Research Article

Some New Inequalities of Hermite-Hadamard Type for (α_1, m_1) -(α_2, m_2)-Convex Functions on Coordinates

Ling Chun

College of Mathematics, Inner Mongolia University for Nationalities, Inner Mongolia Autonomous Region, Tongliao 028043, China

Correspondence should be addressed to Ling Chun; chunling1980@qq.com

Received 22 July 2013; Accepted 12 December 2013; Published 19 January 2014

Academic Editor: Gerassimos Barbatis

Copyright © 2014 Ling Chun. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

The author introduces a new concept “ (α_1, m_1) -(α_2, m_2)-convex functions on coordinates” and establishes some new inequalities of Hermite-Hadamard type for (α_1, m_1) -(α_2, m_2)-convex functions of two variables on co-ordinates.

1. Introduction

The following definitions are well known in the literature.

Definition 1. Let $f : I \subseteq \mathbb{R} = (-\infty, \infty) \rightarrow \mathbb{R}$ be a function on an interval I . If

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y) \quad (1)$$

is valid for all $x, y \in I$ and $\lambda \in [0, 1]$, then we say that f is a convex function on I .

Definition 2 (see [1]). For $f : [0, b] \rightarrow \mathbb{R}$ and $m \in (0, 1]$, if

$$f(tx + m(1 - t)y) \leq tf(x) + m(1 - t)f(y) \quad (2)$$

is valid for all $x, y \in [0, b]$ and $t \in [0, 1]$, then we say that f is an m -convex function on $[0, b]$.

Definition 3 (see [2]). For $f : [0, b] \rightarrow \mathbb{R}$ and $(\alpha, m) \in (0, 1]^2$, if

$$f(tx + m(1 - t)y) \leq t^\alpha f(x) + m(1 - t^\alpha)f(y) \quad (3)$$

is valid for all $x, y \in [0, b]$ and $t \in [0, 1]$, then we say that f is an (α, m) -convex function on $[0, b]$.

Definition 4 (see [3, 4]). For $a < b$ and $c < d$, a function $f : \Delta = [a, b] \times [c, d] \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ is said to be convex on coordinates, if the partial mappings

$$\begin{aligned} f_y : [a, b] \ni u &\mapsto f(u, y) \in \mathbb{R}, \\ f_x : [c, d] \ni v &\mapsto f(x, v) \in \mathbb{R} \end{aligned} \quad (4)$$

are convex for all $x \in (a, b)$ and $y \in (c, d)$.

A formal definition for coordinated convex functions may be stated as follows.

Definition 5. A function $f : \Delta \rightarrow \mathbb{R}$ is said to be convex on coordinates, if

$$\begin{aligned} &f(tx + (1 - t)z, \lambda y + (1 - \lambda)w) \\ &\leq t\lambda f(x, y) + t(1 - \lambda)f(x, w) \\ &\quad + (1 - t)\lambda f(z, y) + (1 - t)(1 - \lambda)f(z, w) \end{aligned} \quad (5)$$

holds for all $t, \lambda \in [0, 1]$, $(x, y), (z, w) \in \Delta$.

Some inequalities of Hermite-Hadamard type for convex functions on coordinates may be recited as follows.

Theorem 6 (see Theorem 1 in [3, 4]). Let $f : \Delta \rightarrow \mathbb{R}$ be a convex function on coordinates. Then

$$\begin{aligned}
 & f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\
 & \leq \frac{1}{2} \left[\frac{1}{b-a} \int_a^b f\left(x, \frac{c+d}{2}\right) dx \right. \\
 & \quad \left. + \frac{1}{d-c} \int_c^d f\left(\frac{a+b}{2}, y\right) dy \right] \\
 & \leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dy dx \\
 & \leq \frac{1}{4} \left[\frac{1}{b-a} \left(\int_a^b f(x, c) dx + \int_a^b f(x, d) dx \right) \right. \\
 & \quad \left. + \frac{1}{d-c} \left(\int_c^d f(a, y) dy + \int_c^d f(b, y) dy \right) \right] \\
 & \leq \frac{1}{4} [f(a, c) + f(b, c) + f(a, d) + f(b, d)].
 \end{aligned} \tag{6}$$

