Research Article

A Gauss-Kuzmin Theorem and Related Questions for θ -Expansions

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Using the natural extension for θ -expansions, we give an infinite-order-chain representation of the sequence of the incomplete quotients of these expansions. Together with the ergodic behavior of a certain homogeneous random system with complete connections, this allows us to solve a variant of Gauss-Kuzmin problem for the previous fraction expansion.

1. Introduction

During the last fifty years, a large amount of research has been devoted to the study of various algorithms for the representation of real numbers by means of sequences of integers. Motivated by problems in random continued fraction expansions (see [1]), Chakraborty and Rao [2] have initiated a systematic study of the continued fraction expansion of a number in terms of an irrational $\theta \in (0, 1)$. This new expansion of positive reals, different from the regular continued fraction expansion, is called θ -expansion.

The purpose of this paper is to solve a Gauss-Kuzmin problem for θ -expansions. In order to solve the problem, we apply the theory of random systems with complete connections extensively studied by Iosifescu and Grigorescu [3]. First we outline the historical framework of this problem. In Section 1.2, we present the current framework. In Section 1.3, we review known results.

1.1. Gauss' Problem. One of the first and still one of the most important results in the metrical theory of continued fractions is the so-called Gauss-Kuzmin theorem. Any irrational $0 < x < 1$ can be written as the infinite regular continued fraction

$$x = \frac{1}{a_1 + (1/(a_2 + (1/a_3 + \dots)))} := [a_1, a_2, a_3, \dots], \quad (1)$$

where $a_n \in \mathbb{N}_+ := \{1, 2, 3, \dots\}$ [4]. Such integers $a_1, a_2, \dots$ are called *incomplete quotients* (or *continued fraction digits*) of x . The metrical theory of continued fraction expansions started on October 25, 1800, with a note by Gauss in his mathematical diary [5]. Define the *regular continued fraction* (or *Gauss transformation* τ on the unit interval $I := [0, 1]$ by

$$\tau(x) = \begin{cases} \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases} \quad (2)$$

where $\lfloor \cdot \rfloor$ denotes the floor (or entire) function. With respect to the asymptotic behavior of iterations $\tau^n = \tau \circ \dots \circ \tau$ (n -times) of τ , Gauss wrote (in modern notation) that

$$\lim_{n \rightarrow \infty} \lambda(\tau^n \leq x) = \frac{\log(1+x)}{\log 2}, \quad x \in I, \quad (3)$$

where λ denotes the Lebesgue measure on I . In 1812, Gauss asked Laplace [5] to estimate the n th error term $e_n(x)$ defined by

$$e_n(x) := \lambda(\tau^{-n}[0, x]) - \frac{\log(1+x)}{\log 2}, \quad n \geq 1, \quad x \in I. \quad (4)$$

This has been called *Gauss' Problem*. It received first solution more than a century later, when Kuzmin [6] showed in 1928 that $e_n(x) = \mathcal{O}(q^{\sqrt{n}})$ as $n \rightarrow \infty$, uniformly in x with some

(unspecified) $0 < q < 1$. This has been called the *Gauss-Kuzmin theorem* or the *Kuzmin theorem*.

One year later, using a different method, Lévy [7] improved Kuzmin's result by showing that $|e_n(x)| \leq q^n$ for $n \in \mathbb{N}_+$, $x \in I$, with $q = 3.5 - 2\sqrt{2} = 0.67157\dots$. For such historical reasons, the *Gauss-Kuzmin-Lévy theorem* is regarded as the first basic result in the rich metrical theory of continued fractions. An advantage of the Gauss-Kuzmin-Lévy theorem relative to the Gauss-Kuzmin theorem is the determination of the value of q .

To this day, the Gauss transformation, on which metrical theory of regular continued fraction is based, has fascinated researchers from various branches of mathematics and science with many applications in computer science, cosmology, and chaos theory [8]. In the last century, mathematicians broke new ground in this area. Apart from the regular continued fraction expansion, many other continued fraction expansions were studied [9, 10].

By such a development, generalizations of these problems for nonregular continued fractions are also called the *Gauss-Kuzmin problem* and the *Gauss-Kuzmin-Lévy problem* [11–15].

1.2. θ -Expansions. For a fixed $\theta \in (0, 1)$, we start with a brief review of continued fraction expansion with respect to θ , analogous to the regular continued fraction expansion which corresponds to the case $\theta = 1$.

For $x \in (0, \infty)$ let

$$a_0 := \max \{n \geq 0 : n\theta \leq x\}. \quad (5)$$

If x equals $a_0\theta$, we write

$$x := [a_0\theta]. \quad (6)$$

If not, define r_1 by

$$x := a_0\theta + \frac{1}{r_1}, \quad (7)$$

where $0 < 1/r_1 < \theta$. Then $r_1 > 1/\theta \geq \theta$ and let

$$a_1 := \max \{n \geq 0 : n\theta \leq r_1\}. \quad (8)$$

If $r_1 = a_1\theta$, then we write

$$x := [a_0\theta, a_1\theta]; \quad (9)$$

that is,

$$x = a_0\theta + \frac{1}{a_1\theta}. \quad (10)$$

If $a_1\theta < r_1$, define r_2 by

$$r_1 := a_1\theta + \frac{1}{r_2}, \quad (11)$$

where $0 < 1/r_2 < \theta$. So, $r_2 > 1/\theta \geq \theta$ and let

$$a_2 := \max \{n \geq 0 : n\theta \leq r_2\}. \quad (12)$$

In this way, either the process terminates after a finite number of steps or it continues indefinitely. Following standard notation, in the first case we write

$$x := [a_0\theta; a_1\theta, \dots, a_n\theta] \quad (13)$$

and we call this the finite continued fraction expansion of x with respect to θ (terminating at the n -stage). In the second case, we write

$$x := [a_0\theta; a_1\theta, a_2\theta, \dots] \quad (14)$$

and we call this the infinite continued fraction expansion of x with respect to θ .

When $0 < x < \theta$, we have $a_0 = 0$ and instead of writing

$$x := [0; a_1\theta, a_2\theta, \dots], \quad (15)$$

we simply write

$$x := [a_1\theta, a_2\theta, \dots] \quad (16)$$

which is the same in the usual notation

$$x = \frac{1}{a_1\theta + (1/(a_2\theta + (1/a_3\theta + \dots)))}. \quad (17)$$

Such a_n 's are also called *incomplete quotients* (or *continued fraction digits*) of x with respect to the expansion in (17).

In general, the θ -expansion of a number $x > 0$ is

$$a_0\theta + [0; a_1\theta, a_2\theta, \dots] := [a_0\theta; a_1\theta, a_2\theta, \dots], \quad (18)$$

where $a_0 = \lfloor x/\theta \rfloor$.

For $x \in [0, \theta]$, the θ -continued fraction expansion of x in (17) leads to an analogous transformation of Gauss map τ in (2). A natural question is whether this new transformation admits an absolutely continuous invariant probability like the Gauss measure in the case $\theta = 1$. Until now, the invariant measure was identified only in the particular case $\theta^2 = 1/m$, m a positive integer [2].

Motivated by this argument, since the invariant measure is a crucial tool in our approach, in the sequel we will consider only the case $\theta^2 = 1/m$ with m being a positive integer. Then $[a_1\theta, a_2\theta, a_3\theta, \dots]$ is the θ -expansion of any $x \in [0, \theta]$ if and only if the following conditions hold:

- (i) $a_n \geq m$ for any $m \in \mathbb{N}_+$;
- (ii) in case when x has a finite expansion, that is, $x = [a_1\theta, a_2\theta, a_3\theta, \dots, a_n\theta]$, then $a_n \geq m + 1$.

This continued fraction is treated as the following dynamical system.

Definition 1. Let $\theta \in (0, 1)$ and $m \in \mathbb{N}_+$ such that $\theta^2 = 1/m$.

