) + f(x - ay) - 2f(x) - a²f(y) - a²f(-y) = 0 \]
in non-Archimedean normed spaces.

1. Introduction and Preliminaries

The study of stability problems has originally been formulated by Ulam [1]: under what condition does there exist a homomorphism near an approximate homomorphism? Hyers [2] had answered affirmatively the question of Ulam for Banach spaces. The theorem of Hyers was generalized by Aoki [3] for additive mappings and by Rassias [4] for linear mappings by considering an unbounded Cauchy difference. The paper work of Rassias [4] has had a lot of influence in the development of what is called the generalized Hyers-Ulam stability of functional equations. Thereafter, many interesting results of the generalized Hyers-Ulam stability to a number of functional equations have been investigated by a number of mathematicians; see [5–13] and references therein.

Now we demonstrate some definitions used in this work.

**Definition 1.** A field \( k \), equipped with a function (valuation) \( | \cdot | \) from \( k \) into \([0, \infty)\), is called a non-Archimedean field if the function \( | \cdot | : k \to [0, \infty) \) satisfies the following conditions:

1. \( |r| = 0 \) if and only if \( r = 0 \);
2. \( |r| = |r| |s| \);
3. the strong triangle inequality, namely, \( |r + s| \leq \max \{|r|, |s|\} \) for all \( r, s \in k \).

Of course, it is easy to see that \( |1| = 1 = |1 - 1| \) and \( |r| \leq 1 \) for all nonzero integer \( n \).

**Definition 2.** Let \( X \) be a vector space over the non-Archimedean field \( k \) with a nontrivial non-Archimedean valuation \( | \cdot | \). A function \( \| \cdot \| : X \to [0, \infty) \) is said to be a non-Archimedean norm (valuation) if it satisfies the following conditions:

1. \( \|x\| = 0 \) if and only if \( x = 0 \);
2. \( \|rx\| = |r|\|x\| \) for all \( x \in X \) and all \( r \in k \);
3. the strong triangle inequality, namely,
\[
\|x + y\| \leq \max \{\|x\|, \|y\|\} \quad \forall x, y \in X.
\] (1)

In this case, \((X, \| \cdot \|)\) is called a non-Archimedean space. Moreover, if every Cauchy sequence is convergent, then \((X, \| \cdot \|)\) is said to be a complete non-Archimedean space.

It follows from the strong triangle inequality that
\[
\|x_n - x_m\| \leq \max \{\|x_{j+1} - x_j\| : m \leq j < n - 1\},
\] (2)
for all \( x_n, x_m \in X \) and all \( n, m \in \mathbb{N} \) with \( n > m \). Therefore a sequence \( \{x_n\} \) is a Cauchy sequence in non-Archimedean space if and only if the sequence \( \{x_{n+1} - x_n\} \) converges to zero in the space.

On the other hand, Moslehian and Rassias [14] discussed the stability of the additive functional equation and the quadratic functional equation in non-Archimedean normed...
spaces. Quite recently, the new results on stability of functional equations in non-Archimedean metric spaces have been investigated (e.g., [6, 12, 15]).

Here and now, we consider a quadratic-additive type functional equation

\[ f(x + ay) + f(x - ay) - 2f(x) - a^2 f(y) - a^2 f(-y) = 0, \]

whose solution is called a quadratic-additive mapping. Quite recently, the second author [16] investigated the Hyers-Ulam stability of the functional equation (3) on restricted domains for the case \( a = 1 \). In particular, hyperstability of the functional equation (3) in the case when \( a = 1 \) was investigated in [17]. The purpose of this work establishes the generalized Hyers-Ulam stability of the functional equation (3) in non-Archimedean normed spaces.

\section{Main Results}

Throughout this section, we assume that \( \mathcal{X} \) is a non-Archimedean normed space and \( \mathcal{Y} \) is a complete non-Archimedean space. For a given mapping \( f : E_1 \to E_2 \) with vector spaces \( E_1 \) and \( E_2 \), we use the abbreviations

\[ \sigma f(x, y) := f(x + y) - f(x) - f(y), \]

\[ \Theta f(x, y) := f(x + y) + f(x - y) - 2f(x) - 2f(y), \]

\[ Df(x, y) := f(x + ay) + f(x - ay) - 2f(x) - a^2 f(y) - a^2 f(-y), \]

for all \( x, y \in E_1 \), where \( a \) is a nonzero integer.

