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Research Article

Stability for Functional Equation of Mixed Type in Non-Archimedean Normed Spaces

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We take into account the stability of the following functional equation $f(x+ay)+f(x-ay)-2f(x)-a^2f(y)-a^2f(-y)=0$ in non-Archimedean normed spaces.

1. Introduction and Preliminaries

The study of stability problems has originally been formulated by Ulam [1]: *under what condition does there exist a homomorphism near an approximate homomorphism?* Hyers [2] had answered affirmatively the question of Ulam for Banach spaces. The theorem of Hyers was generalized by Aoki [3] for additive mappings and by Rassias [4] for linear mappings by considering an unbounded Cauchy difference. The paper work of Rassias [4] has had a lot of influence in the development of what is called the *generalized Hyers-Ulam stability* of functional equations. Thereafter, many interesting results of the generalized Hyers-Ulam stability to a number of functional equations have been investigated by a number of mathematicians; see [5–13] and references therein.

Now we demonstrate some definitions used in this work.

Definition 1. A field $\mathbb{K}$, equipped with a function (valuation) $|\cdot|$ from $\mathbb{K}$ into $[0,\infty)$, is called a *non-Archimedean field* if the function $|\cdot|:\mathbb{K}\to[0,\infty)$ satisfies the following conditions:

(1) $|r|=0$ if and only if $r=0$;

(2) $|rs|=|r||s|$;

(3) the strong triangle inequality, namely, $|r+s|\le\max\{|r|,|s|\}$ for all $r,s\in\mathbb{K}$.

Of course, it is easy to see that $|1|=1=|-1|$ and $|n|\le 1$ for all nonzero integer n.

Definition 2. Let $\mathcal{X}$ be a vector space over the non-Archimedean field $\mathbb{K}$ with a nontrivial non-Archimedean valuation $|\cdot|$. A function $\|\cdot\|:\mathcal{X}\to[0,\infty)$ is said to be a *non-Archimedean norm* (*valuation*) if it satisfies the following conditions:

(1) $\|x\|=0$ if and only if $x=0$;

(2) $\|rx\|=|r|\|x\|$ for all $x\in\mathcal{X}$ and all $r\in\mathbb{K}$;

(3) the strong triangle inequality, namely,

$$\|x+y\|\le\max\{\|x\|,\|y\|\}\quad\forall x,y\in\mathcal{X}. \tag{1}$$

In this case, $(\mathcal{X},\|\cdot\|)$ is called a *non-Archimedean space*. Moreover, if every Cauchy sequence is convergent, then $(\mathcal{X},\|\cdot\|)$ is said to be a *complete non-Archimedean space*.

It follows from the strong triangle inequality that

$$\|x_n-x_m\|\le\max\left\{\left\|x_{j+1}-x_j\right\|:m\le j<n-1\right\}, \tag{2}$$

for all $x_n,x_m\in\mathcal{X}$ and all $m,n\in\mathbb{N}$ with $n>m$. Therefore a sequence $\{x_n\}$ is a Cauchy sequence in non-Archimedean space if and only if the sequence $\{x_{n+1}-x_n\}$ converges to zero in the space.

On the other hand, Moslehian and Rassias [14] discussed the stability of the additive functional equation and the quadratic functional equation in non-Archimedean normed

spaces. Quite recently, the new results on stability of functional equations in non-Archimedean metric spaces have been investigated (e.g., [6, 12, 15]).

Here and now, we consider a *quadratic-additive type functional equation*

$$f(x+ay)+f(x-ay)-2f(x)-a^2f(y)-a^2f(-y)=0, \tag{3}$$

whose solution is called *a quadratic-additive mapping*. Quite recently, the second author [16] investigated the Hyers-Ulam stability of the functional equation (3) on restricted domains for the case $a=1$. In particular, hyperstability of the functional equation (3) in the case when $a=1$ was investigated in [17]. The purpose of this work establishes the generalized Hyers-Ulam stability of the functional equation (3) in non-Archimedean normed spaces.

2. Main Results

Throughout this section, we assume that $\mathcal{X}$ is a non-Archimedean normed space and $\mathcal{Y}$ is a complete non-Archimedean space. For a given mapping $f:E_1\to E_2$ with vector spaces E_1 and E_2, we use the abbreviations

$$\begin{aligned}\mathcal{A}f(x,y)&:=f(x+y)-f(x)-f(y),\\ \mathcal{Q}f(x,y)&:=f(x+y)+f(x-y)-2f(x)-2f(y),\\ Df(x,y)&:=f(x+ay)+f(x-ay)\\ &\quad-2f(x)-a^2f(y)-a^2f(-y),\end{aligned} \tag{4}$$

for all $x,y\in E_1$, where a is a nonzero integer.

Theorem 3. *Let E_1 and E_2 be vector spaces. A mapping $f:E_1\to E_2$ satisfies equation $Df(x,y)=0$ for all $x,y\in E_1$ if and only if there exist a quadratic mapping $g:E_1\to E_2$ and an additive mapping $h:E_1\to E_2$ such that*

$$f(x)=g(x)+h(x), \tag{5}$$

for all $x\in E_1$.

Proof. ($\Rightarrow$) We decompose f into the even part and the odd part by putting

$$\begin{aligned}g(x)&=\frac{f(x)+f(-x)}{2},\\ h(x)&=\frac{f(x)-f(-x)}{2},\end{aligned} \tag{6}$$

for all $x\in E_1$. Notice that $f(0)=-Df(0,0)/2a^2=0$. From the equalities

$$\begin{aligned}\mathcal{Q}g(x,y)&=Dg\left(x,\frac{y}{a}\right)-Dg\left(0,\frac{y}{a}\right)=0,\\ \mathcal{A}h(x,y)&=Dh\left(\frac{x+y}{2},\frac{x+y}{2a}\right)-Dh\left(\frac{x+y}{2},\frac{x-y}{2a}\right)=0,\end{aligned} \tag{7}$$

for all $x,y\in E_1$, we conclude that g is a quadratic mapping and h is an additive mapping.

