Research Article

Blowup of Solution for a Class of Doubly Nonlinear Parabolic Systems

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An initial boundary value problem for a class of doubly parabolic equations is studied. We obtain sufficient conditions for the blowup of solutions under suitable initial data using differential inequalities.

1. Introduction

In this paper, we study the following initial boundary value problem of a class of reaction diffusion equations with multiple nonlinearities:

$$u_t - \Delta u_t - \Delta u + |u|^{k-2} u_t = |u|^{p-2} u, \quad (1)$$

$$u(x, t) = 0, \quad x \in \partial\Omega, \quad (2)$$

$$u(x, 0) = u_0(x), \quad x \in \Omega, \quad (3)$$

where $k > 2$, $p > 2$ are real numbers and Ω is bounded domain in R^n with smooth boundary $\partial\Omega$ so that the divergence theorem can be applied. Here Δ denotes the Laplace operator in Ω .

This type of problems not only is important from the theoretical point of view, but also arises in many physical applications and describes a great deal of models in applied science. It appears in the models of chemical reactions, heat transfer, and population dynamics (see [1] and references therein). Equation (1) can describe an electric breakdown in crystalline semiconductors with allowance for the linear dissipation of bound- and free-charge sources [2, 3].

In the absence of the nonlinear diffusion term $|u|^{k-2} u_t$, (1) reduced to the following equation

$$u_t - \Delta u - \Delta u_t = |u|^{p-2} u, \quad (4)$$

which is called pseudoparabolic equation (see [4] and the references). A related problem to (4) without term $-\Delta u_t$ has attracted a great deal of attention in the last two decades, and many results appeared on the existence, blowup, and asymptotic behavior of solution. It is well known that the nonlinear $|u|^{p-2} u$ reaction term drives the solution of (4) to blowup in finite time. The diffusion term is known to yield existence of global solution if the reaction term is removed from [5]. The more general equation,

$$u_t - \operatorname{div}(|\nabla u|^{m-2} \nabla u) = f(u), \quad (5)$$

has also attracted a great deal of people and the known results show that global existence and nonexistence depend roughly on m , the degree of nonlinearity in f , the dimension n , and the size of the initial data. See, in this regard, the works of Levine [6], Kalantarov and Ladyzhenskaya [7], Levine et al. [8], Messaoudi [9], Liu and Wang [10] and references therein. Pucci and Serrin [11] have discussed the stability of the following equation:

$$|u_t|^{l-2} u_t - \operatorname{div}(|\nabla u|^{m-2} \nabla u) = f(u). \quad (6)$$

Levine et al. [8] got the global existence and nonexistence of solution for (6). Pang et al. [12, 13] and Berrimi and Messaoudi [14] gave the sufficient condition of blowup result for certain solutions of (6) with positive or negative initial energy.

Equation (1) without term $-\Delta u_t$ can also be a special case of doubly nonlinear parabolic-type equations (or the porous medium equation) [8, 15, 16],

$$\beta(u)_t - \Delta u = |u|^{p-2} u, \quad (7)$$

if we take $\beta(u) = u + |u|^{m-2}u$. The authors of [15, 16] took (7) as dynamical systems and studied their attractors. Levine and Sacks [17, 18] and Blanchard and Francfort [19] proved the existence of the solution.

Polat [20] established a blowup result for the solution with vanishing initial energy of the following initial boundary value problem:

$$u_t - u_{xx} + |u|^{k-2} u_t = |u|^{p-2} u. \quad (8)$$

They also gave detailed results of the necessary and sufficient blowup conditions together with blowup rate estimates for the positive solution of the problem,

$$(u^m)_t - \Delta u = f(u), \quad (9)$$

subject to various boundary conditions. Korpusov and Sveshnikov [2, 3] gave the local strong solution and the sufficient close-to-necessary conditions for the blowup of solutions to the problem,

$$u_t - \Delta u_t - \Delta u + |u|^{k-2} u_t = u(u + \alpha)(u - \beta), \quad (10)$$

with initial boundary values (2) and (3) in R^3 for $\alpha, \beta > 0$ by the convex method [6, 7].

In this paper, we will investigate the problem (1)–(3) and there are few results of the problem to our knowledge. We will give sufficient conditions for the blowup of solutions in a finite time interval under suitable initial data using differential inequalities. An essential tool of the proof is an idea used in [21, 22], which was based on an auxiliary function (which is a small perturbation of the total energy), using differential inequalities and obtaining the result. It is different with the result of [2, 3]. This paper is organized as follows. Section 2 is concerned with some notations and statement of assumptions. In Section 3, we give and prove that the result if the initial energy $E(0)$ of our solutions is negative (this means that our initial data are large enough) or the initial energy is $E(0) > 0$.

