Research Article

Norm Comparison Estimates for the Composite Operator

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This paper obtains the Lipschitz and BMO norm estimates for the composite operator $\mathbb{M}_s \circ P$ applied to differential forms. Here, $\mathbb{M}_s$ is the Hardy-Littlewood maximal operator, and P is the potential operator. As applications, we obtain the norm estimates for the Jacobian subdeterminant and the generalized solution of the quasilinear elliptic equation.

1. Introduction

Since the differential form has been proposed by Elie Cartan, its development is obvious. Differential form theory has been applied to many fields, such as partial differential equations, nonlinear analysis, and control theory. Particularly, it is an important part in differential forms to research norm inequalities for an operator or composite operator. L^p norm has been studied very thoroughly [1–3], so we try to establish the Lipschitz norm and BMO norm inequalities for the composition of the Hardy-Littlewood maximal operator and potential operator. It is useful to explore the properties of the weighted norms; see [4–7]. So, we also research the weighted Lipschitz norm and the weighted BMO norm inequalities.

As the traditional notations, we write Ω for a bounded convex domain in $\mathbb{R}^n$, $n \geq 2$, endowed with the usual Lebesgue measure denoted by $|\Omega|$. B and σB are concentric balls, with $\text{diam}(\sigma B) = \sigma \text{diam}(B)$. ω denotes a weight defined by $\omega \in L_{\text{loc}}(\mathbb{R}^n)$, and $\omega > 0$ a.e.. The l -form, denoted by $\Lambda^l = \Lambda^l(\mathbb{R}^n)$, is a l -vector, spanned by exterior products $e_I = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_l}$, for all ordered l -tuples $I = (i_1, i_2, \dots, i_l)$, $1 \leq i_1 < i_2 < \cdots < i_l \leq n$. The l -form $u(x) = \sum_I u_I(x) dx_I$ is called a differential l -form, if u_I is differential. We use $D'(\Omega, \Lambda^l)$ to denote the differential l -form and $L^s(\Omega, \Lambda^l)$ to denote the l -form $u(x)$ on Ω satisfying $\int_{\Omega} |u_I|^s < \infty$. In particular, we know that the 0-form is a function. We

define the exterior derivative $d : D'(\Omega, \Lambda^l) \rightarrow D'(\Omega, \Lambda^{l+1})$ by

$$\begin{aligned} d(u(x)) &= \sum_{k=1}^n \sum_{1 \leq i_1 < i_2 < \cdots < i_l \leq n} \frac{\partial u_{i_1 i_2 \cdots i_l}(x)}{\partial x_k} \\ &\quad \times dx_k \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_l}, \\ &\quad l = 0, 1, \dots, n-1. \end{aligned} \quad (1)$$

We define $\star$ by

$$\star u = \text{sign}(\pi) u_{i_1 i_2 \cdots i_l}(x) dx_{j_1} \wedge \cdots \wedge dx_{j_{n-l}}, \quad (2)$$

where $\pi = (i_1, \dots, i_l, j_1, \dots, j_{n-l})$ is a permutation of $(1, \dots, n)$ and $\text{sign}(\pi)$ is the signature of the permutation. Now, we can define the Hodge codifferential d^* like this

$$\begin{aligned} d^* &= (-1)^{n-l+1} \star d \star : D'(\Omega, \Lambda^{l+1}) \rightarrow D'(\Omega, \Lambda^l), \\ &\quad l = 0, 1, \dots, n-1. \end{aligned} \quad (3)$$

The differential form we research here satisfies the nonhomogeneous A -harmonic equation

$$d^* A(x, du) = B(x, du), \quad (4)$$

where $A : \Omega \times \Lambda^l(\mathbb{R}^n) \rightarrow \Lambda^l(\mathbb{R}^n)$ and $B : \Omega \times \Lambda^{l-1}(\mathbb{R}^n) \rightarrow \Lambda^{l-1}(\mathbb{R}^n)$ satisfy the conditions

$$\begin{aligned} |A(x, \xi)| &\leq a|\xi|^{p-1}, \\ A(x, \xi) \cdot \xi &\geq |\xi|^p, \quad |B(x, \xi)| \leq b|\xi|^{p-1} \end{aligned} \quad (5)$$

for almost every $x \in \Omega$ and all $\xi \in \Lambda^l(\mathbb{R}^n)$. Here $a, b > 0$ are constants and $1 < p < \infty$ is a fixed exponent associated with (4).

2. Lipschitz and BMO Norm Inequalities

In this section, we introduce some definitions. Then, we give main lemmas used in theorems. Finally, we give the norm comparison estimates for the composite operator.