($\Leftarrow$) If there exist a quadratic mapping $g:E_1\to E_2$ and an additive mapping $h:E_1\to E_2$ such that

$$f(x)=g(x)+h(x), \tag{8}$$

for all $x\in E_1$, then we find that

$$\begin{aligned}Df(x,y)&=Dg(x,y)+Dh(x,y)\\ &=\mathcal{Q}g(x,ay)+2g(ay)-a^2g(y)-a^2g(-y)\\ &\quad+\mathcal{A}h(x,x)-\mathcal{A}h(x+ay,x-ay)\\ &=0,\end{aligned} \tag{9}$$

for all $x,y\in E_1$. We arrive at the desired conclusion. □

Theorem 4. *Let $\varphi:\mathcal{X}^2\to[0,\infty)$ be a function such that*

$$\lim_{n\to\infty}\frac{\varphi(2^nx,2^ny)}{|2|^n}=\frac{\varphi(a^nx,a^ny)}{|a|^{2n}}=0, \tag{10}$$

for all $x,y\in\mathcal{X}$ and, for each $x\in\mathcal{X}$, let the limit

$$\begin{aligned}\lim_{n\to\infty}\max\Bigg\{&\frac{\varphi(0,a^jx)}{|2|\,|a|^{2j+2}},\frac{\varphi(2^jx,2^jx/a)}{|2|^{j+2}},\\ &\frac{\varphi(-2^jx,-2^jx/a)}{|2|^{j+2}}:0\le j<n\Bigg\},\end{aligned} \tag{11}$$

denoted by $\tilde{\varphi}(x)$, exist. Suppose that $f:\mathcal{X}\to\mathcal{Y}$ is a mapping satisfying

$$\|Df(x,y)\|\le\varphi(x,y), \tag{12}$$

for all $x,y\in\mathcal{X}$. Then there exists a quadratic-additive mapping $T:\mathcal{X}\to\mathcal{Y}$ such that

$$\|f(x)-T(x)\|\le\tilde{\varphi}(x), \tag{13}$$

for all $x\in\mathcal{X}$, where the mapping T is given by

$$T(x)=\lim_{n\to\infty}\left[\frac{f(a^nx)+f(-a^nx)}{2a^{2n}}+\frac{f(2^nx)-f(-2^nx)}{2^{n+1}}\right], \tag{14}$$

for all $x\in\mathcal{X}$.

Proof. For a given mapping $f:\mathcal{X}\to\mathcal{Y}$ and $n\in\mathbb{N}$, let $J_nf:\mathcal{X}\to\mathcal{Y}$ be a mapping defined by

$$J_nf(x)=\frac{f(a^nx)+f(-a^nx)}{2a^{2n}}+\frac{f(2^nx)-f(-2^nx)}{2^{n+1}}, \tag{15}$$

for all $x \in \mathcal{X}$. Note that $J_0 f(x) = f(x)$ and

$$\begin{aligned}\left\|J_j f(x) - J_{j+1} f(x)\right\| &= \left\|-\frac{Df\left(0, a^j x\right)}{2a^{2j+2}} - \frac{Df\left(2^j x, 2^j x/a\right)}{2^{j+2}} + \frac{Df\left(-2^j x, -2^j x/a\right)}{2^{j+2}}\right\| \\ &\leq \max\left\{\frac{\left\|Df\left(0, a^j x\right)\right\|}{|2|\,|a|^{2j+2}}, \frac{\left\|Df\left(2^j x, 2^j x/a\right)\right\|}{|2|^{j+2}}, \frac{\left\|Df\left(-2^j x, -2^j x/a\right)\right\|}{|2|^{j+2}}\right\} \\ &\leq \max\left\{\frac{\varphi\left(0, a^j x\right)}{|2|\,|a|^{2j+2}}, \frac{\varphi\left(2^j x, 2^j x/a\right)}{|2|^{j+2}}, \frac{\varphi\left(-2^j x, -2^j x/a\right)}{|2|^{j+2}}\right\},\end{aligned} \tag{16}$$

for all $x \in \mathcal{X}$ and $j \geq 0$. It follows from (10) and (16) that the sequence $\{J_n f(x)\}$ is Cauchy. Since $\mathcal{Y}$ is complete, we conclude that $\{J_n f(x)\}$ is convergent. So we can define $T : \mathcal{X} \to \mathcal{Y}$ by

$$T(x) := \lim_{n\to\infty} J_n f(x), \tag{17}$$

for all $x \in \mathcal{X}$. An induction implies that

$$\left\|J_n f(x) - f(x)\right\| \leq \max\left\{\frac{\varphi\left(0, a^j x\right)}{|2|\,|a|^{2j+2}}, \frac{\varphi\left(2^j x, 2^j x/a\right)}{|2|^{j+2}}, \frac{\varphi\left(-2^j x, -2^j x/a\right)}{|2|^{j+2}} : 0 \leq j < n\right\}, \tag{18}$$

for all $n \in \mathbb{N}$ and all $x \in \mathcal{X}$. By taking the limit as $n \to \infty$ in (18) and using (10), we obtain inequality (13).