2. Preliminaries

In this section, we will give some notations and statement of assumptions for m, p, g . We denote $L^p(\Omega)$ by L^p , $H_0^1(\Omega)$ by H_0^1 , the usual Sobolev space. The norm and inner of $L^p(\Omega)$ are denoted by $\|\cdot\|_p = \|\cdot\|_{L^p(\Omega)}$ and $(u, v) = \int_{\Omega} u(x)v(x)dx$, respectively. Particularly, $\|\cdot\| = \|\cdot\|_{L^2(\Omega)}$ for $p = 2$.

For the numbers k and p , we assume that

$$\begin{aligned} 2 < k < p \leq \frac{2(n-1)}{n-2}, \quad & \text{if } n \geq 3, \\ 2 < k < p < +\infty, \quad & \text{if } n = 1, 2. \end{aligned} \quad (11)$$

Similar to [2], we call $u(x, t)$ a solution of problem (1)–(3) on $\Omega \times [0, T)$, if

$$u \in C^1(0, T; H_0^1), \quad |u|^{k-2} u_t \in L^2(\Omega \times [0, T)), \quad (12)$$

satisfying $u(x, 0) = u_0(x)$ and

$$\begin{aligned} \int_{\Omega} [\nabla u(s) \nabla v(s) + \nabla u_t(s) \nabla v(s) + u_t(s) v(s) \\ + |u|^{k-2} u_t v - |u|^{p-2} uv] dx = 0, \end{aligned} \quad (13)$$

$$\forall v \in C(0, T; H_0^1).$$

Now, we introduce two functionals:

$$\begin{aligned} E(t) = E(u) &= \frac{1}{2} \|\nabla u\|^2 - \frac{1}{p} \|u\|_p^p, \\ E(0) &= \frac{1}{2} \|\nabla u_0\|^2 - \frac{1}{p} \|u_0\|_p^p, \end{aligned} \quad (14)$$

where $u \in H_0^1$. Multiplying (1) by u_t and integrating over Ω , we have

$$E'(t) = -\|u_t\|^2 - \|\nabla u_t\|^2 - \int_{\Omega} |u|^{k-2} u_t^2 dx < 0, \quad (15)$$

and then

$$E(t) \leq E(0). \quad (16)$$

3. Blowup of Solution

In this section, we will prove the main result. Our techniques of proof follow very carefully the techniques used in [21, 22].

Theorem 1. *Suppose that the assumption about k, p hold, $u_0 \in H_0^1$ and u is a local solution of the system (1)–(3), and $E(0) < 0$ is sufficient negative. Then the solution of the system (1)–(3) blows up in finite time.*

Proof. We set

$$H(t) = -E(t). \quad (17)$$

By the definition of $H(t)$ and (15)

$$H'(t) = -E'(t) \geq 0. \quad (18)$$

Consequently, by $E(0) < 0$, we have

$$H(0) = -E(0) > 0. \quad (19)$$

It is clear that by (18) and (19)

$$0 < H(0) \leq H(t). \quad (20)$$

By (17) and the expression of $E(t)$,

$$H(t) - \frac{1}{p} \|u\|_p^p = -\frac{1}{2} \|\nabla u\|^2 < 0, \quad (21)$$

One implies

$$0 < H(0) \leq H(t) \leq \frac{1}{p} \|u\|_p^p. \quad (22)$$

Let us define the functional

$$L(t) = H^{1-\sigma}(t) + \frac{\epsilon}{2} \|\nabla u\|^2 + \frac{\epsilon}{2} \|u\|^2, \quad (23)$$

where $\epsilon > 0$ will be fixed in the later and $0 < \sigma \leq (p-k)/p$ (this can be done since $k < p$). By taking the time derivative of (23) and by (1), we have

$$\begin{aligned} L'(t) &= (1-\sigma) H^{-\sigma}(t) H'(t) + \epsilon \int_{\Omega} \nabla u \nabla u_t dx \\ &\quad + \epsilon \int_{\Omega} u u_t dx \\ &= (1-\sigma) H^{-\sigma}(t) H'(t) + 2\epsilon H(t) + 2\epsilon E(t) \\ &\quad - \epsilon \|\nabla u\|^2 + \epsilon \|u\|_p^p - \epsilon \int_{\Omega} |u|^{k-2} u u_t dx \\ &\geq (1-\sigma) H^{-\sigma}(t) H'(t) + 2\epsilon H(t) \\ &\quad + \epsilon \left(1 - \frac{2}{p}\right) \|u\|_p^p - \epsilon \int_{\Omega} |u|^{k-2} u u_t dx. \end{aligned} \quad (24)$$