For $u \in L^1_{\text{loc}}(\Omega, \Lambda^l)$, $l = 0, 1, \dots, n$, we write $u \in \text{loc Lip}_k(\Omega, \Lambda^l)$, $0 \leq k \leq 1$, if

$$\|u\|_{\text{loc Lip}_k, \Omega} = \sup_{\sigma Q \subset \Omega} |Q|^{-(n+k)/n} \|u - u_Q\|_{1, Q} < \infty \quad (6)$$

for some $\sigma > 1$.

For $u \in L^1_{\text{loc}}(\Omega, \Lambda^l)$, $l = 0, 1, \dots, n$, we say $u \in \text{BMO}(\Omega, \Lambda^l)$, if

$$\|u\|_{*, \Omega} = \sup_{\sigma Q \subset \Omega} |Q|^{-1} \|u - u_Q\|_{1, Q} < \infty \quad (7)$$

for some $\sigma > 1$.

Hardy-Littlewood maximal operator is defined by

$$\mathbb{M}_s(u) = \mathbb{M}_s u(x) = \sup_{r>0} \left(\frac{1}{|B(x, r)|} \int_{B(x, r)} |u(y)|^s dy \right)^{1/s}, \quad (8)$$

where $1 \leq s < \infty$, u is a locally L^s -integrable form and $B(x, r)$ is a ball with radius r .

Potential operator is first extended to differential form by Bi in [8], which is defined as follows:

$$Pu = Pu(x) = \sum_I \int_{\Omega} K(x, y) u_I(y) dy dx_I, \quad (9)$$

where the kernel $K(x, y)$ is a nonnegative measurable function defined for $x \neq y$, u is a differential l -form, and the summation is over all ordered l -tuples I .

We need the following two lemmas for the Hardy-Littlewood maximal operator and potential operator to prove Theorem 7.

Lemma 1 (see [9]). *Let $u \in L^t(\Omega, \Lambda^l)$, $l = 0, 1, \dots, n$, be a differential form in a domain Ω and $\mathbb{M}_s$ be the Hardy-Littlewood maximal operator, $1 \leq s < t < \infty$. Then, $\mathbb{M}_s(u) \in L^t(\Omega)$ and there exists a constant C , independent of u , such that*

$$\|\mathbb{M}_s(u)\|_{t, \Omega} \leq C \|u\|_{t, \Omega}. \quad (10)$$

Lemma 2 (see [8]). *Let $u \in D^l(\Omega, \Lambda^l)$, $l = 0, 1, \dots, n$, be a differential form in a domain Ω and let P be the potential*

operator. Then, there exists a constant C , independent of u , such that

$$\|P(u)\|_{s, \Omega} \leq C \|u\|_{s, \Omega}, \quad (11)$$

where $1 < s < \infty$.

The following lemma was established by Iwaniec and Lutoborski in [10].

Lemma 3. *Let $u \in L^s(\Omega, \Lambda^l)$, $l = 0, 1, \dots, n$, $1 < s < \infty$, be a differential form in a domain Ω . Then, there exists a constant C , independent of u , such that*

$$\|u_{\Omega}\|_{s, \Omega} \leq C \|u\|_{s, \Omega}, \quad (12)$$

where u_{Ω} is a closed form.

We will use the following generalized Hölder inequality repeatedly in this paper.

Lemma 4. *Let $0 < q < \infty$, $0 < p < \infty$, and $s^{-1} = q^{-1} + p^{-1}$. If f and g are measurable functions on $\mathbb{R}^n$, then*

$$\|fg\|_{s, \Omega} \leq \|f\|_{q, \Omega} \|g\|_{p, \Omega} \quad (13)$$

for any $\Omega \subset \mathbb{R}^n$.

We also need the Caccioppoli inequality for differential form.

Lemma 5 (see [11]). *Let $u \in D^l(\Omega, \Lambda^l)$, $l = 0, 1, \dots, n$, be a solution of the nonhomogeneous A -harmonic equation (4) in Ω . Then, there exists a constant C , independent of u , such that*

$$\|du\|_{s, B} \leq C \text{diam}(B)^{-1} \|u - c\|_{s, \sigma B} \quad (14)$$

for all balls with $\sigma B \subset \Omega$ and any closed form c , where $1 < s < \infty$, and $\sigma > 1$.

The following Poincaré inequality appears in [5].

Lemma 6. *Let $u \in D^l(\Omega, \Lambda^l)$, $l = 0, 1, \dots, n-1$, be a solution of the nonhomogeneous A -harmonic equation (4) in Ω . Then, there exists a constant C , independent of u , such that*

$$\|u - u_B\|_{s, B} \leq C |B| \text{diam}(B) \|du\|_{s, B} \quad (15)$$

for any ball B in Ω .

First, we establish a Poincaré-type inequality for the composite operator $\mathbb{M}_s \circ P$.