- (i) The measure-theoretical dynamical system $([0, \theta], \mathcal{B}_{[0, \theta]}, T_\theta)$ is defined as follows: $\mathcal{B}_{[0, \theta]}$ denotes the σ -algebra of all Borel subsets of $[0, \theta]$, and T_θ is the transformation

$$T_\theta : [0, \theta] \longrightarrow [0, \theta];$$

$$T_\theta(x) := \begin{cases} \frac{1}{x} - \theta \left\lfloor \frac{1}{x\theta} \right\rfloor, & \text{if } x \in (0, \theta], \\ 0, & \text{if } x = 0. \end{cases} \quad (19)$$

- (ii) In addition to (i), we write $([0, \theta], \mathcal{B}_{[0, \theta]}, \gamma_\theta, T_\theta)$ as $([0, \theta], \mathcal{B}_{[0, \theta]}, T_\theta)$ with the following probability measure γ_θ on $([0, \theta], \mathcal{B}_{[0, \theta]})$:

$$\gamma_\theta(A) := \frac{1}{\log(1 + \theta^2)} \int_A \frac{\theta dx}{1 + \theta x}, \quad A \in \mathcal{B}_{[0, \theta]}. \quad (20)$$

By using T_θ , the sequence $(a_n)_{n \in \mathbb{N}_+}$ in (17) is obtained as follows:

$$a_n = a_n(x) = a_1(T_\theta^{n-1}(x)), \quad n \in \mathbb{N}_+, \quad (21)$$

with $T_\theta^0(x) = x$ and

$$a_1 = a_1(x) = \begin{cases} \left\lfloor \frac{1}{x\theta} \right\rfloor, & \text{if } x \neq 0, \\ \infty, & \text{if } x = 0. \end{cases} \quad (22)$$

In this way, T_θ gives the algorithm of θ -expansion.

Proposition 2. Let $([0, \theta], \mathcal{B}_{[0, \theta]}, \gamma_\theta, T_\theta)$ be as in Definition 1(ii).

- (i) $([0, \theta], \mathcal{B}_{[0, \theta]}, \gamma_\theta, T_\theta)$ is ergodic.
(ii) The measure γ_θ is invariant under T_θ ; that is, $\gamma_\theta(A) = \gamma_\theta(T_\theta^{-1}(A))$ for any $A \in \mathcal{B}_{[0, \theta]}$.

Proof. See Section 8 in [2]. $\square$

By Proposition 2(ii), $([0, \theta], \mathcal{B}_{[0, \theta]}, \gamma_\theta, T_\theta)$ is a “dynamical system” in the sense of Definition 3.1.3 in [16].

1.3. Known Results and Applications. In this subsection, we recall known results and their applications for θ -expansions.

1.3.1. Known Results. Let $0 < \theta < 1$. Define the n th order convergent $[a_1\theta, a_2\theta, \dots, a_n\theta]$ of $x \in [0, \theta]$ by truncating the θ -expansion in (17). Thus, Chakraborty and Rao proved in [2] that

$$[a_1\theta, a_2\theta, \dots, a_n\theta] \longrightarrow x, \quad n \longrightarrow \infty. \quad (23)$$

In what follows the stated identities hold for all n in case x has an infinite θ -expansion and they hold for $n \leq k$ in case x has a finite θ -expansion terminating at the k th stage.

To this end, define real functions $p_n(x)$ and $q_n(x)$, for $n \in \mathbb{N}_+$, by

$$p_n(x) := a_n(x)\theta p_{n-1}(x) + p_{n-2}(x), \quad (24)$$

$$q_n(x) := a_n(x)\theta q_{n-1}(x) + q_{n-2}(x), \quad (25)$$

with $p_{-1}(x) := 1$, $p_0(x) := 0$, $q_{-1}(x) := 0$, and $q_0(x) := 1$. By induction, we have

$$p_{n-1}(x)q_n(x) - p_n(x)q_{n-1}(x) = (-1)^n, \quad (26)$$

$$n \in \mathbb{N} := \mathbb{N}_+ \cup \{0\}.$$

By using (24) and (25), we can verify that

$$x = \frac{p_n(x) + T_\theta^n(x)p_{n-1}(x)}{q_n(x) + T_\theta^n(x)q_{n-1}(x)}, \quad n \geq 1. \quad (27)$$

By taking $T_\theta^n(x) = 0$ in (27), we obtain $[a_1\theta, a_2\theta, \dots, a_n\theta] = p_n(x)/q_n(x)$. Using (26) and (27) we obtain

$$x - \frac{p_n(x)}{q_n(x)} = \frac{(-1)^{n+1}T_\theta^n(x)}{q_n(x)(q_n(x) + T_\theta^n(x)q_{n-1}(x))}, \quad n \geq 1. \quad (28)$$

By applying $0 \leq T_\theta^n \leq \theta$ to (28), we can verify that

$$\frac{1}{q_n(x)(q_{n+1}(x) + \theta q_n(x))} \leq \left| x - \frac{p_n(x)}{q_n(x)} \right| \leq \frac{1}{q_n(x)q_{n+1}(x)}. \quad (29)$$

From (25), we have that $q_n(x) \geq \theta$, $n \in \mathbb{N}_+$. Further, also from (25) and by induction we have that

$$q_n(x) \geq \left\lfloor \frac{n}{2} \right\rfloor \theta^2. \quad (30)$$

Finally, (23) follows from (29) and (30).

1.3.2. Application to Ergodic Theory. Similar to classical results on regular continued fractions, using the ergodicity of T_θ and Birkhoff’s ergodic theorem [17], a number of results were obtained.

For q_n in (25), its asymptotic growth rate β is defined as

$$\beta = \lim_{n \rightarrow \infty} \frac{1}{n} \log q_n. \quad (31)$$

This is a Lévy result and Chakraborty and Rao [2] obtained that β is a finite number

$$\beta = \frac{1}{1 + \theta^2} \int_0^\theta \frac{\theta \log x}{1 + x\theta} dx. \quad (32)$$

They also give a Khintchine result, that is, the asymptotic value of the arithmetic mean of $a_1, a_2, \dots, a_n$, where a_1 and a_n are given in (22) and (21). We have

$$\lim_{n \rightarrow \infty} \frac{a_1 + a_2 + \dots + a_n}{n} = \infty. \quad (33)$$

It should be stressed that the ergodic theorem does not yield any information on the convergence rate in the Gauss problem that amounts to the asymptotic behavior of $\mu(T_\theta^{-n})$ as $n \rightarrow \infty$, where μ is an arbitrary probability measure on $\mathcal{B}_{[0, \theta]}$.

It is only very recent that there has been any investigation of the metrical properties of the θ -expansions. Thus, the results obtained in this paper allow a solution of a Gauss-Kuzmin type problem. We may emphasize that, to our knowledge, Theorem 29 is the first Gauss-Kuzmin result proved for θ -expansions. Our solution presented here is based on the ergodic behavior of a certain random system with complete connections.

The paper is organized as follows. In Section 2, we show the probability structure of $(a_n)_{n \in \mathbb{N}_+}$ under the Lebesgue measure by using the Brodén-Borel-Lévy formula.

In Section 3, we consider the so-called natural extension of $([0, \theta], \mathcal{B}_{[0, \theta]}, \gamma_\theta, T_{[0, \theta]})$ [18]. In Section 4, we derive its Perron-Frobenius operator under different probability measures on $([0, \theta], \mathcal{B}_{[0, \theta]})$. In particular, we derive the asymptotic behavior for the Perron-Frobenius operator of $([0, \theta], \mathcal{B}_{[0, \theta]}, \gamma_\theta, T_{[0, \theta]})$. In Section 5, we study the ergodicity of the associated random system with complete connections (RSCCs for short). In Section 6, we solve a variant of Gauss-Kuzmin problem for θ -expansions. By using the ergodic behavior of the RSCC introduced in Section 5, we determine the limit of the sequence $(\mu(T_\theta^n < x))_{n \geq 1}$ of distributions as $n \rightarrow \infty$.

2. Prerequisites

Roughly speaking, the metrical theory of continued fraction expansions is about the sequence $(a_n)_{n \in \mathbb{N}_+}$ of incomplete quotients and related sequences [4]. As remarked earlier in the introduction we will adopt a similar strategy to that used for regular continued fractions. We begin with a Brodén-Borel-Lévy formula for θ -expansions. Then, some consequences of it to be used in the sequel are also derived.

In this section, let us fix $0 < \theta < 1$, $\theta^2 = 1/m$, $m \in \mathbb{N}_+$.

For $x \in [0, \theta]$, consider $a_n = a_n(x)$, $n \in \mathbb{N}_+$, as in (21) and (22). Putting $\mathbb{N}_m := \{m, m+1, \dots\}$, $m \in \mathbb{N}_+$, the incomplete quotients a_n , $n \in \mathbb{N}_+$, take positive integer values in $\mathbb{N}_m$.