To estimate the last term in the right-hand side of (24), we use the following Young's inequality

$$ab \leq \delta^{-1} a^2 + \delta b^2, \quad (25)$$

for any $\delta > 0$, we have

$$\begin{aligned} \int_{\Omega} |u|^{k-2} u u_t dx &= \int_{\Omega} |u|^{(k-2)/2} u_t |u|^{(k-2)/2} u dx \\ &\leq \delta^{-1} \int_{\Omega} |u|^{k-2} u_t^2 dx + \delta \int_{\Omega} |u|^k dx. \end{aligned} \quad (26)$$

Therefore, we have

$$\begin{aligned} L'(t) &\geq (1-\sigma) H^{-\sigma}(t) H'(t) + 2\epsilon H(t) \\ &\quad + \epsilon \left(1 - \frac{2}{p}\right) \|u\|_p^p - \epsilon \delta \|u\|_k^k \\ &\quad - \epsilon \delta^{-1} \int_{\Omega} |u|^{k-2} u_t^2 dx. \end{aligned} \quad (27)$$

By choosing δ such that $\delta^{-1} = MH^{-\sigma}(t)$ for M enough large constants to be fixed later, and by using (15), we have

$$\begin{aligned} L'(t) &\geq (1-\sigma - M\epsilon) H^{-\sigma}(t) H'(t) + 2\epsilon H(t) \\ &\quad + \epsilon \left(1 - \frac{2}{p}\right) \|u\|_p^p - \epsilon M^{-1} H^{\sigma}(t) \|u\|_k^k. \end{aligned} \quad (28)$$

Since $p > k > 2$ and by embedding theorem, taking into account (22), we obtain, for some positive constants C_1 and C_2 ,

$$H^{\sigma}(t) \|u\|_k^k \leq C_1 \|u\|_p^{p\sigma} \|u\|_k^k \leq C_2 \|u\|_p^{p\sigma+k}. \quad (29)$$

Since $0 < k/p < 1$, now applying the inequality $x^l \leq (x+1) \leq (1+(1/z))(x+z)$, which holds for all $x \geq 0$, $0 \leq l \leq 1$, $z > 0$, in particular, by the choice of σ , taking $x = \|u\|_p^p$, $l = (p\sigma+k)/p$, $z = H(0)$, and by using (22), we have

$$\|u\|_p^{p\sigma+k} \leq \left(1 + \frac{1}{H(0)}\right) (\|u\|_p^p + H(0)) \leq C_3 \|u\|_p^p. \quad (30)$$

Taking into account (28) and (30), we have

$$\begin{aligned} L'(t) &\geq (1-\sigma - M\epsilon) H^{-\sigma}(t) H'(t) + \epsilon H(t) \\ &\quad + \epsilon \left(1 - \frac{2}{p} - C_3 M^{-1}\right) \|u\|_p^p. \end{aligned} \quad (31)$$

For large M such that $1 - (2/p) - C_3 M^{-1} = C_4 > 0$, once M is fixed, we pick ϵ small enough such that $1 - \sigma - M\epsilon > 0$, then there exist $C_5 > 0$ such that (31) become

$$L'(t) \geq C_5 (H(t) + \|u\|_p^p). \quad (32)$$

Then, we have

$$L(t) \geq L(0) \geq 0. \quad (33)$$

On the other hand, by the definition of $L(t)$ and (21), we have

$$\begin{aligned} L(t) &= H^{1-\sigma}(t) - \epsilon \left(H(t) - \frac{1}{p} \|u\|_p^p\right) + \frac{\epsilon}{2} \|u\|^2 \\ &\leq (1-\epsilon) H^{1-\sigma}(t) + \frac{\epsilon}{p} \|u\|_p^p + \frac{\epsilon}{2} \|u\|^2, \end{aligned} \quad (34)$$

where we have used the fact $H(t) \geq H^{1-\sigma}(t)$ (this can be ensured by (19), (20), $0 < \sigma < 1$ and $E(0)$ is sufficient negative). Now, by inequality $x^l \leq (1+(1/z))(x+z)$ again by taking $x = \|u\|_p^{p/(1-\sigma)}$, $l = 1 - \sigma < 1$, $z = H^{1/(1-\sigma)}(0)$, we have

$$\begin{aligned} \|u\|_p^p &\leq \left(1 + \frac{1}{H^{1/(1-\sigma)}(0)}\right) (\|u\|_p^{p/(1-\sigma)} + H^{1/(1-\sigma)}(0)) \\ &\leq C_6 \|u\|_p^{p/(1-\sigma)}. \end{aligned} \quad (35)$$