Theorem 7. *Let $u \in L^t(\Omega, \Lambda^l)$, $l = 1, 2, \dots, n$, $1 \leq s < t < \infty$, be a differential form in a domain Ω . Then, there exists a constant C , independent of u , such that*

$$\|\mathbb{M}_s P(u) - (\mathbb{M}_s P(u))_{\Omega}\|_{t, \Omega} \leq C \|u\|_{t, \Omega}. \quad (16)$$

Proof. From Lemma 3 and Minkowski inequality, we have

$$\begin{aligned}
 \|u - u_\Omega\|_{t,\Omega} &= \left(\int_\Omega |u(x) - u_\Omega(x)|^t dx \right)^{1/t} \\
 &\leq \left(\int_\Omega |u(x)|^t dx \right)^{1/t} + \left(\int_\Omega |u_\Omega(x)|^t dx \right)^{1/t} \\
 &\leq \left(\int_\Omega |u(x)|^t dx \right)^{1/t} + C_1 \left(\int_\Omega |u(x)|^t dx \right)^{1/t} \\
 &\leq C_2 \|u\|_{t,\Omega}.
 \end{aligned} \tag{17}$$

Replacing u with $\mathbb{M}_s P(u)$ and combining Lemmas 1 and 2, we obtain

$$\begin{aligned}
 \|\mathbb{M}_s P(u) - (\mathbb{M}_s P(u))_\Omega\|_{t,\Omega} &\leq C_2 \|\mathbb{M}_s P(u)\|_{t,\Omega} \\
 &\leq C_3 \|P(u)\|_{t,\Omega} \\
 &\leq C_4 \|u\|_{t,\Omega}.
 \end{aligned} \tag{18}$$

Theorem 7 has been completed. $\square$

Then, we estimate the Lipschitz norm of $\mathbb{M}_s P(u)$ in terms of L^t -norm.

Theorem 8. Let $u \in L^t(\Omega, \Lambda^l)$, $l = 1, 2, \dots, n$, $1 \leq s < t < \infty$, be a solution of the nonhomogeneous A -harmonic equation (4) in Ω . Then, there exists a constant C , independent of u , such that

$$\|\mathbb{M}_s P(u)\|_{\text{loc Lip}_k, \Omega} \leq C \|d\mathbb{M}_s P(u)\|_{t,\Omega}, \tag{19}$$

where k is a constant with $0 \leq k \leq 1$.

Proof. Setting $u = \mathbb{M}_s P(u)$ in Lemma 6 we have

$$\|\mathbb{M}_s P(u) - (\mathbb{M}_s P(u))_B\|_{t,B} \leq C_1 |B| \text{diam}(B) \|d\mathbb{M}_s P(u)\|_{t,B} \tag{20}$$

for all balls B with $B \subset \Omega$. Using Hölder inequality, we have

$$\begin{aligned}
 &\|\mathbb{M}_s P(u) - (\mathbb{M}_s P(u))_B\|_{1,B} \\
 &\leq \left(\int_B |\mathbb{M}_s P(u) - (\mathbb{M}_s P(u))_B|^t dx \right)^{1/t} \\
 &\quad \times \left(\int_B 1^{t/(t-1)} dx \right)^{(t-1)/t} \\
 &\leq |B|^{(t-1)/t} \|\mathbb{M}_s P(u) - (\mathbb{M}_s P(u))_B\|_{t,B} \\
 &= |B|^{1-1/t} \|\mathbb{M}_s P(u) - (\mathbb{M}_s P(u))_B\|_{t,B} \\
 &\leq |B|^{1-1/t} C_1 |B| \text{diam}(B) \|d\mathbb{M}_s P(u)\|_{t,B} \\
 &\leq C_2 |B|^{2-1/t+1/n} \|d\mathbb{M}_s P(u)\|_{t,B}.
 \end{aligned} \tag{21}$$

From the definition of Lipschitz norm and (21), it follows that

$$\begin{aligned}
 &\|\mathbb{M}_s P(u)\|_{\text{loc Lip}_k, \Omega} \\
 &= \sup_{\sigma B \subset \Omega} |B|^{-(n+k)/n} \|\mathbb{M}_s P(u) - (\mathbb{M}_s P(u))_B\|_{1,B} \\
 &= \sup_{\sigma B \subset \Omega} |B|^{-1-k/n} \|\mathbb{M}_s P(u) - (\mathbb{M}_s P(u))_B\|_{1,B} \\
 &\leq \sup_{\sigma B \subset \Omega} |B|^{-1-k/n} C_2 |B|^{2-1/t+1/n} \|d\mathbb{M}_s P(u)\|_{t,B} \\
 &= \sup_{\sigma B \subset \Omega} C_2 |B|^{1-k/n-1/t+1/n} \|d\mathbb{M}_s P(u)\|_{t,B} \\
 &\leq \sup_{\sigma B \subset \Omega} C_2 |\Omega|^{1-k/n-1/t+1/n} \|d\mathbb{M}_s P(u)\|_{t,B} \\
 &\leq C_3 \sup_{\sigma B \subset \Omega} \|d\mathbb{M}_s P(u)\|_{t,B} \\
 &\leq C_3 \|d\mathbb{M}_s P(u)\|_{t,\Omega}.
 \end{aligned} \tag{22}$$

Theorem 8 has been completed. $\square$

Now, we establish the norm comparison theorems of the composite operator between Lipschitz norm and BMO norm.