Theorem 7 (see [5, Theorem 2.3]). Let $f : \Delta \rightarrow \mathbb{R}$ be a partial differentiable function. If $|\partial^2 f / \partial t \partial \lambda|$ is a convex function on coordinates, then

$$\begin{aligned}
 & \left| \frac{1}{9} \left[f\left(a, \frac{c+d}{2}\right) + f\left(b, \frac{c+d}{2}\right) + 4f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \right. \right. \\
 & \quad \left. \left. + f\left(\frac{a+b}{2}, c\right) + f\left(\frac{a+b}{2}, d\right) \right] \right. \\
 & \quad \left. + \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{36} \right. \\
 & \quad \left. + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dy dx - A \right| \\
 & \leq \frac{25(b-a)(d-c)}{5184} \\
 & \quad \times \left\{ \left| \frac{\partial^2}{\partial t \partial \lambda} f(a, c) \right| + \left| \frac{\partial^2}{\partial t \partial \lambda} f(a, d) \right| \right. \\
 & \quad \left. + \left| \frac{\partial^2}{\partial t \partial \lambda} f(b, c) \right| + \left| \frac{\partial^2}{\partial t \partial \lambda} f(b, d) \right| \right\},
 \end{aligned} \tag{7}$$

where

$$\begin{aligned}
 A &= \frac{1}{b-a} \\
 & \times \int_a^b \left\{ \frac{1}{6} \left[f(x, c) + 4f\left(x, \frac{c+d}{2}\right) + f(x, d) \right] \right\} dx \\
 & + \frac{1}{d-c} \\
 & \times \int_c^d \left\{ \frac{1}{6} \left[f(a, y) + 4f\left(\frac{a+b}{2}, y\right) + f(b, y) \right] \right\} dy.
 \end{aligned} \tag{8}$$

Theorem 8 (see [6, Theorem 2]). Let $f : \Delta \rightarrow \mathbb{R}$ be a partial differentiable function. If $|\partial^2 f / \partial t \partial \lambda|$ is a convex function on coordinates, then

$$\begin{aligned}
 & \left| \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dy dx \right. \\
 & \quad \left. + f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - A \right| \\
 & \leq \frac{(b-a)(d-c)}{16} \\
 & \quad \cdot \left(\left(\left| \frac{\partial^2}{\partial t \partial \lambda} f(a, c) \right| + \left| \frac{\partial^2}{\partial t \partial \lambda} f(a, d) \right| \right. \right. \\
 & \quad \left. \left. + \left| \frac{\partial^2}{\partial t \partial \lambda} f(b, c) \right| + \left| \frac{\partial^2}{\partial t \partial \lambda} f(b, d) \right| \right) \times (4)^{-1} \right),
 \end{aligned} \tag{9}$$

where

$$A = \frac{1}{b-a} \int_a^b f\left(x, \frac{c+d}{2}\right) dx + \frac{1}{d-c} \int_c^d f\left(\frac{a+b}{2}, y\right) dy. \tag{10}$$

For more information on inequalities of Hermite-Hadamard type for various (s_1, m_1) – (s_2, m_2) -convex functions on coordinates and for extended s -convex functions on coordinates, please refer to the recently published articles [7, 8] and related references therein.

The aim of this paper is to introduce a new concept “ (α_1, m_1) – (α_2, m_2) -convex functions on coordinates” and to establish some new inequalities of Hermite-Hadamard type for (α_1, m_1) – (α_2, m_2) -convex functions of two variables on coordinates.

2. A Definition and a Lemma

We now introduce a new notion “ (α_1, m_1) – (α_2, m_2) -convex functions on coordinates.”

Definition 9. For some $(\alpha_1, m_1), (\alpha_2, m_2) \in (0, 1]^2$, a function $f : [0, b] \times [0, d] \rightarrow \mathbb{R}$ is said to be (α_1, m_1) – (α_2, m_2) -convex on coordinates, if

$$\begin{aligned}
 & f(tx + m_1(1-t)z, \lambda y + m_2(1-\lambda)w) \\
 & \leq t^{\alpha_1} \lambda^{\alpha_2} f(x, y) + m_2 t^{\alpha_1} (1 - \lambda^{\alpha_2}) f(x, w) \\
 & \quad + m_1 (1 - t^{\alpha_1}) \lambda^{\alpha_2} f(z, y) \\
 & \quad + m_1 m_2 (1 - t^{\alpha_1}) (1 - \lambda^{\alpha_2}) f(z, w)
 \end{aligned} \tag{11}$$

holds for all $t, \lambda \in [0, 1]$, $(x, y), (z, w) \in [0, b] \times [0, d]$.

In order to establish some new inequalities of Hermite-Hadamard type for (α_1, m_1) – (α_2, m_2) -convex functions of two variables on coordinates, we need the following lemma.

Lemma 10. Let $f : \Delta \rightarrow \mathbb{R}$ be a partial differentiable function and $L(\Delta)$ denote the set of all Lebesgue integrable functions on Δ . If $\partial^2 f / \partial x \partial y \in L(\Delta)$, then

$$\begin{aligned} I(f) &:= f(a, c) - \frac{1}{d-c} \int_c^d f(a, y) dy - \frac{1}{b-a} \\ &\quad \times \int_a^b f(x, c) dx + \frac{1}{(d-c)(b-a)} \\ &\quad \times \int_c^d \int_a^b f(x, y) dy dx \\ &= (b-a)(d-c) \\ &\quad \times \int_0^1 \int_0^1 t\lambda \frac{\partial^2 f}{\partial x \partial y} \\ &\quad \times (ta + (1-t)b, \lambda c + (1-\lambda)d) dt d\lambda. \end{aligned} \quad (12)$$