\textbf{Theorem 3.} Let \( E_1 \) and \( E_2 \) be vector spaces. A mapping \( f : E_1 \to E_2 \) satisfies equation \( Df(x, y) = 0 \) for all \( x, y \in E_1 \) if and only if there exist a quadratic mapping \( g : E_1 \to E_2 \) and an additive mapping \( h : E_1 \to E_2 \) such that

\[ f(x) = g(x) + h(x), \]

for all \( x \in E_1 \).

\textit{Proof.} (\( \Rightarrow \)) We decompose \( f \) into the even part and the odd part by putting

\[ g(x) = \frac{f(x) + f(-x)}{2}, \]

\[ h(x) = \frac{f(x) - f(-x)}{2}, \]

for all \( x \in E_1 \). Notice that \( f(0) = -Df(0, 0)/2a^2 = 0 \). From the equalities

\[ \Theta g(x, y) = Dg(x, \frac{y}{a}) - Dg(0, \frac{y}{a}) = 0, \]

\[ \sigma h(x, y) = Dh\left(\frac{x + y}{2}, \frac{x + y}{2a}\right) - Dh\left(\frac{x + y}{2}, \frac{x - y}{2a}\right) = 0, \]

for all \( x, y \in E_1 \), we conclude that \( g \) is a quadratic mapping and \( h \) is an additive mapping.

(\( \Leftarrow \)) If there exist a quadratic mapping \( g : E_1 \to E_2 \) and an additive mapping \( h : E_1 \to E_2 \) such that

\[ f(x) = g(x) + h(x), \]

for all \( x \in E_1 \), then we find that

\[ Df(x, y) = Dg(x, y) + Dh(x, y) = 0, \]

for all \( x, y \in E_1 \). We arrive at the desired conclusion. \( \square \)

\textbf{Theorem 4.} Let \( \varphi : \mathcal{X}^2 \to [0, \infty) \) be a function such that

\[ \lim_{n \to \infty} \varphi\left(2^n x, 2^n y\right) = \varphi\left(a^n x, a^n y\right) = 0, \]

for all \( x, y \in \mathcal{X} \) and, for each \( x \in \mathcal{X} \), let the limit

\[ \lim_{n \to \infty} \max\left\{ \frac{\varphi(0, a^j x)}{|2^n|^{j+2}}, \frac{\varphi(2^j x, 2^j x/a)}{|2|^{j+2}}, \frac{\varphi(-2^j x, -2^j x/a)}{|2|^{j+2}} : 0 \leq j < n \right\}, \]

denoted by \( \overline{\varphi}(x) \), exist. Suppose that \( f : \mathcal{X} \to \mathcal{Y} \) is a mapping satisfying

\[ \|Df(x, y)\| \leq \varphi(x, y), \]

for all \( x, y \in \mathcal{X} \). Then there exists a quadratic-additive mapping \( T : \mathcal{X} \to \mathcal{Y} \) such that

\[ \|f(x) - T(x)\| \leq \overline{\varphi}(x), \]

for all \( x \in \mathcal{X} \), where the mapping \( T \) is given by

\[ T(x) = \lim_{n \to \infty} \left[ \frac{f(a^n x) + f(-a^n x)}{2a^{2n}} + \frac{f(2^n x) - f(-2^n x)}{2^{n+1}} \right], \]

for all \( x \in \mathcal{X} \).

\textit{Proof.} For a given mapping \( f : \mathcal{X} \to \mathcal{Y} \) and \( n \in \mathbb{N} \), let \( J_n f : \mathcal{X} \to \mathcal{Y} \) be a mapping defined by

\[ J_n f(x) = \frac{f(a^n x) + f(-a^n x)}{2a^{2n}} + \frac{f(2^n x) - f(-2^n x)}{2^{n+1}}, \]