Theorem 9. Let $u \in L^s(\Omega, \Lambda^l)$, $l = 1, 2, \dots, n$, $1 \leq s < \infty$, be a differential form in Ω . Then, there exists a constant C , independent of u , such that

$$\|\mathbb{M}_s P(u)\|_{*,\Omega} \leq C \|\mathbb{M}_s P(u)\|_{\text{loc Lip}_k, \Omega}, \tag{23}$$

where k is a constant with $0 \leq k \leq 1$.

Proof. From the definition of BMO norm, we have

$$\begin{aligned}
 &\|\mathbb{M}_s P(u)\|_{*,\Omega} \\
 &= \sup_{\sigma B \subset \Omega} |B|^{-1} \|\mathbb{M}_s P(u) - (\mathbb{M}_s P(u))_B\|_{1,B} \\
 &= \sup_{\sigma B \subset \Omega} |B|^{k/n} |B|^{-(n+k)/n} \|\mathbb{M}_s P(u) - (\mathbb{M}_s P(u))_B\|_{1,B} \\
 &\leq \sup_{\sigma B \subset \Omega} |\Omega|^{k/n} |B|^{-(n+k)/n} \|\mathbb{M}_s P(u) - (\mathbb{M}_s P(u))_B\|_{1,B} \\
 &\leq |\Omega|^{k/n} \sup_{\sigma B \subset \Omega} |B|^{-(n+k)/n} \|\mathbb{M}_s P(u) - (\mathbb{M}_s P(u))_B\|_{1,B} \\
 &\leq C \sup_{\sigma B \subset \Omega} |B|^{-(n+k)/n} \|\mathbb{M}_s P(u) - (\mathbb{M}_s P(u))_B\|_{1,B} \\
 &\leq C \|\mathbb{M}_s P(u)\|_{\text{loc Lip}_k, \Omega}.
 \end{aligned} \tag{24}$$

Theorem 9 has been completed. $\square$

Theorem 10. Let $u \in L^s(\Omega, \Lambda^l)$, $l = 1, 2, \dots, n$, $1 < s < \infty$, be a solution of the nonhomogeneous A -harmonic equation (4) in Ω . Then, there exists a constant C , independent of u , such that

$$\|\mathbb{M}_s P(u)\|_{\text{loc Lip}_k, \Omega} \leq C \|\mathbb{M}_s P(u)\|_{*,\Omega}, \tag{25}$$

where k is a constant with $0 \leq k \leq 1$.

Proof. From (21), we obtain

$$\begin{aligned} & \|\mathbb{M}_s P(u) - (\mathbb{M}_s P(u))_B\|_{1,B} \\ & \leq C_1 |B|^{2-1/s+1/n} \|d\mathbb{M}_s P(u)\|_{s,B}. \end{aligned} \quad (26)$$

Based on Lemma 5, we get

$$\|d\mathbb{M}_s P(u)\|_{s,B} \leq C_2 \text{diam}(B)^{-1} \|\mathbb{M}_s P(u) - (\mathbb{M}_s P(u))_B\|_{s,\sigma B}. \quad (27)$$

From the weak reverse Hölder inequality, it follows that

$$\begin{aligned} & \|\mathbb{M}_s P(u) - (\mathbb{M}_s P(u))_B\|_{s,\sigma B} \\ & \leq C_3 |B|^{(1-s)/s} \|\mathbb{M}_s P(u) - (\mathbb{M}_s P(u))_B\|_{1,\sigma' B}, \end{aligned} \quad (28)$$

where $\sigma' > \sigma > 1$. So, we get

$$\begin{aligned} & \|\mathbb{M}_s P(u) - (\mathbb{M}_s P(u))_B\|_{1,B} \\ & \leq C_1 |B|^{2-1/s+1/n} \|d\mathbb{M}_s P(u)\|_{s,B} \\ & \leq C_4 |B| \|\mathbb{M}_s P(u) - (\mathbb{M}_s P(u))_B\|_{1,\sigma' B}. \end{aligned} \quad (29)$$