From (12), we get

$$\begin{aligned}\left\|DJ_n f(x, y)\right\| &= \left\|\frac{Df\left(2^n x, 2^n y\right) - Df\left(-2^n x, -2^n y\right)}{2^{n+1}} + \frac{Df\left(a^n x, a^n y\right) + Df\left(-a^n x, -a^n y\right)}{2a^{2n}}\right\| \\ &\leq \max\left\{\frac{\varphi\left(2^n x, 2^n y\right)}{|2|^{n+1}}, \frac{\varphi\left(-2^n x, -2^n y\right)}{|2|^{n+1}}, \frac{\varphi\left(a^n x, a^n y\right)}{|2|\,|a|^{2n}}, \frac{\varphi\left(-a^n x, -a^n y\right)}{|2|\,|a|^{2n}}\right\}\end{aligned} \tag{19}$$

for all $x, y \in \mathcal{X}$. Take the limit as $n \to \infty$ in the above inequality and then use (10) to have $DT(x, y) = 0$ for all $x, y \in \mathcal{X}$. □

Corollary 5. *Let $r > 2$ be a real number and $|2|, |a| < 1$. If a mapping $f : \mathcal{X} \to \mathcal{Y}$ satisfies the condition*

$$\left\|Df(x, y)\right\| \leq \theta\left(\|x\|^r + \|y\|^r\right), \tag{20}$$

for all $x, y \in \mathcal{X}$, then there exists a unique quadratic-additive mapping $T : \mathcal{X} \to \mathcal{Y}$ such that

$$\left\|f(x) - T(x)\right\| \leq \frac{\left(|a|^r + 1\right)\theta\|x\|^r}{|2|^2|a|^r}, \tag{21}$$

for all $x \in \mathcal{X}$.

Proof. Let $\varphi(x, y) = \theta(\|x\|^r + \|y\|^r)$. Since $|2|^{r-1} < 1$ and $|a|^{r-2} < 1$, we know that

$$\lim_{n\to\infty} \frac{\varphi\left(2^n x, 2^n y\right)}{|2|^n} = \frac{\varphi\left(a^n x, a^n y\right)}{|a|^{2n}} = 0, \tag{22}$$

for all $x, y \in \mathcal{X}$. Therefore the conditions of Theorem 4 are fulfilled. In particular, it is easy to see that

$$\begin{aligned}\widetilde{\varphi}(x) &= \max\left\{\frac{\theta\|x\|^r}{|2|\,|a|^2}, \frac{\left(|a|^r + 1\right)\theta\|x\|^r}{|2|^2|a|^r}\right\} \\ &= \frac{\left(|a|^r + 1\right)\theta\|x\|^r}{|2|^2|a|^r}.\end{aligned} \tag{23}$$

Theorem 4 guarantees that there exists a quadratic-additive mapping $T : \mathcal{X} \to \mathcal{Y}$ with (21).

Now, to show uniqueness of the mapping T, let us assume that T' is another quadratic-additive mapping satisfying (21). Then we have

$$\begin{aligned}T'(x) - J_k T'(x) &= \sum_{j=0}^{k-1}\left(-\frac{DT'\left(0, a^j x\right)}{2a^{2j+2}} - \frac{DT'\left(2^j x, 2^j x/a\right)}{2^{j+2}} + \frac{DT'\left(-2^j x, -2^j x/a\right)}{2^{j+2}}\right) \\ &= 0,\end{aligned} \tag{24}$$

for any $k \in \mathbb{N}$, and thus we feel that

$$\begin{aligned}
&\left\|T(x) - T'(x)\right\| \\
&\quad = \lim_{k\to\infty} \left\|J_k T(x) - J_k T'(x)\right\| \\
&\quad \le \lim_{k\to\infty} \max\left\{\left\|J_k T(x) - J_k f(x)\right\|, \right. \\
&\qquad\qquad \left.\left\|J_k f(x) - J_k T'(x)\right\|\right\} \\
&\quad \le \lim_{k\to\infty} \max\left\{|2|^{-k-1}\left\|T\left(2^k x\right) - f\left(2^k x\right)\right\|, \right. \\
&\qquad |2|^{-k-1}\left\|T\left(-2^k x\right) - f\left(-2^k x\right)\right\|, \\
&\qquad |2|^{-k-1}\left\|f\left(2^k x\right) - T'\left(2^k x\right)\right\|, \\
&\qquad |2|^{-k-1}\left\|f\left(-2^k x\right) - T'\left(-2^k x\right)\right\|, \\
&\qquad |2|^{-1}|a|^{-2k}\left\|T\left(a^k x\right) - f\left(a^k x\right)\right\|, \\
&\qquad |2|^{-1}|a|^{-2k}\left\|T\left(-a^k x\right) - f\left(-a^k x\right)\right\|, \\
&\qquad |2|^{-1}|a|^{-2k}\left\|f\left(a^k x\right) - T'\left(a^k x\right)\right\|, \\
&\qquad \left.|2|^{-1}|a|^{-2k}\left\|f\left(-a^k x\right) - T'\left(-a^k x\right)\right\|\right\} \\
&\quad \le \lim_{k\to\infty} \max\left\{\frac{|a|^{k(r-2)}\left(|a|^r + 1\right)\theta\|x\|^r}{|2|^3|a|^r}, \right. \\
&\qquad \left.\frac{|2|^{k(r-1)}\left(|a|^r + 1\right)\theta\|x\|^r}{|2|^3|a|^r}\right\} = 0,
\end{aligned} \tag{25}$$

for all $x \in \mathcal{X}$, which implies that T is unique. □

Theorem 6. *Let $\varphi : \mathcal{X}^2 \to [0,\infty)$ be a function such that*

$$\lim_{n\to\infty} |2|^n \varphi\left(\frac{x}{2^n}, \frac{x}{2^n}\right) = \lim_{n\to\infty} |a|^{2n} \varphi\left(\frac{x}{a^n}, \frac{y}{a^n}\right) = 0, \tag{26}$$