for all \( x \in \mathcal{X} \).
for all \( x \in X \). Note that \( J_0 f(x) = f(x) \) and
\[
\left\| J_j f(x) - J_{j+1} f(x) \right\| \leq \frac{Df(0,a^j x)}{2^{j+2}} \frac{Df(2^j x, 2^j x/a)}{2^{j+2}} + \frac{Df(-2^j x, -2^j x/a)}{2^{j+2}}
\]
\[
\leq \max \left\{ \frac{\left\| Df(0,a^j x) \right\|}{|2| |a|^{j+2}}, \frac{\left\| Df(2^j x, 2^j x/a) \right\|}{|2|^{j+2}}, \frac{\left\| Df(-2^j x, -2^j x/a) \right\|}{|2|^{j+2}} \right\}
\]
for all \( x \in X \) and \( j \geq 0 \). It follows from (10) and (16) that the sequence \( \{ J_n f(x) \} \) is Cauchy. Since \( Y \) is complete, we conclude that \( \{ J_n f(x) \} \) is convergent. So we can define \( T : X \to Y \) by
\[
T(x) := \lim_{n \to \infty} J_n f(x),
\]
for all \( x \in X \). An induction implies that
\[
\left\| J_n f(x) - f(x) \right\| \leq \max \left\{ \frac{\phi(0,a^j x)}{|2| |a|^{j+2}}, \frac{\phi(2^j x, 2^j x/a)}{|2|^{j+2}}, \frac{\phi(-2^j x, -2^j x/a)}{|2|^{j+2}} : 0 \leq j < n \right\},
\]
for all \( n \in \mathbb{N} \) and all \( x \in X \). By taking the limit as \( n \to \infty \) in (18) and using (10), we obtain inequality (13).

From (12), we get
\[
\left\| D J_n f(x, y) \right\| = \left\| \frac{Df(2^n x, 2^n y) - Df(-2^n x, -2^n y)}{2^{n+1}} \right\|
\]
\[
+ \frac{Df(a^n x, a^n y) + Df(-a^n x, -a^n y)}{2^{n+1}}
\]
\[
\leq \max \left\{ \frac{\phi(2^n x, 2^n y)}{|2|^{n+1}}, \frac{\phi(-2^n x, -2^n y)}{|2|^{n+1}}, \frac{\phi(a^n x, a^n y)}{|2| |a|^{2n}}, \frac{\phi(-a^n x, -a^n y)}{|2| |a|^{2n}} \right\}
\]
(19)
for all \( x, y \in X \). Take the limit as \( n \to \infty \) in the above inequality and then use (10) to have \( DT(x, y) = 0 \) for all \( x, y \in X \).

**Corollary 5.** Let \( r > 2 \) be a real number and \(|2|, |a| < 1\). If a mapping \( f : X \to Y \) satisfies the condition
\[
\left\| Df(x, y) \right\| \leq \theta \left( \|x\|^r + \|y\|^r \right),
\]
for all \( x, y \in X \), then there exists a unique quadratic-additive mapping \( T : X \to Y \) such that
\[
\left\| f(x) - T(x) \right\| \leq \frac{(|a|^r + 1) \theta \|x\|^r}{|2|^r |a|^r},
\]
for all \( x \in X \).

**Proof.** Let \( \phi(x, y) = \theta(\|x\|^r + \|y\|^r) \). Since \(|2|^{r-1} < 1 \) and \(|a|^{r-2} < 1 \), we know that
\[
\lim_{n \to \infty} \frac{\phi(2^n x, 2^n y)}{|2|^n} = \frac{\phi(a^n x, a^n y)}{|a|^{2n}} = 0,
\]
for all \( x, y \in X \). Therefore the conditions of Theorem 4 are fulfilled. In particular, it is easy to see that
\[
\tilde{\phi}(x) = \max \left\{ \frac{\theta \|x\|^r}{|2|^r |a|^r}, \frac{(|a|^r + 1) \theta \|x\|^r}{|2|^r |a|^r} \right\}
\]
(23)
Theorem 4 guarantees that there exists a quadratic-additive mapping \( T : X \to Y \) with (21).