Let $\sigma'' > \sigma'$; we obtain

$$\begin{aligned} & \|\mathbb{M}_s P(u)\|_{\text{loc Lip}_k, \Omega} \\ & = \sup_{\sigma'' B \subset \Omega} |B|^{-(n+k)/n} \|\mathbb{M}_s P(u) - (\mathbb{M}_s P(u))_B\|_{1,B} \\ & \leq \sup_{\sigma'' B \subset \Omega} C_4 |B|^{1-(k/n)} |B|^{-1} \|\mathbb{M}_s P(u) - (\mathbb{M}_s P(u))_B\|_{1,\sigma' B} \\ & \leq C_5 \sup_{\sigma'' B \subset \Omega} C_4 |B|^{-1} \|\mathbb{M}_s P(u) - (\mathbb{M}_s P(u))_B\|_{1,\sigma' B} \\ & \leq C_6 \|\mathbb{M}_s P(u)\|_{*,\Omega}. \end{aligned} \quad (30)$$

Theorem 10 has been completed. $\square$

Combining Theorems 8 and 9, we obtain the following result easily.

Corollary 11. Let $u \in L^t(\Omega, \Lambda^l)$, $l = 1, 2, \dots, n$, $1 \leq s < t < \infty$, be a solution of the nonhomogeneous A -harmonic equation (4) in Ω . Then, there exists a constant C , independent of u , such that

$$\|\mathbb{M}_s P(u)\|_{*,\Omega} \leq C \|d\mathbb{M}_s P(u)\|_{t,\Omega}. \quad (31)$$

3. The Weighted Lipschitz and BMO Norm Inequalities

In this section, we obtain some weighted inequalities for the composition operator $\mathbb{M}_s \circ P$.

The weight function we use here is $A(\alpha, \beta, \gamma, \Omega)$ weight. The $A(\alpha, \beta, \gamma, \Omega)$ -class is a new weight class which was first proposed by Xing in [7]. It contains the well-known $A_r(\Omega)$ -weight as a proper subset.

Definition 12. One says that a measurable function $\omega(x)$ defined on a subset $\Omega \subset \mathbb{R}^n$ satisfies the $A(\alpha, \beta, \gamma, \Omega)$ -condition for some positive constants α, β, γ , if $\omega(x) > 0$ a.e., and

$$\sup_{B \subset \Omega} \left(\frac{1}{|B|} \int_B \omega^\alpha dx \right) \left(\frac{1}{|B|} \int_B \omega^{-\beta} dx \right)^{\gamma/\beta} < \infty. \quad (32)$$

Let $u \in L^1_{\text{loc}}(\Omega, \Lambda^l, \omega)$, $l = 0, 1, 2, \dots, n$. We say $u \in \text{loc Lip}_k(\Omega, \Lambda^l, \omega)$, $0 \leq k \leq 1$, if

$$\|u\|_{\text{loc Lip}_k, \Omega, \omega} = \sup_{\sigma Q \subset \Omega} (\mu(Q))^{-(n+k)/n} \|u - u_Q\|_{1, Q, \omega} < \infty \quad (33)$$

for some $\sigma > 1$, where Ω is bounded in $\mathbb{R}^n$, ω is a weight, and μ is Radon measure defined by $d\mu = \omega(x)dx$.

Let $u \in L^1_{\text{loc}}(\Omega, \Lambda^l, \omega)$, $l = 0, 1, 2, \dots, n$. We say $u \in \text{BMO}(\Omega, \Lambda^l, \omega)$, if

$$\|u\|_{*, \Omega, \omega} = \sup_{\sigma Q \subset \Omega} (\mu(Q))^{-1} \|u - u_Q\|_{1, Q, \omega} < \infty \quad (34)$$

for some $\sigma > 1$, where Ω is bounded in $\mathbb{R}^n$, ω is a weight, and μ is Radon measure defined by $d\mu = \omega(x)dx$.

Now, we estimate the weighted Lipschitz and BMO norm for the composition operator $\mathbb{M}_s \circ P$.

Theorem 13. Let $u \in L^q(\Omega, \Lambda^l, \mu)$, $l = 1, 2, \dots, n$, $1 \leq s < q < \infty$, be a solution of the nonhomogeneous A -harmonic equation (4) in Ω . Radon measure μ is defined by $\omega(x)dx = d\mu$, and $\omega(x) \in A(\alpha, \beta, \gamma, \Omega)$ for some $\alpha > 1$, where $1 < p < \infty$, $\beta = \alpha q / (\alpha p - p - \alpha q)$, $\gamma = \alpha q / p$, and $\alpha p - p - \alpha q > 0$. Then, there exists a constant C , independent of u , such that

$$\|\mathbb{M}_s P(u)\|_{\text{loc Lip}_k, \Omega, \omega} \leq C \|d\mathbb{M}_s P(u)\|_{p, \Omega, \omega}. \quad (35)$$