Proof. Integrating by part yields

$$\begin{aligned} &\int_0^1 \int_0^1 t\lambda \frac{\partial^2 f}{\partial x \partial y} (ta + (1-t)b, \lambda c + (1-\lambda)d) dt d\lambda \\ &= \int_0^1 \lambda \left\{ t \frac{1}{a-b} \frac{\partial f}{\partial y} (ta + (1-t)b, \lambda c + (1-\lambda)d) \right\}_0^1 \\ &\quad - \frac{1}{a-b} \int_0^1 \frac{\partial f}{\partial y} (ta + (1-t)b, \\ &\quad \lambda c + (1-\lambda)d) dt \Big\} d\lambda \\ &= \int_0^1 \lambda \left\{ \frac{1}{a-b} \frac{\partial f}{\partial y} (a, \lambda c + (1-\lambda)d) - \frac{1}{a-b} \right. \\ &\quad \times \int_0^1 \frac{\partial f}{\partial y} (ta + (1-t)b, \\ &\quad \lambda c + (1-\lambda)d) dt \Big\} d\lambda \\ &= \frac{1}{a-b} \left\{ \int_0^1 \lambda \frac{\partial f}{\partial y} (a, \lambda c + (1-\lambda)d) d\lambda \right. \\ &\quad - \int_0^1 \int_0^1 \lambda \frac{\partial f}{\partial y} (ta + (1-t)b, \\ &\quad \lambda c + (1-\lambda)d) dt d\lambda \Big\}, \\ &\int_0^1 \lambda \frac{\partial f}{\partial y} (a, \lambda c + (1-\lambda)d) d\lambda \\ &\quad - \int_0^1 \int_0^1 \lambda \frac{\partial f}{\partial y} (ta + (1-t)b, \lambda c + (1-\lambda)d) dt d\lambda \\ &= \lambda \frac{f(a, \lambda c + (1-\lambda)d)}{c-d} \Big|_0^1 - \frac{1}{c-d} \\ &\quad \times \int_0^1 f(a, \lambda c + (1-\lambda)d) d\lambda \end{aligned}$$

$$\begin{aligned} &- \int_0^1 \left\{ \lambda \frac{f(ta + (1-t)b, \lambda c + (1-\lambda)d)}{c-d} \right\}_0^1 \\ &\quad - \frac{1}{c-d} \int_0^1 f(ta + (1-t)b, \\ &\quad \lambda c + (1-\lambda)d) d\lambda \Big\} dt \\ &= \frac{f(a, c)}{c-d} - \frac{1}{c-d} \int_0^1 f(a, \lambda c + (1-\lambda)d) d\lambda \\ &\quad - \int_0^1 \left\{ \frac{f(ta + (1-t)b, c)}{c-d} - \frac{1}{c-d} \right. \\ &\quad \times \int_0^1 f(ta + (1-t)b, \\ &\quad \lambda c + (1-\lambda)d) d\lambda \Big\} dt \\ &= \frac{f(a, c)}{c-d} - \frac{1}{c-d} \int_0^1 f(a, \lambda c + (1-\lambda)d) d\lambda \\ &\quad - \frac{1}{c-d} \int_0^1 f(ta + (1-t)b, c) dt \\ &\quad + \frac{1}{c-d} \int_0^1 \int_0^1 f(ta + (1-t)b, \lambda c + (1-\lambda)d) d\lambda dt. \end{aligned} \quad (13)$$

Choosing $x = ta + (1-t)b$ and $y = \lambda c + (1-\lambda)d$ for $(t, \lambda) \in [0, 1]^2$ and multiplying by $(b-a)(d-c)$ on both sides of the above equations lead to the identity (12). The proof of Lemma 10 is complete. $\square$

3. Some Integral Inequalities of Hermite-Hadamard Type

We are now in a position to establish new inequalities of Hermite-Hadamard type for (α_1, m_1) -(α_2, m_2)-convex functions on coordinates on a rectangle in the plane $\mathbb{R}_0^2 = [0, \infty) \times [0, \infty)$.

Theorem 11. Let $f : \Delta = [0, b^*] \times [0, d^*] \rightarrow \mathbb{R}$ be a partial differentiable function and $\partial^2 f / \partial x \partial y \in L(\Delta)$. If $|\partial^2 f / \partial x \partial y|^q$ is (α_1, m_1) -(α_2, m_2)-convex on coordinates on $[a, b/m_1] \times [c, d/m_2] \subseteq \Delta$ with $a < b$ and $c < d$ for some $(\alpha_1, m_1), (\alpha_2, m_2) \in (0, 1]^2$ and $q \geq 1$, then