Now, to show uniqueness of the mapping \( T \), let us assume that \( T' \) is another quadratic-additive mapping satisfying (21). Then we have
\[
T'(x) - J_k T'(x)
\]
\[
= \sum_{j=0}^{k-1} \left( \frac{-DT'(0, a^j x)}{2^{j+2}} + \frac{DT'(2^j x, 2^j x/a)}{2^{j+2}} + \frac{DT'(2^j x, -2^j x/a)}{2^{j+2}} \right)
\]
\[
= 0,
\]
(24)
for any \( k \in \mathbb{N} \), and thus we feel that
\[
\left\| T(\mathbf{x}) - T'(\mathbf{x}) \right\| = \lim_{k \to \infty} \left\| J_k T(\mathbf{x}) - J_k T'(\mathbf{x}) \right\|
\leq \lim_{k \to \infty} \max \left\{ \left\| J_k f(\mathbf{x}) - J_k f'(\mathbf{x}) \right\|, \right.
\]
\[
\left. \left\| J_k f(\mathbf{x}) - J_k T'(\mathbf{x}) \right\| \right\}
\leq \lim_{k \to \infty} \max \left\{ |a|^{k-1} \left[ 2^k \right] \frac{\left\| J(\mathbf{x}) - J(x) \right\|}{\left\| J(x) - J(x) \right\|}, \right.
\]
\[
\left. \left\| J_k f(\mathbf{x}) - J_k f'(\mathbf{x}) \right\|, \right\}
\]
\[
\leq \lim_{k \to \infty} \max \left\{ |a|^{k-1} \left[ 2^k \right] \frac{\left\| J(\mathbf{x}) - J(x) \right\|}{\left\| J(x) - J(x) \right\|}, \right.
\]
\[
\left. \left\| J_k f(\mathbf{x}) - J_k f'(\mathbf{x}) \right\| \right\} = 0,
\]
(25)
for all \( \mathbf{x} \in \mathcal{X} \), which implies that \( T \) is unique. \( \square \)

**Theorem 6.** Let \( \varphi : \mathcal{X}^2 \to [0, \infty) \) be a function such that

\[
\lim_{n \to \infty} \left| 2^n \varphi \left( \frac{x}{2^n}, \frac{x}{2^n} \right) = \lim_{n \to \infty} |a|^{2n} \varphi \left( \frac{x}{a^n}, \frac{y}{a^n} \right) = 0, \right. \]
(26)
for all \( x, y \in \mathcal{X} \) and, for each \( x \in \mathcal{X} \), let the limit
\[
\lim_{n \to \infty} \max \left\{ \frac{|a|^{j-1}}{2^j} \varphi \left( 0, \frac{x}{a^{j+1}} \right), \frac{|a|^{j-1}}{2^j} \varphi \left( \frac{x}{a^{j+1}}, \frac{x}{a^{j+1}} \right), \right. \]
\[
\left. \frac{|a|^{j-1}}{2^j} \varphi \left( -\frac{x}{2^{j+1}}, -\frac{x}{2^{j+1}} \right) : 0 \leq j \leq n \right\},
\]
(27)
denoted by \( \bar{\varphi}(x) \), exist. Suppose that \( f : \mathcal{X} \to \mathcal{Y} \) is a mapping satisfying
\[
\left\| Df(\mathbf{x}, \mathbf{y}) \right\| \leq \varphi(\mathbf{x}, \mathbf{y}),
\]
(28)
for all \( \mathbf{x}, \mathbf{y} \in \mathcal{X} \). Then there exists a quadratic-additive mapping \( T : \mathcal{X} \to \mathcal{Y} \) such that
\[
\left\| f(\mathbf{x}) - T(\mathbf{x}) \right\| \leq \bar{\varphi}(\mathbf{x}),
\]
(29)
for all \( x \in \mathcal{X} \), where the mapping \( T \) is given by
\[
T(\mathbf{x}) = \lim_{n \to \infty} \left[ \frac{a^{2n}}{2} \left( f \left( \frac{x}{a^n} \right) + f \left( \frac{-x}{a^n} \right) \right) \right.
\]
\[
+ 2^{n-1} \left( f \left( \frac{x}{2^n} \right) - f \left( \frac{-x}{2^n} \right) \right),
\]
(30)
for all \( x \in \mathcal{X} \).

**Proof.** For a given mapping \( f : \mathcal{X} \to \mathcal{Y} \) and \( n \in \mathbb{N} \), let \( J_n f : \mathcal{X} \to \mathcal{Y} \) be a mapping defined by
\[
J_n f(\mathbf{x}) = \frac{a^{2n}}{2} \left( f \left( \frac{x}{a^n} \right) + f \left( \frac{-x}{a^n} \right) \right)
\]
\[
+ 2^{n-1} \left( f \left( \frac{x}{2^n} \right) - f \left( \frac{-x}{2^n} \right) \right),
\]
(31)
for all \( x \in \mathcal{X} \). Observe that \( J_n f(x) = f(x) \) and
\[
\left\| J_n f(\mathbf{x}) - J_{n+1} f(\mathbf{x}) \right\| = \left\| J_n f(\mathbf{x}) \right\|,
\]
(32)
for all \( x \in \mathcal{X} \). Applying the induction, we yield that
\[
\|J_n f(x) - f(x)\| \leq \max \left\{ \frac{|a|^{2j}}{2^j} \phi \left( 0, \frac{x}{a^{n+1}} \right), |2|^{j-1} \phi \left( \frac{x}{2^{j+1}}, \frac{x}{2^{j+1}a} \right) : 0 \leq j < n \right\},
\] (34)
for all \( n \in \mathbb{N} \) and all \( x \in \mathcal{X} \). Sending the limit as \( n \to \infty \) in (34) with (26), we arrive at inequality (29).