for all $x, y \in \mathcal{X}$ and, for each $x \in \mathcal{X}$, let the limit

$$\begin{aligned}
\lim_{n\to\infty} \max\left\{\frac{|a|^{2j}}{|2|}\varphi\left(0, \frac{x}{a^{j+1}}\right), |2|^{j-1}\varphi\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}a}\right), \right. \\
\left. |2|^{j-1}\varphi\left(\frac{-x}{2^{j+1}}, \frac{-x}{2^{j+1}a}\right) : 0 \le j < n\right\},
\end{aligned} \tag{27}$$

denoted by $\tilde{\varphi}(x)$, exist. Suppose that $f : \mathcal{X} \to \mathcal{Y}$ is a mapping satisfying

$$\left\|Df(x,y)\right\| \le \varphi(x,y), \tag{28}$$

for all $x, y \in \mathcal{X}$. Then there exists a quadratic-additive mapping $T : \mathcal{X} \to \mathcal{Y}$ such that

$$\left\|f(x) - T(x)\right\| \le \tilde{\varphi}(x), \tag{29}$$

for all $x \in \mathcal{X}$, where the mapping T is given by

$$\begin{aligned}
T(x) = \lim_{n\to\infty} \left[\frac{a^{2n}}{2}\left(f\left(\frac{x}{a^n}\right) + f\left(\frac{-x}{a^n}\right)\right) \right. \\
\left. + 2^{n-1}\left(f\left(\frac{x}{2^n}\right) - f\left(\frac{-x}{2^n}\right)\right)\right],
\end{aligned} \tag{30}$$

for all $x \in \mathcal{X}$.

Proof. For a given mapping $f : \mathcal{X} \to \mathcal{Y}$ and $n \in \mathbb{N}$, let $J_n f : \mathcal{X} \to \mathcal{Y}$ be a mapping defined by

$$\begin{aligned}
J_n f(x) = \frac{a^{2n}}{2}\left(f\left(\frac{x}{a^n}\right) + f\left(\frac{-x}{a^n}\right)\right) \\
+ 2^{n-1}\left(f\left(\frac{x}{2^n}\right) - f\left(\frac{-x}{2^n}\right)\right),
\end{aligned} \tag{31}$$

for all $x \in \mathcal{X}$. Observe that $J_0 f(x) = f(x)$ and

$$\begin{aligned}
\left\|J_j f(x) - J_{j+1} f(x)\right\| &= \left\|\frac{a^{2j}}{2} Df\left(0, \frac{x}{a^{j+1}}\right) \right. \\
&\quad + 2^{j-1} Df\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}a}\right) \\
&\quad \left. -2^{j-1} Df\left(\frac{-x}{2^{j+1}}, \frac{-x}{2^{j+1}a}\right)\right\| \\
&\le \max\left\{\frac{|a|^{2j}}{|2|}\left\|Df\left(0, \frac{x}{a^{j+1}}\right)\right\|, \right. \\
&\qquad |2|^{j-1}\left\|Df\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}a}\right)\right\|, \\
&\qquad \left.|2|^{j-1}\left\|Df\left(\frac{-x}{2^{j+1}}, \frac{-x}{2^{j+1}a}\right)\right\|\right\} \\
&\le \max\left\{\frac{|a|^{2j}}{|2|}\varphi\left(0, \frac{x}{a^{j+1}}\right), \right. \\
&\qquad |2|^{j-1}\varphi\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}a}\right), \\
&\qquad \left.|2|^{j-1}\varphi\left(\frac{-x}{2^{j+1}}, \frac{-x}{2^{j+1}a}\right)\right\},
\end{aligned} \tag{32}$$

for all $x \in \mathcal{X}$ and $j \ge 0$. It follows by (26) and (32) that the sequence $\{J_n f(x)\}$ is Cauchy. Since $\mathcal{Y}$ is complete, $\{J_n f(x)\}$ converges and so we can define $T : \mathcal{X} \to \mathcal{Y}$ by

$$T(x) := \lim_{n\to\infty} J_n f(x), \tag{33}$$

for all $x \in \mathcal{X}$. Applying the induction, we yield that

$$\begin{aligned}\|J_n f(x) - f(x)\| &\le \max\left\{\frac{|a|^{2j}}{|2|}\varphi\left(0, \frac{x}{a^{j+1}}\right), |2|^{j-1}\varphi\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}a}\right),\right.\\ &\qquad \left. |2|^{j-1}\varphi\left(\frac{-x}{2^{j+1}}, \frac{-x}{2^{j+1}a}\right) : 0 \le j < n\right\},\end{aligned} \tag{34}$$

for all $n \in \mathbb{N}$ and all $x \in \mathcal{X}$. Sending the limit as $n \to \infty$ in (34) with (26), we arrive at inequality (29).

According to (28), we see that

$$\begin{aligned}&\|DJ_n f(x, y)\|\\ &= \left\|\frac{a^{2n}}{2}\left(Df\left(\frac{x}{a^n}, \frac{y}{a^n}\right) + Df\left(\frac{-x}{a^n}, \frac{-y}{a^n}\right)\right)\right.\\ &\qquad \left. + 2^{n-1}\left(Df\left(\frac{x}{2^n}, \frac{y}{2^n}\right) - Df\left(\frac{-x}{2^n}, \frac{-y}{2^n}\right)\right)\right\|\\ &\le \max\left\{\frac{|a|^{2n}}{|2|}\varphi\left(\frac{x}{a^n}, \frac{y}{a^n}\right), \frac{|a|^{2n}}{|2|}\varphi\left(\frac{-x}{a^n}, \frac{-y}{a^n}\right),\right.\\ &\qquad \left. |2|^{n-1}\varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right), |2|^{n-1}\varphi\left(\frac{-x}{2^n}, \frac{-y}{2^n}\right)\right\},\end{aligned} \tag{35}$$

for all $x, y \in \mathcal{X}$. Taking the limit as $n \to \infty$ and using (26), we get $DT(x, y) = 0$ for all $x, y \in \mathcal{X}$. □

Corollary 7. *Let $r < 1$ be a real number and $|2|, |a| < 1$. If a mapping $f : \mathcal{X} \to \mathcal{Y}$ satisfies the condition*

$$\|Df(x, y)\| \le \theta\left(\|x\|^r + \|y\|^r\right), \tag{36}$$

for all $x, y \in \mathcal{X}$, then there exists a unique quadratic-additive mapping $T : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x) - T(x)\| \le \frac{(|a|^r + 1)\theta\|x\|^r}{|2|^{1+r}|a|^r}, \tag{37}$$

for all $x \in \mathcal{X}$.