$$\begin{aligned} |I(f)| &\leq \frac{(b-a)(d-c)}{4} \left[\frac{1}{(\alpha_1 + 2)(\alpha_2 + 2)} \right]^{1/q} \\ &\quad \times \left[4 \left| \frac{\partial^2 f}{\partial x \partial y} (a, c) \right|^q + 2\alpha_2 m_2 \left| \frac{\partial^2 f}{\partial x \partial y} \left(a, \frac{d}{m_2} \right) \right|^q \right. \\ &\quad + 2\alpha_1 m_1 \left| \frac{\partial^2 f}{\partial x \partial y} \left(\frac{b}{m_1}, c \right) \right|^q \\ &\quad \left. + \alpha_1 \alpha_2 m_1 m_2 \left| \frac{\partial^2 f}{\partial x \partial y} \left(\frac{b}{m_1}, \frac{d}{m_2} \right) \right|^q \right]^{1/q}. \end{aligned} \quad (14)$$

Proof. Using Lemma 10, the coordinated (α_1, m_1) -(α_2, m_2)-convexity of $|\partial^2 f / \partial x \partial y|^q$, and Hölder integral inequality yields

$$\begin{aligned}
 & \frac{|I(f)|}{(b-a)(d-c)} \\
 & \leq \int_0^1 \int_0^1 t\lambda \left| \frac{\partial^2 f}{\partial x \partial y} (ta + (1-t)b, \lambda c + (1-\lambda)d) \right| dt d\lambda \\
 & \leq \left(\int_0^1 \int_0^1 t\lambda dt d\lambda \right)^{1-1/q} \\
 & \quad \times \left\{ \int_0^1 \int_0^1 t\lambda \left[t^{\alpha_1} \lambda^{\alpha_2} \left| \frac{\partial^2 f}{\partial x \partial y} (a, c) \right|^q + m_2 t^{\alpha_1} (1 - \lambda^{\alpha_2}) \right. \right. \\
 & \quad \times \left| \frac{\partial^2 f}{\partial x \partial y} \left(a, \frac{d}{m_2} \right) \right|^q \\
 & \quad + m_1 (1 - t^{\alpha_1}) \lambda^{\alpha_2} \left| \frac{\partial^2 f}{\partial x \partial y} \left(\frac{b}{m_1}, c \right) \right|^q \\
 & \quad + m_1 m_2 (1 - t^{\alpha_1}) (1 - \lambda^{\alpha_2}) \\
 & \quad \times \left. \left| \frac{\partial^2 f}{\partial x \partial y} \left(\frac{b}{m_1}, \frac{d}{m_2} \right) \right|^q \right] dt d\lambda \Big\}^{1/q} \\
 & = \frac{1}{4} \left(\frac{1}{(\alpha_1 + 2)(\alpha_2 + 2)} \right)^{1/q} \\
 & \quad \times \left[4 \left| \frac{\partial^2 f}{\partial x \partial y} (a, c) \right|^q + 2\alpha_2 m_2 \left| \frac{\partial^2 f}{\partial x \partial y} \left(a, \frac{d}{m_2} \right) \right|^q \right. \\
 & \quad + 2\alpha_1 m_1 \left| \frac{\partial^2 f}{\partial x \partial y} \left(\frac{b}{m_1}, c \right) \right|^q \\
 & \quad + \left. \alpha_1 \alpha_2 m_1 m_2 \left| \frac{\partial^2 f}{\partial x \partial y} \left(\frac{b}{m_1}, \frac{d}{m_2} \right) \right|^q \right]^{1/q}. \tag{15}
 \end{aligned}$$

Theorem 11 is thus proved. $\square$

Corollary 12. Under conditions of Theorem 11, when $q = 1$, one has

$$\begin{aligned}
 |I(f)| & \leq \frac{(b-a)(d-c)}{4(\alpha_1 + 2)(\alpha_2 + 2)} \\
 & \quad \times \left[4 \left| \frac{\partial^2 f}{\partial x \partial y} (a, c) \right| + 2\alpha_2 m_2 \left| \frac{\partial^2 f}{\partial x \partial y} \left(a, \frac{d}{m_2} \right) \right| \right. \\
 & \quad + 2\alpha_1 m_1 \left| \frac{\partial^2 f}{\partial x \partial y} \left(\frac{b}{m_1}, c \right) \right| \\
 & \quad + \left. \alpha_1 \alpha_2 m_1 m_2 \left| \frac{\partial^2 f}{\partial x \partial y} \left(\frac{b}{m_1}, \frac{d}{m_2} \right) \right| \right]. \tag{16}
 \end{aligned}$$