According to (28), we see that
\[
\|D_n f(x, y)\| = \left\| \frac{d^n}{dx^n} \left( Df \left( \frac{x}{a^n}, \frac{y}{a^n} \right) + Df \left( -\frac{x}{a^n}, -\frac{y}{a^n} \right) \right) \right\| + 2^{n-1} \left\| Df \left( \frac{x}{2^n}, \frac{y}{2^n} \right) - Df \left( -\frac{x}{2^n}, -\frac{y}{2^n} \right) \right\|
\leq \max \left\{ \frac{|a|^{2n}}{2^n} \phi \left( \frac{x}{a^n}, \frac{y}{a^n} \right), \frac{|a|^{2n}}{2^n} \phi \left( -\frac{x}{a^n}, -\frac{y}{a^n} \right), \right\}
\leq \max \left\{ \frac{|a|^{2n}}{2^n} \phi \left( \frac{x}{a^{n+1}}, \frac{y}{a^{n+1}} \right), \frac{|a|^{2n}}{2^n} \phi \left( -\frac{x}{a^{n+1}}, -\frac{y}{a^{n+1}} \right), \right\},
\] (35)
for all \( x, y \in \mathcal{X} \). Taking the limit as \( n \to \infty \) and using (26), we get \( DT(x, y) = 0 \) for all \( x, y \in \mathcal{X} \).

**Corollary 7.** Let \( r < 1 \) be a real number and \( |2|, |a| < 1 \). If a mapping \( f : \mathcal{X} \to \mathcal{Y} \) satisfies the condition
\[
\|Df(x, y)\| \leq \theta \left( \|x\|^r + \|y\|^r \right),
\] (36)
for all \( x, y \in \mathcal{X} \), then there exists a unique quadratic-additive mapping \( T : \mathcal{X} \to \mathcal{Y} \) such that
\[
\|f(x) - T(x)\| \leq \frac{(|a|^r + 1) \theta \|x\|^r}{|2|^{1+r}|a|^r},
\] (37)
for all \( x \in \mathcal{X} \).

**Proof.** Let \( \varphi(x, y) = \theta(\|x\|^r + \|y\|^r) \). Since \( |a|^{1-r} < 1 \) and \( |a|^{2-r} < 1 \), the mapping \( \varphi \) satisfies equalities (26) for all \( x, y \in \mathcal{X} \). Moreover, we see that
\[
\varphi(x) = \max \left\{ \frac{\theta \|x\|^r}{|2| |a|^r}, \frac{(|a|^r + 1) \theta \|x\|^r}{|2|^{1+r}|a|^r} \right\} = \frac{(|a|^r + 1) \theta \|x\|^r}{|2|^{1+r}|a|^r}.
\] (38)
Then it follows from Theorem 6 that there exists a quadratic-additive mapping \( T : \mathcal{X} \to \mathcal{Y} \) satisfying (37).

In order to prove the uniqueness of \( T \), we assume that \( T' \) is another quadratic-additive mapping satisfying (37). Then
\[
T'(x) - J_k T'(x) = \sum_{j=0}^{k-1} \left( \frac{a^2}{2^n} DT'(0, \frac{x}{a^{n+1}}) \right) + 2^{j-1} DT'(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}a}) - 2^{j-1} DT'(\frac{-x}{2^{j+1}}, \frac{-x}{2^{j+1}a}) = 0,
\] (39)
for any \( k \in \mathbb{N} \), whence we give that
\[
\|T(x) - T'(x)\| = \lim_{k \to \infty} \|J_k T(x) - J_k T'(x)\| \leq \lim_{k \to \infty} \max \left\{ \|J_k T(x) - J_k f(x)\|, \|J_k f(x) - J_k T'(x)\| \right\} \leq \max \left\{ 2^{k-1} \left\| (T - f) \left( \frac{x}{2^k} \right) \right\|, 2^{k-1} \left\| (f - T') \left( \frac{x}{2^k} \right) \right\| \right\}
\leq \max \left\{ \frac{|a|^{2k}}{2^k} \left\| (T - f) \left( \frac{x}{a^k} \right) \right\|, \frac{|a|^{2k}}{2^k} \left\| (f - T') \left( \frac{x}{a^k} \right) \right\| \right\}
\leq \max \left\{ \frac{|a|^{2k}}{2^k} \left( |a|^r + 1 \right) \theta \|x\|^r \right\},
\] (40)
for all \( x \in \mathcal{X} \), which means that \( T \) is unique.