Proof. Using Hölder inequality with $1/q = 1/\alpha q + (\alpha - 1)/\alpha q$, we have

$$\begin{aligned} & \left(\int_B |\mathbb{M}_s P(u) - (\mathbb{M}_s P(u))_B|^q \omega(x) dx \right)^{1/q} \\ & = \left(\int_B |\mathbb{M}_s P(u) - (\mathbb{M}_s P(u))_B| \omega(x)^{1/q} dx \right)^{1/q} \\ & \leq \left(\int_B |\mathbb{M}_s P(u) - (\mathbb{M}_s P(u))_B|^{\alpha q / (\alpha - 1)} dx \right)^{(\alpha - 1)/\alpha q} \\ & \quad \times \left(\int_B (\omega(x)^{1/q})^{\alpha q} dx \right)^{1/\alpha q} \\ & \leq C_1 |B| \text{diam}(B) \left(\int_B |d\mathbb{M}_s P(u)|^{\alpha q / (\alpha - 1)} dx \right)^{(\alpha - 1)/\alpha q} \\ & \quad \times \left(\int_B \omega(x)^\alpha dx \right)^{1/\alpha q}. \end{aligned} \quad (36)$$

Using Hölder inequality with $(\alpha - 1)/\alpha q = (1/p) + (\alpha p - p - \alpha q)/\alpha q p$, we obtain

$$\begin{aligned}
 & \left(\int_B |d\mathbb{M}_s P(u)|^{\alpha q/(\alpha-1)} dx \right)^{(\alpha-1)/\alpha q} \\
 &= \left(\int_B |d\mathbb{M}_s P(u) \omega(x)^{1/p} \omega(x)^{-1/p}|^{\alpha q/(\alpha-1)} dx \right)^{(\alpha-1)/\alpha q} \\
 &\leq \left(\int_B |d\mathbb{M}_s P(u) \omega(x)^{1/p}|^p dx \right)^{1/p} \\
 &\quad \times \left(\int_B (\omega(x)^{-1/p})^{\alpha q p/(\alpha p - p - \alpha q)} dx \right)^{(\alpha p - p - \alpha q)/\alpha q p} \\
 &= \left(\int_B |d\mathbb{M}_s P(u)|^p \omega(x) dx \right)^{1/p} \\
 &\quad \times \left(\int_B \left(\frac{1}{\omega(x)} \right)^{\alpha q/(\alpha p - p - \alpha q)} dx \right)^{(\alpha p - p - \alpha q)/\alpha q p}.
 \end{aligned} \tag{37}$$

Since $\omega(x) \in A(\alpha, \alpha q/(\alpha p - p - \alpha q), \alpha q/p, \Omega)$, we get

$$\begin{aligned}
 & \left(\int_B \omega(x)^\alpha dx \right)^{1/\alpha q} \left(\int_B \left(\frac{1}{\omega(x)} \right)^{\alpha q/(\alpha p - p - \alpha q)} dx \right)^{(\alpha p - p - \alpha q)/\alpha q p} \\
 &= \left(\left(\int_B \omega(x)^\alpha dx \right) \right. \\
 &\quad \times \left. \left(\int_B \left(\frac{1}{\omega(x)} \right)^{\alpha q/(\alpha p - p - \alpha q)} dx \right)^{(\alpha p - p - \alpha q)/p} \right)^{1/\alpha q} \\
 &= |B|^{2/\alpha q} \\
 &\quad \times \left(\left(\frac{1}{|B|} \int_B (\omega(x))^\alpha dx \right) \right. \\
 &\quad \times \left. \left(\frac{1}{|B|} \int_B \left(\frac{1}{\omega(x)} \right)^{\alpha q/(\alpha p - p - \alpha q)} dx \right)^{(\alpha p - p - \alpha q)/p} \right)^{1/\alpha q} \\
 &\leq C_2.
 \end{aligned} \tag{38}$$

From (36), (37), and (38), we know that

$$\begin{aligned}
 & \left(\int_B |\mathbb{M}_s P(u) - (\mathbb{M}_s P(u))_B|^q \omega(x) dx \right)^{1/q} \\
 &\leq C_3 |B| \text{diam}(B) \left(\int_B |d\mathbb{M}_s P(u)|^p \omega(x) dx \right)^{1/p}.
 \end{aligned} \tag{39}$$