For any $n \in \mathbb{N}_+$ and $i^{(n)} = (i_1, \dots, i_n) \in \mathbb{N}_m^n$, define the fundamental interval associated with $i^{(n)}$ by

$$I(i^{(n)}) = \{x \in [0, \theta] : a_k(x) = i_k \text{ for } k = 1, \dots, n\}, \quad (34)$$

where $I(i^{(0)}) = [0, \theta]$. For example, for any $i \in \mathbb{N}_m$, we have

$$I(i) = \{x \in [0, \theta] : a_1(x) = i\} = \left(\frac{1}{(i+1)\theta}, \frac{1}{i\theta} \right). \quad (35)$$

We will write $I(a_1, \dots, a_n) = I(a^{(n)})$, $n \in \mathbb{N}_+$. If $n \geq 1$ and $i_n \in \mathbb{N}_m$, then we have

$$I(a_1, \dots, a_n) = I(i^{(n)}). \quad (36)$$

From the definition of T_θ and (27), we have

$$I(a^{(n)}) = (u(a^{(n)}), v(a^{(n)})), \quad (37)$$

where $u(a^{(n)})$ and $v(a^{(n)})$ are defined as

$$\begin{aligned} u(a^{(n)}) &:= \begin{cases} \frac{p_n + \theta p_{n-1}}{q_n + \theta q_{n-1}}, & \text{if } n \text{ is odd,} \\ \frac{p_n}{q_n}, & \text{if } n \text{ is even,} \end{cases} \\ v(a^{(n)}) &:= \begin{cases} \frac{p_n}{q_n}, & \text{if } n \text{ is odd,} \\ \frac{p_n + \theta p_{n-1}}{q_n + \theta q_{n-1}}, & \text{if } n \text{ is even,} \end{cases} \end{aligned} \quad (38)$$

where $p_n := p_n(x)$ and $q_n := q_n(x)$ are defined in (24) and (25), respectively.

Let λ_θ denote a Lebesgue measure on $[0, \theta]$. Using (26) we get

$$\begin{aligned} \lambda_\theta(I(a^{(n)})) &= \frac{1}{\theta} \left| \frac{p_n}{q_n} - \frac{p_n + \theta p_{n-1}}{q_n + \theta q_{n-1}} \right| \\ &= \frac{1}{q_n(q_n + \theta q_{n-1})}. \end{aligned} \quad (39)$$

To derive the so-called Brodén-Borel-Lévy formula [3, 4] for θ -expansions, let us define $(s_n)_{n \in \mathbb{N}}$ by

$$s_0 := 0, \quad s_n := \frac{q_{n-1}}{q_n}, \quad n \geq 1. \quad (40)$$

From (25), $s_n = 1/(a_n\theta + s_{n-1})$ for $n \geq 1$. Hence,

$$\begin{aligned} s_n &= \frac{1}{a_n\theta + (1/(a_{n-1}\theta + \dots + (1/a_1\theta)))} \\ &:= [a_n\theta, a_{n-1}\theta, \dots, a_1\theta], \end{aligned} \quad (41)$$

for $n \geq 1$.

Proposition 3 (Brodén-Borel-Lévy-type formula). *Let λ_θ denote the Lebesgue measure on $[0, \theta]$. For any $n \in \mathbb{N}_+$, the conditional probability $\lambda_\theta(T_\theta^n < x \mid a_1, \dots, a_n)$ is given as follows:*

$$\lambda_\theta(T_\theta^n < x \mid a_1, \dots, a_n) = \frac{(s_n\theta + 1)x}{\theta(s_nx + 1)}, \quad x \in [0, \theta], \quad (42)$$

where s_n is defined in (40) and $a_1, \dots, a_n$ are as in (21) and (22).

Proof. By definition, we have

$$\begin{aligned} \lambda_\theta(T_\theta^n < x \mid a_1, \dots, a_n) &= \frac{\lambda_\theta((T_\theta^n < x) \cap I(a_1, \dots, a_n))}{\lambda_\theta(I(a_1, \dots, a_n))}, \end{aligned} \quad (43)$$

for any $n \in \mathbb{N}_+$ and $x \in [0, \theta]$. Using (27) and (37), we get

$$\begin{aligned} \lambda_\theta((T_\theta^n < x) \cap I(a_1, \dots, a_n)) &= \frac{1}{\theta} \left| \frac{p_n}{q_n} - \frac{p_n + x p_{n-1}}{q_n + x q_{n-1}} \right| \\ &= \frac{x}{q_n(q_n + x q_{n-1})\theta}. \end{aligned} \quad (44)$$

From this and (39) it follows that

$$\begin{aligned} \lambda_\theta(T_\theta^n < x \mid a_1, \dots, a_n) &= \frac{\lambda_\theta((T_\theta^n < x) \cap I(a_1, \dots, a_n))}{\lambda_\theta(I(a_1, \dots, a_n))} \\ &= \frac{x(q_n + \theta q_{n-1})}{(q_n + x q_{n-1})\theta} = \frac{x(s_n\theta + 1)}{(s_nx + 1)\theta}, \end{aligned} \quad (45)$$

for any $n \in \mathbb{N}_+$ and $x \in [0, \theta]$. $\square$

The Brodén-Borel-Lévy formula allows us to determine the probability structure of incomplete quotients $(a_n)_{n \in \mathbb{N}_+}$ under λ_θ .

Proposition 4. For any $i \in \mathbb{N}_m$ and $n \in \mathbb{N}_+$, one has

$$\begin{aligned}\lambda_\theta(a_1 = i) &= \frac{m}{i(i+1)}, \\ \lambda_\theta(a_{n+1} = i \mid a_1, \dots, a_n) &= P_i(s_n),\end{aligned}\quad (46)$$

where $(s_n)_{n \in \mathbb{N}_+}$ is defined in (40), and

$$P_i(x) := \frac{x\theta + 1}{(x + i\theta)(x + (i+1)\theta)}. \quad (47)$$

Proof. From (35), the case $\lambda_\theta(a_1 = i)$ holds. For $n \in \mathbb{N}_+$ and $x \in [0, \theta]$, we have $T_\theta^n(x) = [a_{n+1}\theta, a_{n+2}\theta, \dots]$. Using (42), we obtain

$$\begin{aligned}\lambda_\theta(a_{n+1} = i \mid a_1, \dots, a_n) &= \lambda_\theta\left(T_\theta^n \in \left(\frac{1}{(i+1)\theta}, \frac{1}{i\theta}\right] \mid a_1, \dots, a_n\right) \\ &= \frac{(s_n\theta + 1)(1/i\theta)}{\theta(s_n(1/i\theta) + 1)} - \frac{(s_n\theta + 1)(1/(i+1)\theta)}{\theta(s_n(1/(i+1)\theta) + 1)} \\ &= P_i(s_n).\end{aligned}\quad (48)$$

□

Remark 5. (i) It is easy to check that

$$\sum_{i=m}^{\infty} P_i(x) = 1, \quad \text{for any } x \in [0, \theta]. \quad (49)$$

(ii) Proposition 4 is the starting point of an approach to the metrical theory of θ -expansions via dependence with complete connections (see [3, Section 5.2]).

Corollary 6. The sequence, $(s_n)_{n \in \mathbb{N}_+}$ with $s_0 = 0$ is an $[0, \theta]$ -Markov chain on $([0, \theta], \mathcal{B}_{[0, \theta]}, \lambda_\theta)$ with the following transition mechanism: from state s the possible transitions are to any state $1/(s + i\theta)$ with corresponding transition probability $P_i(s)$, $i \in \mathbb{N}_m$.

3. An Infinite-Order-Chain Representation

In this section we introduce the natural extension $\overline{T}_\theta$ of T_θ in (19) and define extended random variables according to Chapter 1.3.3 of [4]. Then we give an infinite-order-chain representation of the sequence of the incomplete quotients for θ -expansions.

3.1. Natural Extension. Let $([0, \theta], \mathcal{B}_{[0, \theta]}, T_\theta)$ be as in Definition 1(i). Let $\mathcal{B}_{[0, \theta]}^2 := \mathcal{B}_{[0, \theta]} \otimes \mathcal{B}_{[0, \theta]}$ denote the σ -algebra of Borel sets in $[0, \theta]^2 := [0, \theta] \times [0, \theta]$.