**Theorem 8.** Let \( \mathcal{X}^2 \to [0, \infty) \) be a function such that
\[
\lim_{n \to \infty} 2^n \varphi \left( \frac{x}{2^n}, \frac{y}{2^n} \right) = \lim_{n \to \infty} \varphi \left( \frac{a^nx}{a^n}, \frac{a^ny}{a^n} \right),
\] (41)
for all \( x \in \mathcal{X} \), which means that \( T \) is unique.
for all \( x, y \in \mathcal{X} \) and, for each \( x \in \mathcal{X} \), let \( \lim_{n \to \infty} \max \{ \varphi(0, a_j x) \mid 2 |a_j| x \} \),
\[
\text{denoted by } \tilde{\varphi}(x), \text{exist. Suppose that } f: \mathcal{X} \to \mathcal{Y} \text{ is a mapping satisfying}
\]
\[
\left\| Df(x, y) \right\| \leq \varphi(x, y),
\]
for all \( x, y \in \mathcal{X} \). Then there exists a quadratic-additive mapping \( T: \mathcal{X} \to \mathcal{Y} \) such that
\[
\left\| f(x) - T(x) \right\| \leq \tilde{\varphi}(x),
\]
for all \( x \in \mathcal{X} \), where the mapping \( T \) is given by
\[
T(x) = \lim_{n \to \infty} \left\{ \frac{f(a^n x) + f(-a^n x)}{2a^{2n}} + 2^{n-1} \left( f\left( \frac{x}{2^n} \right) - f\left( \frac{-x}{2^n} \right) \right) \right\},
\]
for all \( x \in \mathcal{X} \).

Proof. For a given mapping \( f: \mathcal{X} \to \mathcal{Y} \) and \( n \in \mathbb{N} \), let \( J_n f: \mathcal{X} \to \mathcal{Y} \) be a mapping defined by
\[
J_n f(x) = \frac{f(a^n x) + f(-a^n x)}{2a^{2n}} + 2^{n-1} \left( f\left( \frac{x}{2^n} \right) - f\left( \frac{-x}{2^n} \right) \right),
\]
for all \( x \in \mathcal{X} \). Now we note that \( J_0 f(x) = f(x) \) and
\[
\left\| J_n f(x) - J_{n+1} f(x) \right\| = \left\| -Df(0, a^n x) \right\| + 2^{n-1} \left( f\left( \frac{x}{2^n} \right) - f\left( \frac{x}{2^{n+1}} \right) \right),
\]
for all \( x \in \mathcal{X} \) and \( j \geq 0 \). Hence it follows by (41) and (47) that the sequence \( \{J_n f(x)\} \) is Cauchy. Since \( \mathcal{Y} \) is complete, we conclude that \( \{J_n f(x)\} \) is convergent. Thus we can define \( T: \mathcal{X} \to \mathcal{Y} \) by
\[
T(x) = \lim_{n \to \infty} J_n f(x),
\]
for all \( x \in \mathcal{X} \). Using the induction argument, we prove that
\[
\left\| J_n f(x) - f(x) \right\| \leq \max \left\{ \varphi(0, a_j x) \mid 2 |a_j| x \right\},
\]
for all \( n \in \mathbb{N} \) and all \( x \in \mathcal{X} \). Taking the limit as \( n \to \infty \) in (49) and using (41) lead to inequality (44).

By virtue of (43), we get
\[
\left\| Df(x, y) \right\| \leq \varphi(x, y),
\]
for all \( x, y \in \mathcal{X} \). Send the limit as \( n \to \infty \) and then use (41) to find \( DT(x, y) = 0 \) for all \( x, y \in \mathcal{X} \).