Corollary 13. Under conditions of Theorem 11,

(1) if $\alpha_1 = \alpha_2 = \alpha$ and $m_1 = m_2 = m$, then

$$\begin{aligned}
 |I(f)| & \leq \frac{(b-a)(d-c)}{4} \left(\frac{1}{(\alpha+2)^2} \right)^{1/q} \\
 & \quad \times \left[4 \left| \frac{\partial^2 f}{\partial x \partial y} (a, c) \right|^q + 2\alpha m \left| \frac{\partial^2 f}{\partial x \partial y} \left(a, \frac{d}{m} \right) \right|^q \right. \\
 & \quad + 2\alpha m \left| \frac{\partial^2 f}{\partial x \partial y} \left(\frac{b}{m}, c \right) \right|^q \\
 & \quad + \left. \alpha^2 m^2 \left| \frac{\partial^2 f}{\partial x \partial y} \left(\frac{b}{m}, \frac{d}{m} \right) \right|^q \right]^{1/q}; \tag{17}
 \end{aligned}$$

(2) if $\alpha_1 = \alpha_2 = m_1 = m_2 = 1$, then

$$\begin{aligned}
 |I(f)| & \leq \frac{(b-a)(d-c)}{4} \left(\frac{1}{9} \right)^{1/q} \\
 & \quad \times \left[4 \left| \frac{\partial^2 f}{\partial x \partial y} (a, c) \right|^q + 2 \left| \frac{\partial^2 f}{\partial x \partial y} (a, d) \right|^q \right. \\
 & \quad + 2 \left| \frac{\partial^2 f}{\partial x \partial y} (b, c) \right|^q + \left. \left| \frac{\partial^2 f}{\partial x \partial y} (b, d) \right|^q \right]^{1/q}. \tag{18}
 \end{aligned}$$

Theorem 14. Let $f : \Delta = [0, b^*] \times [0, d^*] \rightarrow \mathbb{R}$ be a partial differentiable function and $\partial^2 f / \partial x \partial y \in L(\Delta)$. If $|\partial^2 f / \partial x \partial y|^q$ is (α_1, m_1) -(α_2, m_2)-convex on coordinates on $[a, b/m_1] \times [c, d/m_2] \subseteq \Delta$ with $a < b$ and $c < d$ for $(\alpha_1, m_1), (\alpha_2, m_2) \in (0, 1)^2$ and $q > 1$, then

$$\begin{aligned}
 |I(f)| & \leq (b-a)(d-c) \left(\frac{q-1}{2q-1} \right)^{2(1-1/q)} \\
 & \quad \times \left(\frac{1}{(\alpha_1 + 1)(\alpha_2 + 1)} \right)^{1/q} \\
 & \quad \times \left[\left| \frac{\partial^2 f}{\partial x \partial y} (a, c) \right|^q + \alpha_2 m_2 \left| \frac{\partial^2 f}{\partial x \partial y} \left(a, \frac{d}{m_2} \right) \right|^q \right. \\
 & \quad + \alpha_1 m_1 \left| \frac{\partial^2 f}{\partial x \partial y} \left(\frac{b}{m_1}, c \right) \right|^q \\
 & \quad + \left. \alpha_1 \alpha_2 m_1 m_2 \left| \frac{\partial^2 f}{\partial x \partial y} \left(\frac{b}{m_1}, \frac{d}{m_2} \right) \right|^q \right]^{1/q}. \tag{19}
 \end{aligned}$$

Proof. By Lemma 10, the coordinated (α_1, m_1) -(α_2, m_2)-convexity of $|\partial^2 f / \partial x \partial y|^q$, and Hölder integral inequality, we have

$$\begin{aligned}
 & \frac{|I(f)|}{(b-a)(d-c)} \\
 & \leq \int_0^1 \int_0^1 t\lambda \left| \frac{\partial^2 f}{\partial x \partial y} (ta + (1-t)b, \lambda c + (1-\lambda)d) \right| dt d\lambda
 \end{aligned}$$

$$\begin{aligned}
& \leq \left(\int_0^1 \int_0^1 (t\lambda)^{q/(q-1)} dt d\lambda \right)^{1-1/q} \\
& \quad \times \left\{ \int_0^1 \int_0^1 \left[t^{\alpha_1} \lambda^{\alpha_2} \left| \frac{\partial^2 f}{\partial x \partial y} (a, c) \right|^q \right. \right. \\
& \quad \quad + m_2 t^{\alpha_1} (1 - \lambda^{\alpha_2}) \left| \frac{\partial^2 f}{\partial x \partial y} \left(a, \frac{d}{m_2} \right) \right|^q \\
& \quad \quad + m_1 (1 - t^{\alpha_1}) \lambda^{\alpha_2} \left| \frac{\partial^2 f}{\partial x \partial y} \left(\frac{b}{m_1}, c \right) \right|^q \\
& \quad \quad \left. \left. + m_1 m_2 (1 - t^{\alpha_1}) (1 - \lambda^{\alpha_2}) \right| \frac{\partial^2 f}{\partial x \partial y} \left(\frac{b}{m_1}, \frac{d}{m_2} \right) \right|^q \right] dt d\lambda \right\}^{1/q} \\
& = \left(\frac{q-1}{2q-1} \right)^{2(1-1/q)} \left(\frac{1}{(\alpha_1+1)(\alpha_2+1)} \right)^{1/q} \\
& \quad \times \left[\left| \frac{\partial^2 f}{\partial x \partial y} (a, c) \right|^q + \alpha_2 m_2 \left| \frac{\partial^2 f}{\partial x \partial y} \left(a, \frac{d}{m_2} \right) \right|^q \right. \\
& \quad \quad + \alpha_1 m_1 \left| \frac{\partial^2 f}{\partial x \partial y} \left(\frac{b}{m_1}, c \right) \right|^q \\
& \quad \quad \left. + \alpha_1 \alpha_2 m_1 m_2 \left| \frac{\partial^2 f}{\partial x \partial y} \left(\frac{b}{m_1}, \frac{d}{m_2} \right) \right|^q \right]^{1/q}. \quad (20)
\end{aligned}$$