Therefore, we get

$$L(t) \leq (1-\epsilon) H^{1-\sigma}(t) + C_6 \|u\|_p^{p/(1-\sigma)} + \frac{\epsilon}{2} \|u\|^2. \quad (36)$$

Then, by embedding theorem since $p > 2$, we have, for fixed ϵ sufficient small,

$$L^{1/(1-\sigma)}(t) \leq C_7 [H(t) + \|u\|_p^p + \|u\|_p^{2/(1-\sigma)}]. \quad (37)$$

Now, by inequality $x^l \leq (1+(1/z))(x+z)$ again by taking $x = \|u\|_p^p$, $l = 2/p(1-\sigma) < 1$, since $\sigma < (p-k)/p < (p-2)/p$, $z = H(0)$, we have

$$\begin{aligned} \|u\|_p^{2/(1-\sigma)} &= (\|u\|_p^p)^{2/p(1-\sigma)} \leq \left(1 + \frac{1}{H(0)}\right) (\|u\|_p^p + H(0)) \\ &\leq C_8 \|u\|_p^p. \end{aligned} \quad (38)$$

From (37) and (38), we obtain

$$L^{1/(1-\sigma)}(t) \leq C_9 [H(t) + \|u\|_p^p]. \quad (39)$$

Combining with (32) and (39), we arrive to

$$L'(t) \geq C_{10} L^{1/(1-\sigma)}(t). \quad (40)$$

Integration of (40) between 0 and t gives the desired results.

In the following, we will prove that the energy will grow up as an exponential function as time goes to infinity, provided that the initial energy $E(0) > 0$.

The following lemma will play an essential role in the proof of our main result, and it is similar to a Lemma used firstly by Vitillaro [23]. In order to give the result and for the sake of simplicity, we set

$$\lambda_1 = C_*^{-p/(p-2)}, \quad E_1 = \left(\frac{1}{2} - \frac{1}{p}\right) \lambda_1^2, \quad (41)$$

where C_* is the best Poincaré's constant. $\square$

Lemma 2 (see [22]). *Let u be a solution of (1)–(3). Suppose that the assumption of k, p hold. Assume further that $E(0) < E_1$ and $\|\nabla u_0\| > \lambda_1$. Then there exists a constant $\lambda_2 > \lambda_1$ such that $\|\nabla u\| > \lambda_2$.*

Theorem 3. *Suppose that the assumption about k, p hold, $u_0 \in H_0^1$ and u is a local solution of the system (1)–(3), $\|\nabla u_0\| > \lambda_1$ and $E(0) < E_1$. Then the solution of the system (1)–(3) blows up.*

Proof. We set

$$H(t) = E_2 - E(t), \quad (42)$$

where E_2 is a constant and $E(0) < E_2 < E_1$. By the definition of $H(t)$ and (15)

$$H'(t) = -E'(t) \geq 0. \quad (43)$$

Consequently,

$$H(0) = E_2 - E(0) > 0. \quad (44)$$

It is clear that by (43) and (44)

$$0 < H(0) \leq H(t). \quad (45)$$

By (42), the expression of $E(t)$, and Lemma 2

$$\begin{aligned} H(t) &= E_2 - \frac{1}{2} \|\nabla u\|^2 + \frac{1}{p} \|u\|_p^p \\ &\leq E_1 - \frac{1}{2} \lambda_1^2 + \frac{1}{p} \|u\|_p^p \\ &= -\frac{1}{p} \lambda_1^2 + \frac{1}{p} \|u\|_p^p < \frac{1}{p} \|u\|_p^p. \end{aligned} \quad (46)$$

One implies

$$0 < H(0) \leq H(t) \leq \frac{1}{p} \|u\|_p^p. \quad (47)$$

Then we can prove the theorem similar to the proof of Theorem 1. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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