Definition 7. The natural extension [18] of $([0, \theta], \mathcal{B}_{[0, \theta]}, T_\theta)$ is $([0, \theta]^2, \mathcal{B}_{[0, \theta]}^2, \overline{T}_\theta)$ where the transformation $\overline{T}_\theta$ of $([0, \theta]^2, \mathcal{B}_{[0, \theta]}^2)$ is defined by

$$\begin{aligned}\overline{T}_\theta : [0, \theta]^2 &\longrightarrow [0, \theta]^2; \\ \overline{T}_\theta(x, y) &:= \left(T_\theta(x), \frac{1}{a_1(x)\theta + y}\right), \quad (x, y) \in [0, \theta]^2.\end{aligned}\quad (50)$$

This is a one-to-one transformation of $[0, \theta]^2$ with the inverse

$$(\overline{T}_\theta)^{-1}(x, y) = \left(\frac{1}{a_1(y)\theta + x}, T_\theta(y)\right), \quad (x, y) \in [0, \theta]^2. \quad (51)$$

Iterations of (50) and (51) are given as follows for each $n \geq 2$:

$$\begin{aligned}(\overline{T}_\theta)^n(x, y) &= (T_\theta^n(x), [a_n(x)\theta, a_{n-1}(x)\theta, \dots, a_2(x)\theta, a_1(x)\theta + y]), \\ (\overline{T}_\theta)^{-n}(x, y) &= ([a_n(y)\theta, a_{n-1}(y)\theta, \dots, a_2(y)\theta, a_1(y)\theta + x], T_\theta^n(y)).\end{aligned}\quad (52)$$

For γ_θ in (20), define its extended measure $\overline{\gamma}_\theta$ on $([0, \theta]^2, \mathcal{B}_{[0, \theta]}^2)$ as

$$\overline{\gamma}_\theta(B) := \frac{1}{\log(1 + \theta^2)} \iint_B \frac{dx dy}{(1 + xy)^2}, \quad B \in \mathcal{B}_{[0, \theta]}^2. \quad (53)$$

Then

$$\overline{\gamma}_\theta(A \times [0, \theta]) = \overline{\gamma}_\theta([0, \theta] \times A) = \gamma_\theta(A), \quad (54)$$

for any $A \in \mathcal{B}_{[0, \theta]}$.

Proposition 8. The measure $\overline{\gamma}_\theta$ is preserved by $\overline{T}_\theta$.

Proof. We show that $\overline{\gamma}_\theta((\overline{T}_\theta)^{-1}(B)) = \overline{\gamma}_\theta(B)$ for any $B \in \mathcal{B}_{[0, \theta]}^2$. Since $\overline{T}_\theta$ is invertible on $[0, \theta]^2$, the last equation is equivalent to

$$\overline{\gamma}_\theta(\overline{T}_\theta(B)) = \overline{\gamma}_\theta(B), \quad \text{for any } B \in \mathcal{B}_{[0, \theta]}^2. \quad (55)$$

Recall fundamental interval in (34). Since the collection of Cartesian products $I(i^{(n)}) \times I(j^{(t)})$, $i^{(n)} \in \mathbb{N}_m^n$, $j^{(t)} \in \mathbb{N}_m^t$, $n, t \in \mathbb{N}$, generates the σ -algebra $\mathcal{B}_{[0, \theta]}^2$, it is enough to show that

$$\overline{\gamma}_\theta(\overline{T}_\theta(I(i^{(n)}) \times I(j^{(t)}))) = \overline{\gamma}_\theta(I(i^{(n)}) \times I(j^{(t)})), \quad (56)$$

for any $i^{(n)} \in \mathbb{N}_m^n$, $j^{(t)} \in \mathbb{N}_m^t$, $n, t \in \mathbb{N}$. It follows from (54) and Proposition 2(ii) that (56) holds for $n = 0$ and $t \in \mathbb{N}$. If $n \in \mathbb{N}_+$, then it is easy to see that

$$\begin{aligned}\overline{T}_\theta(I(i^{(n)}) \times I(j^{(t)})) &= I(i_2, \dots, i_n) \times I(i_1, j_1, \dots, j_t), \quad t \in \mathbb{N}_+, \\ &= I(i_2, \dots, i_n) \times I(i_1, j_1, \dots, j_t),\end{aligned}\quad (57)$$

where $I(i_2, \dots, i_n)$ equals $[0, \theta]$ for $n = 1$. Also, if $I(i^{(n)}) = (a, b) \subset [0, \theta]$ and $I(j^{(t)}) = (c, d) \subset [0, \theta]$, then

$$\begin{aligned}I(i_2, \dots, i_n) &= \left(\frac{1}{b} - i_1\theta, \frac{1}{a} - i_1\theta\right), \\ I(i_1, j_1, \dots, j_t) &= \left(\frac{1}{d + i_1\theta}, \frac{1}{c + i_1\theta}\right).\end{aligned}\quad (58)$$

A simple computation yields

$$\overline{\gamma}_\theta((a, b) \times (c, d)) = \frac{1}{\log(1 + \theta^2)} \log \frac{(ac + 1)(bd + 1)}{(ad + 1)(bc + 1)}, \quad (59)$$

and then

$$\begin{aligned} & \overline{\gamma}_\theta \left(\left(\frac{1}{b} - i_1 \theta, \frac{1}{a} - i_1 \theta \right) \times \left(\frac{1}{d + i_1 \theta}, \frac{1}{c + i_1 \theta} \right) \right) \\ &= \frac{1}{\log(1 + \theta^2)} \\ & \times \log \left(\left(\left(\frac{(1/b - i_1 \theta)}{(d + i_1 \theta)} + 1 \right) \left(\frac{(1/a - i_1 \theta)}{(c + i_1 \theta)} + 1 \right) \right) \right. \\ & \quad \left. \times \left(\left(\frac{(1/b - i_1 \theta)}{(c + i_1 \theta)} + 1 \right) \left(\frac{(1/a - i_1 \theta)}{(d + i_1 \theta)} + 1 \right) \right)^{-1} \right) \\ &= \frac{1}{\log(1 + \theta^2)} \log \frac{(ac + 1)(bd + 1)}{(ad + 1)(bc + 1)}, \end{aligned} \quad (60)$$

that is, (56) holds. $\square$

3.2. Extended Random Variables. With respect to $\overline{T}_\theta$ in (50), define *extended incomplete quotients* $\overline{a}_l(x, y)$, $l \in \mathbb{Z} := \{\dots, -2, -1, 0, 1, 2, \dots\}$, $(x, y) \in [0, \theta]^2$ by

$$\overline{a}_{l+1}(x, y) := \overline{a}_1 \left(\left(\overline{T}_\theta \right)^l (x, y) \right), \quad l \in \mathbb{Z}, \quad (61)$$

with

$$\overline{a}_1(x, y) = a_1(x), \quad (x, y) \in [0, \theta]^2. \quad (62)$$

Remark 9. (i) Since $\overline{T}_\theta$ is invertible, it follows that $\overline{a}_l(x, y)$ in (61) is also well-defined for $l \leq 0$. By (52), we have

$$\begin{aligned} \overline{a}_n(x, y) &= a_n(x), \quad \overline{a}_0(x, y) = a_1(y), \\ \overline{a}_{-n}(x, y) &= a_{n+1}(y), \end{aligned} \quad (63)$$

for any $n \in \mathbb{N}_+$ and $(x, y) \in [0, \theta]^2$.

(ii) From Proposition 8, the doubly infinite sequence $(\overline{a}_l(x, y))_{l \in \mathbb{Z}}$ is strictly stationary (i.e., its distribution is invariant under a shift of the indices) under $\overline{\gamma}_\theta$.

The following theorem will play a key role in the sequel.

Theorem 10. For any $x \in [0, \theta]$, one has

$$\overline{\gamma}_\theta([0, x] \times [0, \theta] \mid \overline{a}_0, \overline{a}_{-1}, \dots) = \frac{(a\theta + 1)x}{(ax + 1)\theta} \quad \overline{\gamma}_\theta\text{-a.s.}, \quad (64)$$

where $a := [\overline{a}_0\theta, \overline{a}_{-1}\theta, \dots]$ (= the θ -expansion with incomplete quotients $\overline{a}_0, \overline{a}_{-1}, \dots$).