Theorem 9. Let \( \phi: \mathcal{X}^2 \to [0, \infty) \) be a function such that
\[
\lim_{n \to \infty} \phi(2^n x, 2^n y) = 0,
\]
for all \( x, y \in \mathcal{X} \) and, for each \( x \in \mathcal{X} \), let the limit
\[
\lim_{n \to \infty} \max \left\{ \varphi(0, a_j x) \mid 2 |a_j| x \right\},
\]
for all \( x \in \mathcal{X} \) and \( j \geq 0 \). Hence it follows by (41) and (47) that the sequence \( \{J_n f(x)\} \) is Cauchy. Since \( \mathcal{Y} \) is complete, we conclude that \( \{J_n f(x)\} \) is convergent. Thus we can define \( T: \mathcal{X} \to \mathcal{Y} \) by
\[
T(x) = \lim_{n \to \infty} J_n f(x),
\]
for all \( x \in \mathcal{X} \). Using the induction argument, we prove that
\[
\left\| J_n f(x) - f(x) \right\| \leq \max \left\{ \varphi(0, a_j x) \mid 2 |a_j| x \right\},
\]
for all \( n \in \mathbb{N} \) and all \( x \in \mathcal{X} \). Taking the limit as \( n \to \infty \) in (49) and using (41) lead to inequality (44).

By virtue of (43), we get
\[
\left\| Df(x, y) \right\| \leq \varphi(x, y),
\]
for all \( x, y \in \mathcal{X} \). Send the limit as \( n \to \infty \) and then use (41) to find \( DT(x, y) = 0 \) for all \( x, y \in \mathcal{X} \).
for all $x, y \in \mathcal{X}$. Then there exists a quadratic-additive mapping $T: \mathcal{X} \to \mathcal{Y}$ such that
\[ \| f(x) - T(x) \| \leq \overline{q}(x), \quad (54) \]
for all $x \in \mathcal{X}$, where the mapping $T$ is given by
\[ T(x) = \lim_{n \to \infty} \left[ \frac{a^{2n}}{2} \left( f \left( \frac{x}{a^n} \right) + f \left( \frac{-x}{a^n} \right) \right) + \frac{f \left( 2^n x \right) - f \left( -2^n x \right)}{2^{n+1}} \right], \quad (55) \]
for all $x \in \mathcal{X}$.

Proof. For a given mapping $f: \mathcal{X} \to \mathcal{Y}$ and $n \in \mathbb{N}$, let $J_n f: \mathcal{X} \to \mathcal{Y}$ be a mapping defined by
\[ J_n f(x) = \frac{a^{2n}}{2} \left( f \left( \frac{x}{a^n} \right) + f \left( \frac{-x}{a^n} \right) \right) + \frac{f \left( 2^n x \right) - f \left( -2^n x \right)}{2^{n+1}}, \quad (56) \]
for all $x \in \mathcal{X}$. We remark that $J_0 f(x) = f(x)$ and
\[ \| J_1 f(x) - J_{n+1} f(x) \| = \frac{a^{2j}}{2} Df \left( 0, \frac{x}{a^{j+1}} \right) \]
\[ - \frac{Df \left( 2^j x, 2^j x/a \right)}{2^{j+2}} \]
\[ + \frac{Df \left( -2^j x, -2^j x/a \right)}{2^{j+2}} \]
\[ \leq \max \left\{ \frac{|a|^{2j}}{2} \| Df \left( 0, \frac{x}{a^{j+1}} \right) \|, \right. \]
\[ \left. \left\| Df \left( 2^j x, 2^j x/a \right) \right\| \right\}, \quad (57) \]
for all $x, y \in \mathcal{X}$ and $j \geq 0$. So it follows from (51) and (57) that the sequence $\{J_n f(x)\}$ is Cauchy. Due to the completeness of $\mathcal{Y}$, this sequence is convergent. Let $T: \mathcal{X} \to \mathcal{Y}$ be a mapping defined by
\[ T(x) := \lim_{n \to \infty} J_n f(x), \quad (58) \]
for all $x \in \mathcal{X}$. An induction implies that
\[ \| J_n f(x) - f(x) \| \leq \max \left\{ \frac{|a|^{2j}}{2} \phi \left( 0, \frac{x}{a^{j+1}} \right), \right. \]
\[ \phi \left( 2^j x, 2^j x/a \right) \frac{|2|^{j+2}}{|2|^{j+2}}, \]
\[ \left. \left. \phi \left( -2^j x, -2^j x/a \right) \frac{|2|^{j+2}}{|2|^{j+2}} \right\} : 0 \leq j < n \right\}, \quad (59) \]
for all $n \in \mathbb{N}$ and all $x \in \mathcal{X}$. By passing the limit as $n \to \infty$ in (59) with (51), we obtain the relation (54).