Theorem 14 is thus proved. $\square$

Corollary 15. Under conditions of Theorem 14, if $\alpha_1 = \alpha_2 = \alpha$ and $m_1 = m_2 = m$, then

$$\begin{aligned}
& |I(f)| \\
& \leq (b-a)(d-c) \left(\frac{q-1}{2q-1} \right)^{2(1-1/q)} \left(\frac{1}{(\alpha+1)^2} \right)^{1/q} \\
& \quad \times \left[\left| \frac{\partial^2 f}{\partial x \partial y} (a, c) \right|^q + \alpha m \left| \frac{\partial^2 f}{\partial x \partial y} \left(a, \frac{d}{m} \right) \right|^q \right. \\
& \quad \quad + \alpha m \left| \frac{\partial^2 f}{\partial x \partial y} \left(\frac{b}{m}, c \right) \right|^q \\
& \quad \quad \left. + \alpha^2 m^2 \left| \frac{\partial^2 f}{\partial x \partial y} \left(\frac{b}{m}, \frac{d}{m} \right) \right|^q \right]^{1/q}; \quad (21)
\end{aligned}$$

if $\alpha_1 = \alpha_2 = m_1 = m_2 = 1$, then

$$\begin{aligned}
& |I(f)| \\
& \leq (b-a)(d-c) \left(\frac{q-1}{2q-1} \right)^{2(1-1/q)} \left(\frac{1}{4} \right)^{1/q} \\
& \quad \times \left[\left| \frac{\partial^2 f}{\partial x \partial y} (a, c) \right|^q + \left| \frac{\partial^2 f}{\partial x \partial y} (a, d) \right|^q \right. \\
& \quad \quad \left. + \left| \frac{\partial^2 f}{\partial x \partial y} (b, c) \right|^q + \left| \frac{\partial^2 f}{\partial x \partial y} (b, d) \right|^q \right]^{1/q}. \quad (22)
\end{aligned}$$

Theorem 16. Let $f : \Delta = [0, b^*] \times [0, d^*] \rightarrow \mathbb{R}$ be a partial differentiable function and $\partial^2 f / \partial x \partial y \in L(\Delta)$. If $|\partial^2 f / \partial x \partial y|^q$ is (α_1, m_1) - (α_2, m_2) -convex on co-ordinates on $[a, b/m_1] \times [c, d/m_2] \subseteq \Delta$ with $a < b$ and $c < d$ for $(\alpha_1, m_1), (\alpha_2, m_2) \in (0, 1]^2$, $q > 1$, and $q \geq r, s > 0$, then

$$\begin{aligned}
& |I(f)| \\
& \leq (b-a)(d-c) \left[\frac{(q-1)^2}{(2q-r-1)(2q-s-1)} \right]^{1-1/q} \\
& \quad \times \left[\frac{1}{(r+1)(\alpha_1+r+1)(s+1)(\alpha_2+s+1)} \right]^{1/q} \\
& \quad \times \left[(r+1)(s+1) \left| \frac{\partial^2 f}{\partial x \partial y} (a, c) \right|^q \right. \\
& \quad \quad + \alpha_2 m_2 (r+1) \left| \frac{\partial^2 f}{\partial x \partial y} \left(a, \frac{d}{m_2} \right) \right|^q \\
& \quad \quad + \alpha_1 m_1 (s+1) \left| \frac{\partial^2 f}{\partial x \partial y} \left(\frac{b}{m_1}, c \right) \right|^q \\
& \quad \quad \left. + \alpha_1 \alpha_2 m_1 m_2 \left| \frac{\partial^2 f}{\partial x \partial y} \left(\frac{b}{m_1}, \frac{d}{m_2} \right) \right|^q \right]^{1/q}. \quad (23)
\end{aligned}$$