Proof. Let I_n denote the fundamental interval $I(\overline{a}_0, \overline{a}_{-1}, \dots, \overline{a}_{-n})$ for $n \in \mathbb{N}$. We have

$$\begin{aligned} & \overline{\gamma}_\theta([0, x] \times [0, \theta] \mid \overline{a}_0, \overline{a}_{-1}, \dots) \\ &= \lim_{n \rightarrow \infty} \overline{\gamma}_\theta([0, x] \times [0, \theta] \mid \overline{a}_0, \dots, \overline{a}_{-n}) \quad \overline{\gamma}_\theta\text{-a.s.}, \\ & \overline{\gamma}_\theta([0, x] \times [0, \theta] \mid \overline{a}_0, \dots, \overline{a}_{-n}) \\ &= \frac{\overline{\gamma}_\theta([0, x] \times I_n)}{\overline{\gamma}_\theta([0, \theta] \times I_n)} \\ &= \frac{1}{\gamma_\theta(I_n)} \frac{1}{\log(1 + \theta^2)} \int_{I_n} dy \int_0^x \frac{du}{(uy + 1)^2} \\ &= \frac{1}{\gamma_\theta(I_n)} \frac{1}{\theta} \int_{I_n} \frac{x(1 + y\theta)}{1 + xy} d\gamma_\theta(y) \\ &= \frac{x(y_n\theta + 1)}{(xy_n + 1)\theta}, \end{aligned} \quad (65)$$

for some $y_n \in I_n$. Since

$$\lim_{n \rightarrow \infty} y_n = [\overline{a}_0\theta, \overline{a}_{-1}\theta, \dots] = a, \quad (66)$$

the proof is completed. $\square$

The probability structure of $(\overline{a}_l)_{l \in \mathbb{Z}}$ under $\overline{\gamma}_\theta$ is given as follows.

Corollary 11. For any $i \in \mathbb{N}_m$, one has

$$\overline{\gamma}_\theta(\overline{a}_1 = i \mid \overline{a}_0, \overline{a}_{-1}, \dots) = P_i(a) \quad \overline{\gamma}_\theta\text{-a.s.}, \quad (67)$$

where $a = [\overline{a}_0\theta, \overline{a}_{-1}\theta, \dots]$ and the functions P_i , $i \in \mathbb{N}_m$, are defined by (47).

Proof. Let I_n be as in the proof of Theorem 10. We have

$$\overline{\gamma}_\theta(\overline{a}_1 = i \mid \overline{a}_0, \overline{a}_{-1}, \dots) = \lim_{n \rightarrow \infty} \overline{\gamma}_\theta(\overline{a}_1 = i \mid I_n). \quad (68)$$

We have

$$(\overline{a}_1 = i) = \left(\frac{1}{(i+1)\theta}, \frac{1}{i\theta} \right] \times [0, \theta], \quad i \in \mathbb{N}_m. \quad (69)$$

Now

$$\begin{aligned} & \overline{\gamma}_\theta \left(\left(\frac{1}{(i+1)\theta}, \frac{1}{i\theta} \right] \times [0, \theta] \mid I_n \right) \\ &= \frac{\overline{\gamma}_\theta((1/(i+1)\theta, 1/i\theta] \times I_n)}{\overline{\gamma}_\theta([0, \theta] \times I_n)} \\ &= \frac{1}{\gamma_\theta(I_n)} \int_{I_n} P_i(y) d\gamma_\theta(y) = P_i(y_n), \end{aligned} \quad (70)$$

for some $y_n \in I_n$. From (66), the proof is completed. $\square$

Remark 12. The strict stationarity of $(\overline{a}_l)_{l \in \mathbb{Z}}$ under $\overline{\gamma}_\theta$ implies that

$$\overline{\gamma}_\theta(\overline{a}_{l+1} = i \mid \overline{a}_l, \overline{a}_{l-1}, \dots) = P_i(a) \quad \overline{\gamma}_\theta\text{-a.s.}, \quad (71)$$

for any $i \in \mathbb{N}_m$ and $l \in \mathbb{Z}$, where $a = [\bar{a}_l \theta, \bar{a}_{l-1} \theta, \dots]$. The last equation emphasizes that $(\bar{a}_l)_{l \in \mathbb{Z}}$ is an infinite-order-chain representation in the theory of dependence with complete connections (see [3, Section 5.5]).

Define extended random variables $(\bar{s}_l)_{l \in \mathbb{Z}}$ as $\bar{s}_l := [\bar{a}_l \theta, \bar{a}_{l-1} \theta, \dots]$, $l \in \mathbb{Z}$. Clearly, $\bar{s}_l = \bar{s}_0 \circ (\bar{T}_\theta)^l$, $l \in \mathbb{Z}$. It follows from Proposition 8 and Corollary 11 that $(\bar{s}_l)_{l \in \mathbb{Z}}$ is a strictly stationary $[0, \theta]$ -valued Markov process on $([0, \theta]^2, \mathcal{B}_{[0, \theta]}^2, \bar{\gamma}_\theta)$ with the following transition mechanism: from state $\bar{s} \in [0, \theta]$ the possible transitions are to any state $1/(\bar{s} + i\theta)$ with corresponding transition probability $P_i(\bar{s})$, $i \in \mathbb{N}_m$. Clearly, for any $l \in \mathbb{Z}$, we have

$$\begin{aligned} \bar{\gamma}_\theta(\bar{s}_l < x) &= \bar{\gamma}_\theta(\bar{s}_0 < x) \\ &= \bar{\gamma}_\theta([0, \theta] \times [0, x]) = \gamma_\theta([0, x]), \quad x \in [0, \theta]. \end{aligned} \quad (72)$$

Motivated by Theorem 10, we will consider the one-parameter family $\{\gamma_{\theta, a} : a \in [0, \theta]\}$ of (conditional) probability measures on $([0, \theta], \mathcal{B}_{[0, \theta]})$ defined by their distribution functions

$$\gamma_{\theta, a}([0, x]) := \frac{(a\theta + 1)x}{(ax + 1)\theta}, \quad x, a \in [0, \theta]. \quad (73)$$

Note that $\gamma_{\theta, 0} = \lambda_\theta$.

For any $a \in [0, \theta]$, put

$$s_{0, a} := a, \quad s_{n, a} := \frac{1}{a_n \theta + s_{n-1, a}}, \quad n \in \mathbb{N}_+. \quad (74)$$

Remark 13. It follows from the properties just described of the process $(\bar{s}_l)_{l \in \mathbb{Z}}$ that the sequence $(s_{n, a})_{n \in \mathbb{N}_+}$ is an $[0, \theta]$ -valued Markov chain on $([0, \theta], \mathcal{B}_{[0, \theta]}, \gamma_{\theta, a})$ which starts at $s_{0, a} := a$ and has the following transition mechanism: from state $s \in [0, \theta]$ the possible transitions are to any state $1/(s + i\theta)$ with corresponding transition probability $P_i(s)$, $i \in \mathbb{N}_m$.

4. An Operatorial Treatment

Let $([0, \theta], \mathcal{B}_{[0, \theta]}, \gamma_\theta, T_\theta)$ be as in Definition 1(ii). The Gauss-Kuzmin problem for the transformation T_θ can be approached in terms of the associated Perron-Frobenius operator.