Proof. Let $\varphi(x, y) = \theta(\|x\|^r + \|y\|^r)$. Since $|a|^{1-r} < 1$ and $|a|^{2-r} < 1$, the mapping φ satisfies equalities (26) for all $x, y \in \mathcal{X}$. Moreover, we see that

$$\widetilde{\varphi}(x) = \max\left\{\frac{\theta\|x\|^r}{|2|\,|a|^r}, \frac{(|a|^r + 1)\theta\|x\|^r}{|2|^{1+r}|a|^r}\right\} = \frac{(|a|^r + 1)\theta\|x\|^r}{|2|^{1+r}|a|^r}. \tag{38}$$

Then it follows from Theorem 6 that there exists a quadratic-additive mapping $T : \mathcal{X} \to \mathcal{Y}$ satisfying (37).

In order to prove the uniqueness of T, we assume that T' is another quadratic-additive mapping satisfying (37). Then

$$\begin{aligned}T'(x) - J_k T'(x) = \sum_{j=0}^{k-1}\left(\frac{a^{2j}}{2}DT'\left(0, \frac{x}{a^{j+1}}\right)\right.&\\ + 2^{j-1}DT'\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}a}\right)&\\ \left. - 2^{j-1}DT'\left(\frac{-x}{2^{j+1}}, \frac{-x}{2^{j+1}a}\right)\right) &= 0,\end{aligned} \tag{39}$$

for any $k \in \mathbb{N}$, whence we give that

$$\begin{aligned}&\|T(x) - T'(x)\|\\ &= \lim_{k\to\infty}\|J_k T(x) - J_k T'(x)\|\\ &\le \lim_{k\to\infty}\max\left\{\|J_k T(x) - J_k f(x)\|, \|J_k f(x) - J_k T'(x)\|\right\}\\ &\le \lim_{k\to\infty}\max\left\{|2|^{k-1}\left\|(T - f)\left(\frac{x}{2^k}\right)\right\|,\right.\\ &\qquad |2|^{k-1}\left\|(T - f)\left(\frac{-x}{2^k}\right)\right\|,\\ &\qquad |2|^{k-1}\left\|(f - T')\left(\frac{x}{2^k}\right)\right\|,\\ &\qquad |2|^{k-1}\left\|(f - T')\left(\frac{-x}{2^k}\right)\right\|,\\ &\qquad \frac{|a|^{2k}}{|2|}\left\|(T - f)\left(\frac{x}{a^k}\right)\right\|,\\ &\qquad \frac{|a|^{2k}}{|2|}\left\|(T - f)\left(\frac{-x}{a^k}\right)\right\|,\\ &\qquad \frac{|a|^{2k}}{|2|}\left\|(f - T')\left(\frac{x}{a^k}\right)\right\|,\\ &\qquad \left.\frac{|a|^{2k}}{|2|}\left\|(f - T')\left(\frac{-x}{a^k}\right)\right\|\right\}\\ &\le \lim_{k\to\infty}\max\left\{\frac{|a|^{(2-r)k}(|a|^r + 1)\theta\|x\|^r}{|2|^{2+r}|a|^r},\right.\\ &\qquad \left.\frac{|2|^{k(1-r)}(|a|^r + 1)\theta\|x\|^r}{|2|^{2+r}|a|^r}\right\} = 0,\end{aligned} \tag{40}$$

for all $x \in \mathcal{X}$, which means that T is unique. □

Theorem 8. *Let $\varphi : \mathcal{X}^2 \to [0, \infty)$ be a function such that*

$$\lim_{n\to\infty}|2|^n\varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = \lim_{n\to\infty}\frac{\varphi(a^n x, a^n y)}{|a|^{2n}} = 0, \tag{41}$$

for all $x, y \in \mathcal{X}$ and, for each $x \in \mathcal{X}$, let the limit

$$\lim_{n\to\infty} \max\left\{\frac{\varphi\left(0, a^j x\right)}{|2|\,|a|^{2j+2}}, |2|^{j-1}\varphi\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}a}\right), |2|^{j-1}\varphi\left(\frac{-x}{2^{j+1}}, \frac{-x}{2^{j+1}a}\right) : 0 \le j < n\right\}, \tag{42}$$

denoted by $\tilde{\varphi}(x)$, exist. Suppose that $f : \mathcal{X} \to \mathcal{Y}$ is a mapping satisfying

$$\|Df(x, y)\| \le \varphi(x, y), \tag{43}$$

for all $x, y \in \mathcal{X}$. Then there exists a quadratic-additive mapping $T : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x) - T(x)\| \le \tilde{\varphi}(x), \tag{44}$$

for all $x \in \mathcal{X}$, where the mapping T is given by

$$T(x) = \lim_{n\to\infty}\left[\frac{f(a^n x) + f(-a^n x)}{2a^{2n}} + 2^{n-1}\left(f\left(\frac{x}{2^n}\right) - f\left(\frac{-x}{2^n}\right)\right)\right], \tag{45}$$

for all $x \in \mathcal{X}$.