So, we obtain

$$\begin{aligned}
 & \|\mathbb{M}_s P(u) - (\mathbb{M}_s P(u))_B\|_{1,B,\omega} \\
 &= \int_B |\mathbb{M}_s P(u) - (\mathbb{M}_s P(u))_B| d\mu \\
 &\leq \left(\int_B |\mathbb{M}_s P(u) - (\mathbb{M}_s P(u))_B|^q \omega(x) dx \right)^{1/q} \\
 &\quad \times \left(\int_B 1^{q/(q-1)} d\mu \right)^{(q-1)/q} \\
 &= (\mu(B))^{(q-1)/q} \|\mathbb{M}_s P(u) - (\mathbb{M}_s P(u))_B\|_{q,B,\omega} \\
 &\leq C_4 (\mu(B))^{(q-1)/q} |B| \text{diam}(B) \|d\mathbb{M}_s P(u)\|_{p,B,\omega}.
 \end{aligned} \tag{40}$$

Based on the definition of the weighted Lipschitz norm and (40), we have

$$\begin{aligned}
 & \|\mathbb{M}_s P(u)\|_{\text{loc Lip}_k, \Omega, \omega} \\
 &= \sup_{\sigma B \subset \Omega} (\mu(B))^{-(n+k)/n} \|\mathbb{M}_s P(u) - (\mathbb{M}_s P(u))_B\|_{1,B,\omega} \\
 &\leq C_5 \sup_{\sigma B \subset \Omega} (\mu(B))^{-(n+k)/n + (q-1)/q + 1 + 1/n} \\
 &\quad \times \|d\mathbb{M}_s P(u)\|_{p,B,\omega} \\
 &\leq C_5 \sup_{\sigma B \subset \Omega} (\mu(\Omega))^{-(n+k)/n + (q-1)/q + 1 + 1/n} \\
 &\quad \times \|d\mathbb{M}_s P(u)\|_{p,B,\omega} \\
 &\leq C \|d\mathbb{M}_s P(u)\|_{p,\Omega,\omega}.
 \end{aligned} \tag{41}$$

Theorem 13 has been completed. $\square$

Theorem 14. Let $u \in L^s(\Omega, \Lambda^l, \mu)$, $l = 1, 2, \dots, n$, $1 \leq s < \infty$, be a differential form in Ω . Radon measure μ is defined by $\omega(x)dx = d\mu$, and $\omega(x) \in A(\alpha, \beta, \gamma, \Omega)$. Then, there exists a constant C , independent of u , such that

$$\|\mathbb{M}_s P(u)\|_{*, \Omega, \omega} \leq C \|\mathbb{M}_s P(u)\|_{\text{loc Lip}_k, \Omega, \omega}, \tag{42}$$

where α, β, γ are some positive constants.

Proof. From the definition of the weighted BMO norm, we have

$$\begin{aligned}
 & \|\mathbb{M}_s P(u)\|_{*, \Omega, \omega} \\
 &= \sup_{\sigma B \subset \Omega} (\mu(B))^{-1} \|\mathbb{M}_s P(u) - (\mathbb{M}_s P(u))_B\|_{1,B,\omega} \\
 &= \sup_{\sigma B \subset \Omega} (\mu(B))^{k/n} (\mu(B))^{-(n+k)/n} \\
 &\quad \times \|\mathbb{M}_s P(u) - (\mathbb{M}_s P(u))_B\|_{1,B,\omega} \\
 &\leq \sup_{\sigma B \subset \Omega} (\mu(\Omega))^{k/n} (\mu(B))^{-(n+k)/n} \\
 &\quad \times \|\mathbb{M}_s P(u) - (\mathbb{M}_s P(u))_B\|_{1,B,\omega}
 \end{aligned}$$

$$\begin{aligned}
&\leq (\mu(\Omega))^{k/n} \sup_{\sigma B \subset \Omega} (\mu(B))^{-(n+k)/n} \\
&\quad \times \|\mathbb{M}_s P(u) - (\mathbb{M}_s P(u))_B\|_{1,B,\omega} \\
&\leq C \sup_{\sigma B \subset \Omega} (\mu(B))^{-(n+k)/n} \\
&\quad \times \|\mathbb{M}_s P(u) - (\mathbb{M}_s P(u))_B\|_{1,B,\omega} \\
&\leq C \|\mathbb{M}_s P(u)\|_{\text{loc Lip}_k, \Omega, \omega}. \tag{43}
\end{aligned}$$

Theorem 14 has been completed. $\square$

Combining Theorems 13 and 14, we obtain the following corollary.