Let μ be a probability measure on $([0, \theta], \mathcal{B}_{[0, \theta]})$ such that $\mu((T_\theta)^{-1}(A)) = 0$ whenever $\mu(A) = 0$ for any $A \in \mathcal{B}_{[0, \theta]}$. For example, this condition is satisfied if T_θ is μ -preserving; that is, $\mu(T_\theta)^{-1} = \mu$. Let

$$L_\mu^1 := \left\{ f : [0, \theta] \longrightarrow \mathbb{C} : \int_0^\theta |f| d\mu < \infty \right\}. \quad (75)$$

The *Perron-Frobenius operator* of T_θ under μ is defined as the bounded linear operator U_μ which takes the Banach space L_μ^1 into itself and satisfies the equation

$$\int_A U_\mu f d\mu = \int_{(T_\theta)^{-1}(A)} f d\mu \quad \forall A \in \mathcal{B}_{[0, \theta]}, \quad f \in L_\mu^1. \quad (76)$$

Proposition 14. (i) The Perron-Frobenius operator $U := U_{\gamma_\theta}$ of T_θ under the invariant probability measure γ_θ is given a.e. in $[0, \theta]$ by the equation

$$Uf(x) = \sum_{i \geq m} P_i(x) f(u_i(x)), \quad m \in \mathbb{N}_+, \quad f \in L_{\gamma_\theta}^1, \quad (77)$$

where P_i , $i \geq m$, is as in (47) and u_i , $i \geq m$, is defined by

$$u_i : [0, \theta] \longrightarrow [0, \theta]; \quad u_i(x) := \frac{1}{x + i\theta}. \quad (78)$$

(ii) Let μ be a probability measure on $([0, \theta], \mathcal{B}_{[0, \theta]})$ such that μ is absolutely continuous with respect to the Lebesgue measure λ_θ and let $h := d\mu/d\lambda_\theta$ a.e. in $[0, \theta]$. For any $n \in \mathbb{N}$ and $A \in \mathcal{B}_{[0, \theta]}$, one has

$$\mu((T_\theta)^{-n}(A)) = \int_A U^n f(x) d\gamma_\theta(x), \quad (79)$$

where $f(x) := (\log(1 + \theta^2))((x\theta + 1)/\theta)h(x)$, $x \in [0, \theta]$.

Proof. (i) Let $T_{\theta, i}$ denote the restriction of T_θ to the subinterval $I(i) := (1/(i + 1)\theta, 1/i\theta]$, $i \geq m$, that is,

$$T_{\theta, i}(x) = \frac{1}{x} - i\theta, \quad x \in I(i). \quad (80)$$

Let $C(A) := (T_\theta)^{-1}(A)$ and $C_i(A) := (T_{\theta, i})^{-1}(A)$ for any $A \in \mathcal{B}_{[0, \theta]}$. Since $C(A) = \bigcup_i C_i(A)$ and $C_i \cap C_j$ is a null set when $i \neq j$, we have

$$\int_{C(A)} f d\gamma_\theta = \sum_{i \geq m} \int_{C_i(A)} f d\gamma_\theta, \quad f \in L_{\gamma_\theta}^1, \quad A \in \mathcal{B}_{[0, \theta]}. \quad (81)$$

For any $i \geq m$, by the change of variable $x = (T_{\theta, i})^{-1}(y) = u_i(y)$, we successively obtain

$$\begin{aligned} & \int_{C_i(A)} f(x) d\gamma_\theta(x) \\ &= \frac{\theta}{\log(1 + \theta^2)} \int_{C_i(A)} \frac{f(x)}{1 + x\theta} dx \\ &= \frac{1}{\log(1 + \theta^2)} \int_A \frac{f(u_i(y))}{1 + u_i(y)\theta} \frac{\theta dy}{(y + i\theta)^2} \\ &= \int_A P_i(y) f(u_i(y)) d\gamma_\theta(y). \end{aligned} \quad (82)$$

Now, (77) follows from (81) and (82).

(ii) We will use mathematical induction. For $n = 0$, (80) holds by definitions of f and h . Assume that (80) holds for some $n \in \mathbb{N}$. Then

$$\begin{aligned} \mu((T_\theta)^{-(n+1)}(A)) &= \mu((T_\theta)^{-n}((T_\theta)^{-1}(A))) \\ &= \int_{C(A)} U^n f(x) d\gamma_\theta(x). \end{aligned} \quad (83)$$

By the very definition of the Perron-Frobenius operator U , we have

$$\int_{C(A)} U^n f(x) d\gamma_\theta(x) = \int_A U^{n+1} f(x) d\gamma_\theta(x). \quad (84)$$

Therefore,

$$\mu\left((T_\theta)^{-(n+1)}(A)\right) = \int_A U^{n+1} f(x) d\gamma_\theta(x), \quad (85)$$

which ends the proof. $\square$

Let $B([0, \theta])$ denote the collection of all bounded measurable functions $f : [0, \theta] \rightarrow \mathbb{C}$. A different interpretation is available for the operator U restricted to $B([0, \theta])$, which is a Banach space under the supremum norm.

Proposition 15. *The operator $U : B([0, \theta]) \rightarrow B([0, \theta])$ is the transition operator of both the Markov chain $(s_{n,a})_{n \in \mathbb{N}_+}$ on $([0, \theta], \mathcal{B}_{[0, \theta]}, \gamma_{\theta,a})$, for any $a \in [0, \theta]$, and the Markov chain $(\bar{s}_l)_{l \in \mathbb{Z}}$ on $([0, \theta]^2, \mathcal{B}_{[0, \theta]}^2, \bar{\gamma}_\theta)$.*

Proof. The transition operator of $(s_{n,a})_{n \in \mathbb{N}_+}$ takes $f \in B([0, \theta])$ to the function defined by

$$\begin{aligned} E_a(f(s_{n+1,a}) \mid s_{n,a} = s) \\ = \sum_{i \geq m} P_i(s) f(u_i(s)) = Uf(s) \quad \text{for any } s \in [0, \theta], \end{aligned} \quad (86)$$

where E_a stands for the mean-value operator with respect to the probability measure $\gamma_{\theta,a}$, whatever $a \in [0, \theta]$. $\square$

A similar reasoning is valid for the case of the Markov chain $(\bar{s}_l)_{l \in \mathbb{Z}}$.

Remark 16. In hypothesis of Proposition 14(ii) it follows that

$$\mu(T_\theta^{-n}(A)) - \gamma_\theta(A) = \int_A (U^n f(x) - 1) d\gamma_\theta(x), \quad (87)$$

for any $n \in \mathbb{N}$ and $A \in \mathcal{B}_{[0, \theta]}$, where $f(x) := (\log(1 + \theta^2))((x\theta + 1)/\theta)h(x)$, $x \in [0, \theta]$. The last equation shows that the asymptotic behavior of $\mu(T_\theta^{-n}(A)) - \gamma_\theta(A)$ as $n \rightarrow \infty$ is given by the asymptotic behavior of the n th power of the Perron-Frobenius U on $L^1_{\gamma_\theta}$ or on smaller Banach spaces.

5. Ergodicity of the Associated RSCC

The facts presented in the previous sections lead us to a certain random system with complete connections associated with the θ -expansion. To study the ergodicity of this RSCC, it becomes necessary to recall some definitions and results from [3].

According to the general theory, we have the following statement.

Definition 17. A homogeneous RSCC is a quadruple $\{(W, \mathcal{W}), (X, \mathcal{X}), u, P\}$, where

- (i) $(W, \mathcal{W})$ and $(X, \mathcal{X})$ are arbitrary measurable spaces;
- (ii) $u : W \times X \rightarrow W$ is a $(\mathcal{W} \otimes \mathcal{X}, \mathcal{W})$ -measurable map;
- (iii) P is a transition probability function from $(W, \mathcal{W})$ to $(X, \mathcal{X})$.

For any $n \in \mathbb{N}_+$, consider the maps $u^{(n)} : W \times X^n \rightarrow W$, defined by

$$\begin{aligned} u^{(1)}(w, x) &:= u(w, x), \\ u^{(n+1)}(w, x^{(n+1)}) &:= u(u^{(n)}(w, x^{(n)}), x_{n+1}), \quad n \geq 1, \end{aligned} \quad (88)$$

where $x^{(n)} = (x_1, \dots, x_n) \in X^n$. We will simply write $w x^{(n)}$ for $u^{(n)}(w, x^{(n)})$. For every $w \in W$, $r \in \mathbb{N}_+$ and $A \in \mathcal{X}^r$, define

$$\begin{aligned} P_1(w, A) &:= P(w, A), \\ P_r(w, A) &:= \int_X P(w, dx_1) \\ &\quad \times \int_X P(wx_1, dx_2) \dots \int_X P(wx^{(r-1)}, dx_r) \chi_A(x^{(r)}), \\ &\quad r \geq 2, \end{aligned} \quad (89)$$

where χ_A is the indicator function of the set A . Obviously, for $n \in \mathbb{N}_+$ fixed, P_r is a transition probability function from $(W, \mathcal{W})$ to $(X^r, \mathcal{X}^r)$.