Proof. For a given mapping $f : \mathcal{X} \to \mathcal{Y}$ and $n \in \mathbb{N}$, let $J_n f : \mathcal{X} \to \mathcal{Y}$ be a mapping defined by

$$J_n f(x) = \frac{f(a^n x) + f(-a^n x)}{2a^{2n}} + 2^{n-1}\left(f\left(\frac{x}{2^n}\right) - f\left(\frac{-x}{2^n}\right)\right), \tag{46}$$

for all $x \in \mathcal{X}$. Now we note that $J_0 f(x) = f(x)$ and

$$\begin{aligned}
&\|J_j f(x) - J_{j+1} f(x)\| \\
&= \left\|-\frac{Df\left(0, a^j x\right)}{2a^{2j+2}} + 2^{j-1} Df\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}a}\right) - 2^{j-1} Df\left(\frac{-x}{2^{j+1}}, \frac{-x}{2^{j+1}a}\right)\right\| \\
&\le \max\left\{\frac{\|Df\left(0, a^j x\right)\|}{|2|\,|a|^{2j+2}}, |2|^{j-1}\left\|Df\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}a}\right)\right\|, |2|^{j-1}\left\|Df\left(\frac{-x}{2^{j+1}}, \frac{-x}{2^{j+1}a}\right)\right\|\right\} \\
&\le \max\left\{\frac{\varphi\left(0, a^j x\right)}{|2|\,|a|^{2j+2}}, |2|^{j-1}\varphi\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}a}\right), |2|^{j-1}\varphi\left(\frac{-x}{2^{j+1}}, \frac{-x}{2^{j+1}a}\right)\right\},
\end{aligned} \tag{47}$$

for all $x \in \mathcal{X}$ and $j \ge 0$. Hence it follows by (41) and (47) that the sequence $\{J_n f(x)\}$ is Cauchy. Since $\mathcal{Y}$ is complete, we conclude that $\{J_n f(x)\}$ is convergent. Thus we can define $T : \mathcal{X} \to \mathcal{Y}$ by

$$T(x) := \lim_{n\to\infty} J_n f(x), \tag{48}$$

for all $x \in \mathcal{X}$. Using the induction argument, we prove that

$$\begin{aligned}
&\|J_n f(x) - f(x)\| \\
&\le \max\left\{\frac{\varphi\left(0, a^j x\right)}{|2|\,|a|^{2j+2}}, |2|^{j-1}\varphi\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}a}\right), |2|^{j-1}\varphi\left(\frac{-x}{2^{j+1}}, \frac{-x}{2^{j+1}a}\right) : 0 \le j < n\right\},
\end{aligned} \tag{49}$$

for all $n \in \mathbb{N}$ and all $x \in \mathcal{X}$. Taking the limit as $n \to \infty$ in (49) and using (41) lead to inequality (44).

By virtue of (43), we get

$$\begin{aligned}
\|DJ_n f(x, y)\| &= \left\|\frac{Df(a^n x, a^n y) + Df(-a^n x, -a^n y)}{2a^{2n}} + 2^{n-1}\left(Df\left(\frac{x}{2^n}, \frac{y}{2^n}\right) - Df\left(\frac{-x}{2^n}, \frac{-y}{2^n}\right)\right)\right\| \\
&\le \max\left\{\frac{\varphi(a^n x, a^n y)}{|2|\,|a|^{2n}}, \frac{\varphi(-a^n x, -a^n y)}{|2|\,|a|^{2n}}, |2|^{n-1}\varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right), |2|^{n-1}\varphi\left(\frac{-x}{2^n}, \frac{-y}{2^n}\right)\right\},
\end{aligned} \tag{50}$$

for all $x, y \in \mathcal{X}$. Send the limit as $n \to \infty$ and then use (41) to find $DT(x, y) = 0$ for all $x, y \in \mathcal{X}$. □

Theorem 9. *Let $\varphi : \mathcal{X}^2 \to [0, \infty)$ be a function such that*

$$\lim_{n\to\infty}\frac{\varphi(2^n x, 2^n y)}{|2|^n} = \lim_{n\to\infty}|a|^{2n}\varphi\left(\frac{x}{a^n}, \frac{y}{a^n}\right) = 0, \tag{51}$$

for all $x, y \in \mathcal{X}$ and, for each $x \in \mathcal{X}$, let the limit

$$\lim_{n\to\infty}\max\left\{\frac{|a|^{2j}}{|2|}\varphi\left(0, \frac{x}{a^{j+1}}\right), \frac{\varphi\left(2^j x, 2^j x/a\right)}{|2|^{j+2}}, \frac{\varphi\left(-2^j x, -2^j x/a\right)}{|2|^{j+2}} : 0 \le j < n\right\}, \tag{52}$$

denoted by $\tilde{\varphi}(x)$, exist. Suppose that $f : \mathcal{X} \to \mathcal{Y}$ is a mapping satisfying

$$\|Df(x, y)\| \le \varphi(x, y), \tag{53}$$

for all $x, y \in \mathcal{X}$. Then there exists a quadratic-additive mapping $T : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x) - T(x)\| \le \tilde{\varphi}(x), \tag{54}$$

for all $x \in \mathcal{X}$, where the mapping T is given by

$$T(x) = \lim_{n\to\infty}\left[\frac{a^{2n}}{2}\left(f\left(\frac{x}{a^n}\right) + f\left(\frac{-x}{a^n}\right)\right) + \frac{f(2^n x) - f(-2^n x)}{2^{n+1}}\right], \tag{55}$$

for all $x \in \mathcal{X}$.