From (53), we find that
\[ \| Df_n(x,y) \| = \left\| \frac{a^{2n}}{2} \left( Df \left( \frac{x}{a^n}, \frac{y}{a^n} \right) + Df \left( -\frac{x}{a^n}, -\frac{y}{a^n} \right) \right) \right\| \]
\[ + \frac{Df \left( 2^n x, 2^n y \right) - 2^n Df \left( -2^n x, -2^n y \right)}{2^{n+1}} \]
\[ \leq \max \left\{ \frac{|a|^{2n}}{2} \phi \left( \frac{x}{a^n}, \frac{y}{a^n} \right), \frac{|a|^{2n}}{2} \phi \left( -\frac{x}{a^n}, -\frac{y}{a^n} \right), \right. \]
\[ \left. \phi \left( 2^n x, 2^n y \right), \phi \left( -2^n x, -2^n y \right) \right\} \frac{|2|^{n+1}}{|2|^{n+1}}, \quad (60) \]
for all $x, y \in \mathcal{X}$. Taking the limit as $n \to \infty$ and using (51), we arrive at $DT(x,y) = 0$ for all $x, y \in \mathcal{X}$.

Corollary 10. Let $|a|, |2| < 1$ and let $r$ be a real number such that $1 < r < 2$. If a mapping $f: \mathcal{X} \to \mathcal{Y}$ satisfies the condition
\[ \| Df \left( x, y \right) \| \leq \theta \left( \|x\|^r + \|y\|^r \right), \quad (61) \]
for all $x, y \in \mathcal{X}$, then there exists a unique quadratic-additive mapping $T: \mathcal{X} \to \mathcal{Y}$ such that
\[ \| f(x) - T(x) \| \leq \frac{(|a|^r + 1) \theta \|x\|^r}{|2|^r |a|^r}, \quad (62) \]
for all $x \in \mathcal{X}$.

Proof. We consider $\phi(x,y) = \theta(\|x\|^r + \|y\|^r)$. Based on the fact that $|a|^{2-r} < 1$ and $|2|^{r-1} < 1$, the mapping $\phi$ satisfies conditions (51). In fact, it is easy to see that
\[ \phi(x) = \left\{ \theta \|x\|^r \left( \frac{|a|^r + 1}{|2|^{r-1} |a|^r} \right) \right\}, \quad (63) \]
On account of Theorem 9, there is a quadratic-additive mapping $T: \mathcal{X} \to \mathcal{Y}$ satisfying (62).
To show uniqueness of the mapping $T$, we suppose that $T'$ is another quadratic-additive mapping satisfying (62). Then
\[
T'(x) - J_k T'(x) = \sum_{j=0}^{k-1} \left( \frac{a^{2j}}{2} Df \left( 0, \frac{x}{a^{j+1}} \right) - \frac{Df \left( 2^j x, 2^j x/a \right)}{2^{j+1}} + \frac{Df \left( -2^j x, -2^j x/a \right)}{2^{j+1}} \right) = 0,
\]
for any $k \in \mathbb{N}$ and so we figure out the following:
\[
\|T(x) - T'(x)\| = \lim_{k \to \infty} \|J_k T(x) - J_k T'(x)\| \\
\leq \lim_{k \to \infty} \max \left\{ \left\| J_k f(x) - J_k T'(x) \right\|, \left\| J_k f(x) - J_k f(x) \right\| \right\} \\
\leq \lim_{k \to \infty} \max \left\{ \left\| J_k f(x) - J_k T'(x) \right\| \right\} \\
\leq \lim_{k \to \infty} \max \left\{ \left\| J_k f(x) - J_k T'(x) \right\| \right\} \\
(64)
\]
for all $x \in X$. This implies that $T$ is unique. $\square$

The problem is whether or not the previous corollaries hold the cases when $r = 1$ or $r = 2$.

**Conflict of Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.

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