Corollary 15. Let $u \in L^q(\Omega, \Lambda^l, \mu)$, $l = 1, 2, \dots, n$, $1 \leq s < q < \infty$, be a solution of the nonhomogeneous A -harmonic equation (4) in Ω . Radon measure μ is defined by $\omega(x)dx = d\mu$, and $\omega(x) \in A(\alpha, \beta, \gamma, \Omega)$ for some $\alpha > 1$, where $1 < p < \infty$, $\beta = \alpha q/(\alpha p - p - \alpha q)$, $\gamma = \alpha q/p$, and $\alpha p - p - \alpha q > 0$. Then, there exists a constant C , independent of u , such that

$$\|\mathbb{M}_s P(u)\|_{*, \Omega, \omega} \leq C \|d\mathbb{M}_s P(u)\|_{p, \Omega, \omega}. \tag{44}$$

4. Applications

In this section, we use the theorems we obtain to estimate the norms of Jacobian subdeterminant and the generalized solution of the quasilinear elliptic equation.

Example 16. Let $u = (u^1, \dots, u^n)$ be a map from Ω to $\mathbb{R}^n$. $J(x, u)$ is the Jacobian determinant. Now choosing the subdeterminant of the Jacobian determinant,

$$J(x_{j_1}, x_{j_2}; u^{i_1}, u^{i_2}) = \begin{vmatrix} \frac{\partial u^{i_1}}{\partial x_{j_1}} & \frac{\partial u^{i_1}}{\partial x_{j_2}} \\ \frac{\partial u^{i_2}}{\partial x_{j_1}} & \frac{\partial u^{i_2}}{\partial x_{j_2}} \end{vmatrix}. \tag{45}$$

We know that $J(x_{j_1}, x_{j_2}; u^{i_1}, u^{i_2})dx_{j_1} \wedge dx_{j_2}$ is a 2-form. Let $v = J(x_{j_1}, x_{j_2}; u^{i_1}, u^{i_2})dx_{j_1} \wedge dx_{j_2}$. If $v \in L^s(\Omega, \Lambda^2)$, $1 \leq s < \infty$, from Theorems 9 and 14, we obtain the following results:

$$\begin{aligned}
\|\mathbb{M}_s P(v)\|_{*, \Omega} &\leq C \|\mathbb{M}_s P(v)\|_{\text{loc Lip}_k, \Omega}, \\
\|\mathbb{M}_s P(v)\|_{*, \Omega, \omega} &\leq C \|\mathbb{M}_s P(v)\|_{\text{loc Lip}_k, \Omega, \omega}, \tag{46}
\end{aligned}$$

where C is a constant, $0 \leq k \leq 1$, $\mathbb{M}_s$ is the Hardy-Littlewood maximal operator, P is the potential operator, and ω is a $A(\alpha, \beta, \gamma, \Omega)$ weight satisfying $\alpha > 1$, $\beta = \alpha q/(\alpha p - p - \alpha q)$, $\gamma = \alpha q/p$, and $\alpha p - p - \alpha q > 0$.

Example 17. Let $u(x) = (u^1, u^2, \dots, u^n) : \Omega \rightarrow \mathbb{R}^n$ be a K -quasiregular mapping, $K \geq 1$. Then,

$$v = u^i(x) \quad (i = 1, 2, \dots, n) \tag{47}$$

or

$$v = \log |u(x)| \tag{48}$$

is a generalized solution of the following equation:

$$\operatorname{div} A(x, \nabla v) = 0, \quad A = (A^1, A^2, \dots, A^n). \tag{49}$$

Here, $A^i(x, \eta)$ can be expressed as

$$A^i(x, \eta) = \frac{\partial}{\partial \eta_i} \left(\sum_{j=1}^n \alpha_{i,j}(x) \eta_j \right)^{n/2}. \tag{50}$$

$\alpha_{i,j}$ in the formula above are some functions, which can be expressed in terms of the differential matrix $Du(x)$, and satisfy

$$C_1(K) |\eta|^2 \leq \sum_{i,j=1}^n \alpha_{i,j} \eta_i \eta_j \leq C_2(K) |\eta|^2, \tag{51}$$

where $C_1(K), C_2(K) > 0$. If we assume that $\mathbb{M}_s$ is the Hardy-Littlewood maximal operator, P is the potential operator, $0 \leq k \leq 1$, and ω is a $A(\alpha, \beta, \gamma, \Omega)$ weight satisfying $\alpha > 1$, $\beta = \alpha q/(\alpha p - p - \alpha q)$, $\gamma = \alpha q/p$, and $\alpha p - p - \alpha q > 0$, according to Theorem 13 and Corollary 15, we obtain the following inequalities:

$$\begin{aligned}
\|\mathbb{M}_s P(v)\|_{\text{loc Lip}_k, \Omega, \omega} &\leq C \|d\mathbb{M}_s P(v)\|_{p, \Omega, \omega}, \\
\|\mathbb{M}_s P(v)\|_{*, \Omega, \omega} &\leq C \|\mathbb{M}_s P(v)\|_{p, \Omega, \omega}, \tag{52}
\end{aligned}$$

where C is a constant.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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