Proof. For a given mapping $f : \mathcal{X} \to \mathcal{Y}$ and $n \in \mathbb{N}$, let $J_n f : \mathcal{X} \to \mathcal{Y}$ be a mapping defined by

$$J_n f(x) = \frac{a^{2n}}{2}\left(f\left(\frac{x}{a^n}\right) + f\left(\frac{-x}{a^n}\right)\right) + \frac{f(2^n x) - f(-2^n x)}{2^{n+1}}, \tag{56}$$

for all $x \in \mathcal{X}$. We remark that $J_0 f(x) = f(x)$ and

$$\begin{aligned}\|J_j f(x) - J_{j+1} f(x)\| &= \left\|\frac{a^{2j}}{2} Df\left(0, \frac{x}{a^{j+1}}\right) - \frac{Df(2^j x, 2^j x/a)}{2^{j+2}} + \frac{Df(-2^j x, -2^j x/a)}{2^{j+2}}\right\| \\ &\le \max\left\{\frac{|a|^{2j}}{|2|}\left\|Df\left(0, \frac{x}{a^{j+1}}\right)\right\|, \frac{\|Df(2^j x, 2^j x/a)\|}{|2|^{j+2}}, \frac{\|Df(-2^j x, -2^j x/a)\|}{|2|^{j+2}}\right\} \\ &\le \max\left\{\frac{|a|^{2j}}{|2|}\varphi\left(0, \frac{x}{a^{j+1}}\right), \frac{\varphi(2^j x, 2^j x/a)}{|2|^{j+2}}, \frac{\varphi(-2^j x, -2^j x/a)}{|2|^{j+2}}\right\},\end{aligned} \tag{57}$$

for all $x \in \mathcal{X}$ and $j \ge 0$. So it follows from (51) and (57) that the sequence $\{J_n f(x)\}$ is Cauchy. Due to the completeness of $\mathcal{Y}$, this sequence is convergent. Let $T : \mathcal{X} \to \mathcal{Y}$ be a mapping defined by

$$T(x) := \lim_{n\to\infty} J_n f(x), \tag{58}$$

for all $x \in \mathcal{X}$. An induction implies that

$$\|J_n f(x) - f(x)\| \le \max\left\{\frac{|a|^{2j}}{|2|}\varphi\left(0, \frac{x}{a^{j+1}}\right), \frac{\varphi(2^j x, 2^j x/a)}{|2|^{j+2}}, \frac{\varphi(-2^j x, -2^j x/a)}{|2|^{j+2}} : 0 \le j < n\right\}, \tag{59}$$

for all $n \in \mathbb{N}$ and all $x \in \mathcal{X}$. By passing the limit as $n \to \infty$ in (59) with (51), we obtain the relation (54).

From (53), we find that

$$\begin{aligned}\|DJ_n f(x, y)\| &= \left\|\frac{a^{2n}}{2}\left(Df\left(\frac{x}{a^n}, \frac{y}{a^n}\right) + Df\left(\frac{-x}{a^n}, \frac{-y}{a^n}\right)\right) + \frac{Df(2^n x, 2^n y) - 2^n Df(-2^n x, -2^n y)}{2^{n+1}}\right\| \\ &\le \max\left\{\frac{|a|^{2n}}{|2|}\varphi\left(\frac{x}{a^n}, \frac{y}{a^n}\right), \frac{|a|^{2n}}{|2|}\varphi\left(\frac{-x}{a^n}, \frac{-y}{a^n}\right), \frac{\varphi(2^n x, 2^n y)}{|2|^{n+1}}, \frac{\varphi(-2^n x, -2^n y)}{|2|^{n+1}}\right\},\end{aligned} \tag{60}$$

for all $x, y \in \mathcal{X}$. Taking the limit as $n \to \infty$ and using (51), we arrive at $DT(x, y) = 0$ for all $x, y \in \mathcal{X}$. □

Corollary 10. *Let $|a|, |2| < 1$ and let r be a real number such that $1 < r < 2$. If a mapping $f : \mathcal{X} \to \mathcal{Y}$ satisfies the condition*

$$\|Df(x, y)\| \le \theta\left(\|x\|^r + \|y\|^r\right), \tag{61}$$

for all $x, y \in \mathcal{X}$, then there exists a unique quadratic-additive mapping $T : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x) - T(x)\| \le \frac{(|a|^r + 1)\theta\|x\|^r}{|2|^2|a|^r}, \tag{62}$$

for all $x \in \mathcal{X}$.

Proof. We consider $\varphi(x, y) = \theta(\|x\|^r + \|y\|^r)$. Based on the fact that $|a|^{2-r} < 1$ and $|2|^{r-1} < 1$, the mapping φ satisfies conditions (51). In fact, it is easy to see that

$$\begin{aligned}\tilde{\varphi}(x) &= \max\left\{\frac{\theta\|x\|^r}{|2|\,|a|^r}, \frac{(|a|^r + 1)\theta\|x\|^r}{|2|^2|a|^r}\right\} \\ &= \frac{(|a|^r + 1)\theta\|x\|^r}{|2|^2|a|^r}.\end{aligned} \tag{63}$$

On account of Theorem 9, there is a quadratic-additive mapping $T : \mathcal{X} \to \mathcal{Y}$ satisfying (62).