Proof. Using Lemma 10, the coordinated (α_1, m_1) - (α_2, m_2) -convexity of $|\partial^2 f / \partial x \partial y|^q$, and Hölder integral inequality reveals

$$\begin{aligned}
& \frac{|I(f)|}{(b-a)(d-c)} \\
& \leq \int_0^1 \int_0^1 t\lambda \left| \frac{\partial^2 f}{\partial x \partial y} (ta + (1-t)b, \lambda c + (1-\lambda)d) \right| dt d\lambda \\
& \leq \left(\int_0^1 \int_0^1 t^{(q-r)/(q-1)} \lambda^{(q-s)/(q-1)} dt d\lambda \right)^{1-1/q} \\
& \quad \times \left\{ \int_0^1 \int_0^1 t^r \lambda^s \left[t^{\alpha_1} \lambda^{\alpha_2} \left| \frac{\partial^2 f}{\partial x \partial y} (a, c) \right|^q \right. \right. \\
& \quad \quad + m_2 t^{\alpha_1} (1 - \lambda^{\alpha_2}) \left| \frac{\partial^2 f}{\partial x \partial y} \left(a, \frac{d}{m_2} \right) \right|^q \\
& \quad \quad + m_1 (1 - t^{\alpha_1}) \lambda^{\alpha_2} \left| \frac{\partial^2 f}{\partial x \partial y} \left(\frac{b}{m_1}, c \right) \right|^q \\
& \quad \quad \left. \left. + m_1 m_2 (1 - t^{\alpha_1}) (1 - \lambda^{\alpha_2}) \right| \frac{\partial^2 f}{\partial x \partial y} \left(\frac{b}{m_1}, \frac{d}{m_2} \right) \right|^q \right] dt d\lambda \right\}^{1/q} \\
& = \left[\frac{(q-1)^2}{(2q-r-1)(2q-s-1)} \right]^{1-1/q}
\end{aligned}$$

$$\begin{aligned}
& \times \left[\frac{1}{(\alpha_1 + r + 1)(\alpha_2 + s + 1)} \left| \frac{\partial^2 f}{\partial x \partial y}(a, c) \right|^q \right. \\
& + \frac{\alpha_2 m_2}{(\alpha_1 + r + 1)(s + 1)(\alpha_2 + s + 1)} \left| \frac{\partial^2 f}{\partial x \partial y} \left(a, \frac{d}{m_2} \right) \right|^q \\
& + \frac{\alpha_1 m_1}{(r + 1)(\alpha_1 + r + 1)(\alpha_2 + s + 1)} \left| \frac{\partial^2 f}{\partial x \partial y} \left(\frac{b}{m_1}, c \right) \right|^q \\
& + \frac{\alpha_1 \alpha_2 m_1 m_2}{(r + 1)(\alpha_1 + r + 1)(s + 1)(\alpha_2 + s + 1)} \\
& \left. \times \left| \frac{\partial^2 f}{\partial x \partial y} \left(\frac{b}{m_1}, \frac{d}{m_2} \right) \right|^q \right]^{1/q} \\
& = \left[\frac{(q-1)^2}{(2q-r-1)(2q-s-1)} \right]^{1-1/q} \\
& \times \left[\frac{1}{(r+1)(\alpha_1+r+1)(s+1)(\alpha_2+s+1)} \right]^{1/q} \\
& \times \left[(r+1)(s+1) \left| \frac{\partial^2 f}{\partial x \partial y}(a, c) \right|^q \right. \\
& + \alpha_2 m_2 (r+1) \left| \frac{\partial^2 f}{\partial x \partial y} \left(a, \frac{d}{m_2} \right) \right|^q \\
& + \alpha_1 m_1 (s+1) \left| \frac{\partial^2 f}{\partial x \partial y} \left(\frac{b}{m_1}, c \right) \right|^q \\
& \left. + \alpha_1 \alpha_2 m_1 m_2 \left| \frac{\partial^2 f}{\partial x \partial y} \left(\frac{b}{m_1}, \frac{d}{m_2} \right) \right|^q \right]^{1/q}.
\end{aligned} \tag{24}$$

Hence, the proof of Theorem 16 is completed. $\square$

Corollary 17. Under conditions of Theorem 16,

(1) if $r = s = q$, then

$$\begin{aligned}
& |I(f)| \\
& \leq (b-a)(d-c) \left[\frac{1}{(q+1)^2(\alpha_1+q+1)(\alpha_2+q+1)} \right]^{1/q} \\
& \times \left[(q+1)^2 \left| \frac{\partial^2 f}{\partial x \partial y}(a, c) \right|^q \right. \\
& + \alpha_2 m_2 (q+1) \left| \frac{\partial^2 f}{\partial x \partial y} \left(a, \frac{d}{m_2} \right) \right|^q \\
& + \alpha_1 m_1 (q+1) \left| \frac{\partial^2 f}{\partial x \partial y} \left(\frac{b}{m_1}, c \right) \right|^q \\
& \left. + \alpha_1 \alpha_2 m_1 m_2 \left| \frac{\partial^2 f}{\partial x \partial y} \left(\frac{b}{m_1}, \frac{d}{m_2} \right) \right|^q \right]^{1/q};
\end{aligned} \tag{25}$$