By virtue of the existence theorem ([3, Theorem 1.1.2]), for a given RSCC $\{(W, \mathcal{W}), (X, \mathcal{X}), u, P\}$, there exists an associated Markov chain with the transition operator U defined by

$$Uf(w) := \int_X P(w, dx) f(wx), \quad f \in B(W, \mathcal{W}), \quad (90)$$

where $B(W, \mathcal{W})$ is the Banach space of all bounded $\mathcal{W}$ -measurable complex-valued functions defined on W . Moreover, the transition probability function of the associated Markov chain is

$$Q(w, B) := \int_X P(w, dx) \chi_B(wx) = P(w, B_w), \quad (91)$$

where $B_w = \{x \in X : wx \in B\}$, $w \in W$, $B \in \mathcal{W}$. The iterates of the operator U are given by

$$\begin{aligned} U^n f(w) &= \int_{X^n} P_n(w, dx^{(n)}) f(wx^{(n)}), \\ &\quad f \in B(W, \mathcal{W}), \quad n \in \mathbb{N}_+. \end{aligned} \quad (92)$$

It follows that the n -step transition probability function is given by

$$Q^n(w, B) = P_n(w, B_w^{(n)}), \quad w \in W, B \in \mathcal{W}, n \in \mathbb{N}_+, \quad (93)$$

where $B_w^{(n)} = \{x^{(n)} : wx^{(n)} \in B\}$. Hence, the transition operator associated with the Markov chain with state space $(W, \mathcal{W})$ and transition probability function Q is defined by

$$Uf(\cdot) := \int_W Q(\cdot, dw) f(w), \quad f \in B(W, \mathcal{W}). \quad (94)$$

Its iterates are given by

$$U^n f(\cdot) = \int_W Q^n(\cdot, dw) f(w), \quad n \in \mathbb{N}_+, \quad (95)$$

where Q^n is the n -step transition probability function.

Putting

$$Q_n(w, B) = \frac{1}{n} \sum_{k=1}^n Q^k(w, B), \quad (96)$$

for all $n \in \mathbb{N}_+$, $w \in W$ and $B \in \mathcal{W}$, it is clear that Q_n is a transition probability function on $(W, \mathcal{W})$. Let U_n be the Markov operator associated with Q_n .

Let (W, d) be a metric space and let $L(W)$ denote the Banach space of all complex-valued Lipschitz continuous functions on W with the following norm:

$$\|f\|_L := \|f\| + s(f), \quad (97)$$

where

$$\begin{aligned} \|f\| &:= \sup_{w \in W} |f(w)|, \\ s(f) &:= \sup_{w' \neq w''} \frac{|f(w') - f(w'')|}{d(w', w'')}. \end{aligned} \quad (98)$$

Definition 18. (i) The operator U is said to be orderly with respect to $L(W)$ if and only if there exists a bounded linear operator U^∞ on $L(W)$ such that

$$\lim_{n \rightarrow \infty} \|U_n - U^\infty\|_L = 0. \quad (99)$$

(ii) The operator U is said to be aperiodic with respect to $L(W)$ if and only if there exists a bounded linear operator U^∞ on $L(W)$ such that

$$\lim_{n \rightarrow \infty} \|U^n - U^\infty\|_L = 0. \quad (100)$$

(iii) The operator U is said to be ergodic with respect to $L(W)$ if and only if it is orderly and the range $U^\infty(L(W))$ is one-dimensional.

(iv) The operator U is regular with respect to $L(W)$ if and only if it is ergodic and aperiodic.

Definition 19. The transition operator U of a Markov chain with state space W is said to be a Doeblin-Fortet operator if and only if U takes $L(W)$ into $L(W)$ boundedly with respect to $\|\cdot\|_L$ and there exist $k \in \mathbb{N}_+$, $r \in [0, 1)$ and $R < \infty$ such that

$$s(U^k f) \leq rs(f) + R\|f\|, \quad f \in L(W). \quad (101)$$

Alternatively, the Markov chain itself is said to be a Doeblin-Fortet chain.

The definition below isolates a class of RSCCs, called RSCCs with contraction, for which the associated Markov chains are Doeblin-Fortet chains.

Definition 20. An RSCC $\{(W, \mathcal{W}), (X, \mathcal{X}), u, P\}$ is said to be with contraction if and only if (W, d) is a separable metric space, $r_1 < \infty$, $R_1 < \infty$ and there exists $j \in \mathbb{N}_+$ such that $r_j < 1$. Here

$$\begin{aligned} r_j &:= \sup_{w' \neq w''} \int_{X^j} P_j(w', dx^{(j)}) \frac{d(w'x^{(j)}, w''x^{(j)})}{d(w', w'')}, \\ R_j &:= \sup_{A \in \mathcal{X}^j} s(P_j(\cdot, A)), \quad j \in \mathbb{N}_+, \end{aligned} \quad (102)$$

where P_j is a transition probability function from $(W, \mathcal{W})$ to $(X^j, \mathcal{X}^j)$ defined in (89).

We will also need the following results.

Theorem 21. The Markov chain associated with an RSCC with contraction is a Doeblin-Fortet chain.

Lemma 22. Assume that the Markov operator U is aperiodic with respect to $L(W)$. Put $T = U - U^\infty$. Then one has $U^\infty U^\infty = U^\infty$, $TU^\infty = U^\infty T = 0$, $U^n = U^\infty + T^n$, $n \in \mathbb{N}_+$. Moreover, there exist positive constants $q < 1$ and K such that

$$\|T^n\|_L \leq Kq^n, \quad n \in \mathbb{N}_+. \quad (103)$$

This immediately implies the validity of

$$\|U^n f - U^\infty f\|_L \leq Kq^n \|f\|_L, \quad f \in L(W), \quad n \in \mathbb{N}_+. \quad (104)$$

Definition 23. A Markov chain is said to be compact if and only if its state space is a compact metric space (W, d) and its transition operator is a Doeblin-Fortet operator.

Theorem 24. A compact Markov chain is orderly with respect to $L(W)$ and there exists a transition probability function Q^∞ on $(W, \mathcal{W})$ such that

$$U^\infty f(\cdot) = \int_W Q^\infty(\cdot, dw) f(w), \quad f \in L(W). \quad (105)$$

Moreover, $Q^\infty(\cdot, B) \in E(1)$, for any $B \in \mathcal{W}$, and, for any $w \in W$, $Q^\infty(w, \cdot)$ is a stationary probability for the chain. (Here $E(1)$ is the set of the eigenvalues of modulus 1 of the operator U .)

A criterion of regularity for a compact Markov chain is expressed in Theorem 25 in terms of the supports $\sigma_n(w)$ of the transition probability functions $Q^n(w, \cdot)$, $n \in \mathbb{N}_+$.

Theorem 25. A compact Markov chain is regular with respect to $L(W)$ if and only if there exists a point $w^* \in W$ such that

$$\lim_{n \rightarrow \infty} d(\sigma_n(w), w^*) = 0, \quad w \in W. \quad (106)$$

The application of this criterion is facilitated by the interrelationship among the sets $\sigma_n(w)$, $n \in \mathbb{N}_+$, which is given in the next lemma.

Lemma 26. For all $m, n \in \mathbb{N}$ and $w \in W$, one has

$$\sigma_{m+n}(w) = \overline{\bigcup_{w' \in \sigma_m(w)} \sigma_n(w')}, \quad (107)$$

where the overline means topological closure in W .

Now, we are able to study the following RSCC:

$$\{([0, \theta], \mathcal{B}_{[0, \theta]}), (\mathbb{N}_m, \mathcal{P}(\mathbb{N}_m)), u, P\}, \quad (108)$$

where $u : [0, \theta] \times \mathbb{N}_m \rightarrow [0, \theta]$, $u(s, i) = u_i(s)$ is given in (78) and the function $P(s, i) = P_i(s)$ given in (47) defines a transition probability from $([0, \theta], \mathcal{B}_{[0, \theta]})$ to $(\mathbb{N}_m, \mathcal{P}(\mathbb{N}_m))$. Here $\mathbb{N}_m = \{m, m+1, \dots\}$, $m \in \mathbb{N}_+$ and $\mathcal{P}(\mathbb{N}_m)$ denotes the power set of $\mathbb{N}_m$.

Whatever $a \in [0, \theta]$ the Markov chain $(s_{n,a})_{n \in \mathbb{N}}$ associated with the RSCC (89) has the transition operator U , with the transition probability function

$$Q(s, B) = \sum_{\{i \geq m | u_i(s) \in B\}} P_i(s), \quad s \in [0, \theta], \quad B \in \mathcal{B}_{[0, \theta]}. \quad (109)$$

Then $Q^n(\cdot, \cdot)$ will denote the n -step transition probability function of the same Markov chain.