To show uniqueness of the mapping T, we suppose that T' is another quadratic-additive mapping satisfying (62). Then

$$\begin{aligned} T'(x) - J_k T'(x) &= \sum_{j=0}^{k-1} \left(\frac{a^{2j}}{2} Df\left(0, \frac{x}{a^{j+1}}\right) \right. \\ &\quad \left. - \frac{Df\left(2^j x, 2^j x/a\right)}{2^{j+2}} + \frac{Df\left(-2^j x, -2^j x/a\right)}{2^{j+2}} \right) = 0, \end{aligned} \tag{64}$$

for any $k \in \mathbb{N}$ and so we figure out the following:

$$\begin{aligned} \left\| T(x) - T'(x) \right\| &= \lim_{k\to\infty} \left\| J_k T(x) - J_k T'(x) \right\| \\ &\le \lim_{k\to\infty} \max\left\{ \left\| J_k T(x) - J_k f(x) \right\|, \right. \\ &\qquad \left. \left\| J_k f(x) - J_k T'(x) \right\| \right\} \\ &\le \lim_{k\to\infty} \max\left\{ |2|^{-k-1} \left\| T\left(2^k x\right) - f\left(2^k x\right) \right\|, \right. \\ &\qquad |2|^{-k-1} \\ &\qquad \times \left\| T\left(-2^k x\right) - f\left(-2^k x\right) \right\|, \\ &\qquad |2|^{-k-1} \left\| f\left(2^k x\right) - T'\left(2^k x\right) \right\|, \\ &\qquad |2|^{-k-1} \\ &\qquad \times \left\| f\left(-2^k x\right) - T'\left(-2^k x\right) \right\|, \\ &\qquad \frac{|a|^{2k}}{|2|} \left\| (T - f)\left(\frac{x}{a^k}\right) \right\|, \\ &\qquad \frac{|a|^{2k}}{|2|} \left\| (T - f)\left(\frac{-x}{a^k}\right) \right\|, \\ &\qquad \frac{|a|^{2k}}{|2|} \left\| \left(f - T'\right)\left(\frac{x}{a^k}\right) \right\|, \\ &\qquad \left. \frac{|a|^{2k}}{|2|} \left\| \left(f - T'\right)\left(\frac{-x}{a^k}\right) \right\| \right\} \\ &\le \lim_{k\to\infty} \max\left\{ \frac{|a|^{k(2-r)} \left(|a|^r + 1\right) \theta \|x\|^r}{|2|^3 |a|^r}, \right. \\ &\qquad \left. \frac{|2|^{k(r-1)} \left(|a|^r + 1\right) \theta \|x\|^r}{|2|^3 |a|^r} \right\} \\ &= 0, \end{aligned} \tag{65}$$

for all $x \in \mathcal{X}$. This implies that T is unique. □

The problem is whether or not the previous corollaries hold the cases when $r = 1$ or $r = 2$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

[1] S. M. Ulam, *A Collection of the Mathematical Problems*, Interscience, New York, NY, USA, 1960.

[2] D. H. Hyers, "On the stability of the linear functional equation," *The Proceedings of the National Academy of Sciences of the United States of America*, vol. 27, pp. 222–224, 1941.

[3] T. Aoki, "On the stability of the linear transformation in Banach spaces," *Journal of the Mathematical Society of Japan*, vol. 2, pp. 64–66, 1950.

[4] T. M. Rassias, "On the stability of the linear mapping in Banach spaces," *Proceedings of the American Mathematical Society*, vol. 72, pp. 297–300, 1978.

[5] N. Brillouet-Belluot, J. Brzdęk, K. Cieplinski, and J. Szczawiriska, "On some recent developments in Ulam's type stability," *Abstract and Applied Analysis*, vol. 2012, Article ID 716936, 41 pages, 2012.

[6] K. Cieplinski, "Stability of multi-Jensen mappings in non-Archimedean normed spaces," in *Fuctional Equations in Mathematical Anaysis*, T. M. Rassias and J. Brzdęk, Eds., vol. 52 of *Springer Optimization and Its Applications*, pp. 79–86, Springer, New York, NY, USA, 2012.

[7] P. Gavruta, "A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings," *Journal of Mathematical Analysis and Applications*, vol. 184, pp. 431–436, 1994.

[8] S. Jung, *On the Hyers-Ulam-Rassias Stability of Functional Equation in Nonlinear Analysis*, Springer, New York, NY, USA, 2011.

[9] A. Najati and M. B. Moghimi, "Stability of a functional equation deriving from quadratic and additive functions in quasi-Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 337, no. 1, pp. 399–415, 2008.

[10] T. Z. Xu, "On the stability of multi-Jesen mappings in non-Archimedean normed spaces," *Journal of Mathematical Physics*, vol. 25, Article ID 023507, 9 pages, 2012.

[11] T. Z. Xu, J. M. Rassias, and W. X. Xu, "A fixed point approach to the stability of a general mixed AQCQ-functional equation in non-archimedean normed spaces," *Discrete Dynamics in Nature and Society*, vol. 2010, Article ID 812545, 24 pages, 2010.

[12] J. Brzdęk and K. Ciepliński, "A fixed point approach to the stability of functional equations in non-Archimedean metric spaces," *Nonlinear Analysis, Theory, Methods and Applications*, vol. 74, no. 18, pp. 6861–6867, 2011.

[13] J. Brzdęk and K. Cieplinski, "A fixed point theorem and the Hyers-Ulam stability in non- Archimedean spaces," *Journal of Mathematical Analysis and Applications*, vol. 400, pp. 68–75, 2013.

[14] M. S. Moslehian and T. M. Rassias, "Stability of functional equations in non-archimedean spaces," *Applicable Analysis and Discrete Mathematics*, vol. 1, no. 2, pp. 325–334, 2007.

[15] A. Bahyrycz, J. Brzdęk, and M. Piszczek :, "On approximately p-Wright affine functions in ultra metric spaces," *Journal of Function Spaces and Applications*, vol. 2013, Article ID 723545, 7 pages, 2013.

[16] Y. Lee, "Hyers-Ulam-Rassias stability of a quadratic-additive type functional equation on a restricted domain," *International Journal of Mathematical Analysis*, vol. 7, pp. 2745–2752, 2013.

[17] M. Piszczek and J. Szczawiska, "Hyperstability of the Drygas functional equations," *Journal of Function Spaces and Applications*, vol. 2013, Article ID 912718, 4 pages, 2013.