(2) if $r = s = q$, $\alpha_1 = \alpha_2 = \alpha$, and $m_1 = m_2 = m$, one has

$$\begin{aligned}
& |I(f)| \\
& \leq (b-a)(d-c) \left[\frac{1}{(q+1)^2(\alpha+q+1)^2} \right]^{1/q} \\
& \times \left[(q+1)^2 \left| \frac{\partial^2 f}{\partial x \partial y}(a, c) \right|^q \right. \\
& + \alpha m (q+1) \left| \frac{\partial^2 f}{\partial x \partial y} \left(a, \frac{d}{m} \right) \right|^q \\
& + \alpha m (q+1) \left| \frac{\partial^2 f}{\partial x \partial y} \left(\frac{b}{m}, c \right) \right|^q \\
& \left. + \alpha^2 m^2 \left| \frac{\partial^2 f}{\partial x \partial y} \left(\frac{b}{m}, \frac{d}{m} \right) \right|^q \right]^{1/q};
\end{aligned} \tag{26}$$

(3) if $r = s = q$ and $\alpha_1 = \alpha_2 = m_1 = m_2 = 1$, one has

$$\begin{aligned}
& |I(f)| \\
& \leq (b-a)(d-c) \left[\frac{1}{(q+1)^2(q+2)^2} \right]^{1/q} \\
& \times \left[(q+1)^2 \left| \frac{\partial^2 f}{\partial x \partial y}(a, c) \right|^q + (q+1) \left| \frac{\partial^2 f}{\partial x \partial y}(a, d) \right|^q \right. \\
& \left. + (q+1) \left| \frac{\partial^2 f}{\partial x \partial y}(b, c) \right|^q + \left| \frac{\partial^2 f}{\partial x \partial y}(b, d) \right|^q \right]^{1/q}.
\end{aligned} \tag{27}$$

Corollary 18. Under conditions of Theorem 16, if $\alpha_1 = \alpha_2 = m_1 = m_2 = 1$, then

$$\begin{aligned}
& |I(f)| \\
& \leq (b-a)(d-c) \left[\frac{(q-1)^2}{(2q-r-1)(2q-s-1)} \right]^{1-1/q} \\
& \times \left[\frac{1}{(r+1)(r+2)(s+1)(s+2)} \right]^{1/q} \\
& \times \left[(r+1)(s+1) \left| \frac{\partial^2 f}{\partial x \partial y}(a, c) \right|^q + (r+1) \left| \frac{\partial^2 f}{\partial x \partial y}(a, d) \right|^q \right. \\
& \left. + (s+1) \left| \frac{\partial^2 f}{\partial x \partial y}(b, c) \right|^q + \left| \frac{\partial^2 f}{\partial x \partial y}(b, d) \right|^q \right]^{1/q}.
\end{aligned} \tag{28}$$

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The author appreciates the anonymous referee for his/her careful corrections to and valuable comments on the original version of this paper. This work was partially supported by the NNSF under Grant no. 11361038 of China, by the Foundation of the Research Program under Grant no. NJZY13159 of Science and Technology at Universities of Inner Mongolia Autonomous Region, and by the Science Research Funding under Grant no. NMD1225 of the Inner Mongolia University for Nationalities, China.

References

- [1] G. Toader, "Some generalizations of the convexity," in *Proceedings of the Colloquium on Approximation and Optimization*, pp. 329–338, University of Cluj-Napoca, Cluj-Napoca, Romania, 1985.
- [2] V. G. Miheșan, "A generalization of the convexity," in *Proceedings of the Seminar on Functional Equations, Approximation and Convexity*, Cluj-Napoca, Romania, 1993.
- [3] S. S. Dragomir, "On the Hadamard's inequality for convex functions on the co-ordinates in a rectangle from the plane," *Taiwanese Journal of Mathematics*, vol. 5, no. 4, pp. 775–788, 2001.
- [4] S. S. Dragomir and C. E. M. Pearce, "Selected topics on Hermite-Hadamard inequalities and applications," RGMIA Monographs, Victoria University, 2000, <http://rgmia.org/monographs/hermite-hadamard.html>.
- [5] M. E. Özdemir, A. O. Akdemir, H. Kavurmaci, and M. Avci, "On the Simpson's inequality for co-ordinated convex functions," <http://arxiv.org/abs/1101.0075>.
- [6] M. A. Latif and S. S. Dragomir, "On some new inequalities for differentiable co-ordinated convex functions," *Journal of Inequalities and Applications*, vol. 2012, article 28, 2012.
- [7] S.-P. Bai and F. Qi, "Some inequalities for (s_1, m_1) – (s_2, m_2) -convex functions on co-ordinates," *Global Journal of Mathematical Analysis*, vol. 1, no. 1, pp. 22–28, 2013.
- [8] B.-Y. Xi, J. Hua, and F. Qi, "Hermite-Hadamard type inequalities for extended s-convex functions on co-ordinates in a rectangle," *Journal of Applied Analysis*, vol. 20, no. 1, 2014.