Proposition 27. *RSCC (108) is regular with respect to $L([0, \theta])$. Moreover, there exist a stationary probability measure $Q^\infty = \gamma_\theta$ and two positive constants $q < 1$ and K such that*

$$\left\| U^n f - \int_0^\theta f d\gamma_\theta \right\|_L \leq K q^n \|f\|_L, \quad n \in \mathbb{N}_+, \quad f \in L([0, \theta]), \quad (110)$$

where

$$U^n f(\cdot) := \int_0^\theta Q^n(\cdot, ds) f(s). \quad (111)$$

Proof. Since

$$\begin{aligned} \frac{d}{ds} u_i(s) &= \frac{-1}{(s + i\theta)^2}, \\ \frac{d}{ds} P_i(s) &= \frac{i\theta - 1/\theta}{(s + i\theta)^2} - \frac{(i+1)\theta - 1/\theta}{(s + (i+1)\theta)^2}, \end{aligned} \quad (112)$$

for any $s \in [0, \theta]$ and $i \geq m$, it follows that

$$\begin{aligned} \sup_{s \in [0, \theta]} \left| \frac{d}{ds} u_i(s) \right| &= \frac{1}{(i\theta)^2}, \\ \sup_{s \in [0, \theta]} \left| \frac{d}{ds} P_i(s) \right| &< \infty. \end{aligned} \quad (113)$$

Hence, the requirements of Definition 20 of an RSCC with contraction are met with $j = 1$. By Theorem 21 it follows that the Markov chain $(s_{n,a})_{n \in \mathbb{N}}$ associated with this RSCC with contraction is a Doeblin-Fortet chain and its transition operator U is a Doeblin-Fortet operator. It remains to prove the regularity of U with respect to $L([0, \theta])$. For this we have to prove the existence of a point $s^* \in [0, \theta]$ such that $\lim_{n \rightarrow \infty} |\sigma_n(s) - s^*| = 0$, for any $s \in [0, \theta]$, where $\sigma_n(s)$ is the support of measure $Q^n(s, \cdot)$, $n \in \mathbb{N}_+$.

Let $s \in [0, \theta]$ be an arbitrarily fixed number and define

$$w_1 := s, \quad w_{n+1} := \frac{1}{w_n + m\theta}, \quad n \in \mathbb{N}_+. \quad (114)$$

We have $w_n \in [0, \theta]$, and letting $n \rightarrow \infty$ in (114), we get

$$w_n \rightarrow s^* := \frac{-1 + \sqrt{1 + 4\theta^2}}{2\theta}. \quad (115)$$

Clearly, $w_{n+1} \in \sigma_1(w_n)$ and Lemma 26 and an induction argument show that $w_n \in \sigma_n(s)$, $n \in \mathbb{N}_+$. Thus,

$$d(\sigma_n(s), s^*) \leq |w_n - s^*| \rightarrow 0, \quad n \rightarrow \infty, \quad (116)$$

where d stands for the Euclidian distance on the line. Now, the regularity of U with respect to $L([0, \theta])$ follows from Theorem 25.

From (104) and Theorem 24, there exist a stationary probability measure Q^∞ and two constants $q < 1$ and K such that

$$\|U^n f - U^\infty f\|_L \leq K q^n \|f\|_L, \quad n \in \mathbb{N}_+, \quad f \in L([0, \theta]), \quad (117)$$

where $U^n f$ is as in (111) and

$$U^\infty f = \int_0^\theta f(x) Q^\infty(dx). \quad (118)$$

Here $Q^\infty = \gamma_\theta$ is the invariant probability measure of the transformation T_θ in (19); that is, Q^∞ has the density $\rho_\theta = 1/(x + m\theta)$, $x \in [0, \theta]$, with the normalizing factor $1/\log(1 + \theta^2)$. $\square$

Remark 28. Another way to put this is that ρ_θ is the eigenfunction of eigenvalue 1 of the Perron-Frobenius operator U .

6. A Gauss-Kuzmin-Type Theorem

Now, we may determine the limit of the sequence $(\mu(T_\theta^n < x))_{n \in \mathbb{N}_+}$ as $n \rightarrow \infty$ and give the rate of this convergence.

Theorem 29 (Gauss-Kuzmin-type theorem for T_θ). *Let $([0, \theta], \mathcal{B}_{[0, \theta]}, T_\theta)$ be as in Definition 1(i).*

(i) *For a probability measure μ on $([0, \theta], \mathcal{B}_{[0, \theta]})$, let assumption (A) as follows:*

(A) μ is nonatomic and has a Riemann-integrable density.

Then, for any probability measure μ which satisfies (A), the following holds:

$$\lim_{n \rightarrow \infty} \mu(T_\theta^n < x) = \frac{1}{\log(1 + \theta^2)} \log((m\theta + x)\theta), \quad (119)$$

$x \in [0, \theta].$

(ii) *In addition to assumption of μ in (i), if the density of $[0, \theta] \ni x \mapsto \mu([0, x])$ is Lipschitz continuous, then there exist two positive constants $q < 1$ and K such that for any $x \in [0, \theta]$ and $n \in \mathbb{N}_+$, the following holds:*

$$\lim_{n \rightarrow \infty} \mu(T_\theta^n < x) = \frac{1 + \alpha q^n}{\log(1 + \theta^2)} \log((m\theta + x)\theta), \quad (120)$$

where $\alpha := \alpha(\mu, n, x)$ with $|\alpha| \leq K$.

As a consequence, the n th error term $e_n(\theta, \mu; x)$ of the Gauss-Kuzmin problem is obtained as follows:

$$e_n(\theta, \mu; x) = \frac{\alpha q^n}{\log(1 + \theta^2)} \log((m\theta + x)\theta). \quad (121)$$

Proof. Let T_θ be as in (19). By Proposition 14(ii), we have

$$\mu((T_\theta)^{-n}(A)) = \int_A U^n f_0(x) \rho_\theta(x) dx, \quad (122)$$

for any $n \in \mathbb{N}$, $A \in \mathcal{B}_{[0, \theta]}$,

where $f_0(x) = ((x\theta + 1)/\theta)(d\mu/d\lambda_\theta)(x)$, for $x \in [0, \theta]$. If $d\mu/d\lambda_\theta \in L([0, \theta])$, by (118) we have

$$\begin{aligned} U^\infty f_0 &= \int_0^\theta f_0(x) Q^\infty(dx) \\ &= \int_0^\theta f_0(x) \gamma_\theta(dx) = \frac{1}{\log(1 + \theta^2)}. \end{aligned} \quad (123)$$

Taking into account (117), there exist two constants $q < 1$ and K such that

$$\|U^n f_0 - U^\infty f_0\|_L \leq K q^n \|f_0\|_L, \quad n \in \mathbb{N}_+. \quad (124)$$

Furthermore, consider the Banach space $C([0, \theta])$ of all real-valued continuous functions on $[0, \theta]$ with the norm $\|f\| := \sup_{x \in [0, \theta]} |f(x)|$. Since $L([0, \theta])$ is a dense subspace of $C([0, \theta])$, we have

$$\lim_{n \rightarrow \infty} \|(U^n - U^\infty)f\| = 0 \quad \forall f \in C([0, \theta]). \quad (125)$$

Therefore, (125) is valid for a measurable function f_0 which is Q^∞ -almost surely continuous, that is, for a Riemann-integrable function f_0 . Thus, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \mu(T_\theta^n < x) &= \lim_{n \rightarrow \infty} \int_0^x U^n f_0(u) \rho_\theta(u) du \\ &= \frac{1}{\log(1 + \theta^2)} \int_0^x \rho_\theta(u) du \\ &= \frac{1}{\log(1 + \theta^2)} \log((m\theta + x)\theta). \end{aligned} \quad (126)$$

Hence, (119) is proved. $\square$

Remark 30. Since the Lebesgue measure λ satisfies assumptions in both (i) and (ii) of Theorem 29, (119) and (120) hold for the case $\mu = \lambda$. Hence, Theorem 29 gives the solution of the Gauss-Kuzmin problem for the pair (T_θ, μ) instead of (τ, λ) in (3).

Remark 31. Until now, the estimate of the convergence rate remains an open question. To obtain a better estimate of the convergence rate involved, we may use a Wirsing-type approach as in [19–21].

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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