Research Article

A Continuous Characterization of Triebel-Lizorkin Spaces Associated with Hermite Expansions

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We study the properties of the Triebel-Lizorkin spaces associated with the multidimensional Hermite expansions on $\mathbb{R}^d$ ($d \geq 1$). In addition to the endpoint estimates, we give the continuous characterizations of these spaces via harmonic and thermic extensions from $\mathbb{R}^d$ into $\mathbb{R}_+^{d+1}$. Based on this result, we obtain the boundedness of Riesz transform associated with the Hermite expansions.

1. Introduction

For a smooth functional series $\{\varphi_j\}_{j=-\infty}^{\infty}$, if it satisfies the following conditions:

- (i) $\text{supp } \varphi_j \subset \{x : 2^{j-1} \leq |x| \leq 2^{j+1}\}$, $j = 1, 2, 3, \dots$,
- (ii) $2^{jk} |D^{(k)} \varphi_j(x)| \leq C_k$, $j \in \mathbb{Z}$, $k \in \mathbb{N}_0$,
- (iii) $\sum_{j=-\infty}^{\infty} \varphi_j(x) = 1$, $x \in \mathbb{R}^d$,

then we call $\{\varphi_j\}_{j=-\infty}^{\infty}$ a smooth dyadic resolution of unity in $\mathbb{R}^d$ ($d \geq 1$). As proved in [1], it is easy to construct such a resolution of unity.

Let D be a self-adjoint operator acting on $L^2(\mathbb{R}^d)$, and let dE be its spectral resolution which is defined by

$$Df = \int_0^{\infty} \lambda dE(\lambda) f. \quad (1)$$

For $\alpha \in \mathbb{R}$, $0 < p < \infty$, $0 < q \leq \infty$, and a C^∞ function $\varphi \in \{\varphi_j\}_{j=-\infty}^{\infty}$, we define the Triebel-Lizorkin norm associated with D by

$$\|f\|_{F_p^{\alpha,q}(D)} = \left\| \left(\sum_{j=-\infty}^{\infty} (2^{j\alpha} |\varphi_j(D) f(x)|)^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^d)}, \quad (2)$$

where

$$\varphi_j(D) f = \varphi(2^{-j} D) f = \int_0^{\infty} \varphi(2^{-j} \lambda) dE(\lambda) f. \quad (3)$$

Note that if $D = \Delta = -\sum_{j=1}^d (\partial^2 / \partial x_j^2)$ is the Laplacian on $\mathbb{R}^d$, then the norm $\|f\|_{F_p^{\alpha,q}(D)}$ in (2) is equivalent to the classical Triebel-Lizorkin norm $\|f\|_{F_p^{\alpha,q}(\Delta)}$; see [2]. The main goal of this paper is to study the Triebel-Lizorkin space $F_p^{\alpha,q}(D)$ when D is the Hermite operator.

The d -dimensional Hermite functions on the d -dimensional Euclidean space $\mathbb{R}^d$ ($d \geq 1$) are defined by

$$h_\alpha(x) = h_{\alpha_1}(x_1) \times h_{\alpha_2}(x_2) \times \cdots \times h_{\alpha_d}(x_d), \quad (4)$$

where $\alpha = (\alpha_1, \dots, \alpha_d)$, $\alpha_i \in \{0, 1, \dots\}$, $x = (x_1, \dots, x_d)$, and

$$h_k(s) = (\pi^{1/2} 2^k k!)^{-1/2} H_k(s) \exp\left(-\frac{s^2}{2}\right), \quad (5)$$

$$k = 0, 1, \dots,$$

are the one-dimensional Hermite functions; $H_k(s)$ denotes the k th Hermite polynomial. We know that the Hermite functions form an orthonormal basis for $L^2(\mathbb{R}^d)$ and that h_α

are eigenfunctions of the Hermite operator $H = -\Delta + |x|^2$; that is,

$$Hh_\alpha = (2|\alpha| + d)h_\alpha, \quad (6)$$

where $|\alpha| = \alpha_1 + \dots + \alpha_d$. Moreover, the operator H is positive and symmetric in L^2 on the domain $C_c^\infty(\mathbb{R}^d)$. It is easy to check that the operator H given by

$$H\left(\sum \langle f, h_\alpha \rangle h_\alpha\right) = \sum (2|\alpha| + d) \langle f, h_\alpha \rangle h_\alpha \quad (7)$$

is a self-adjoint extension of H on the domain

$$\begin{aligned} \text{Dom}(H) \\ = \{f \in L^2(\mathbb{R}^d) : \sum |(2|\alpha| + d) \langle f, h_\alpha \rangle|^2 < \infty\}. \end{aligned} \quad (8)$$

The operator H has the discrete spectrum $\{2k + d : k = 0, 1, \dots\}$ and admits the spectral decomposition

$$Hf = \sum_{k=0}^{\infty} (2k + d) P_k f, \quad f \in \text{Dom}(H), \quad (9)$$

where P_k is a spectral projection that is defined to be $P_k f = \sum_{|\alpha|=k} \langle f, h_\alpha \rangle h_\alpha$. As to the other properties of the Hermite functions, the interested readers can refer to Thangavelu's monograph [3] for a detailed description.

In the past few years, many authors have studied the Triebel-Lizorkin spaces associated with the Hermite operators. In [4], Epperson studied the Triebel-Lizorkin spaces associated with the one-dimensional Hermite operators for

$$\text{supp } \varphi \subset \left[\frac{1}{2}, 2\right], \quad |\varphi(\lambda)| > c > 0, \quad \lambda \in \left[\frac{3}{4}, \frac{7}{4}\right]. \quad (10)$$

Using Mehler's formula, Epperson proved that the definition of the corresponding space $\|f\|_{F_p^{\alpha,q}(H)}$ is independent of the particular choice of the function φ . In [5], Dziubański continued this study and proved that the results in [4] also hold for the multidimensional Hermite expansions. Besides, Petrushev and Xu [6] showed that the Triebel-Lizorkin on $\mathbb{R}^d$ induced by Hermite expansions can be characterized in terms of the needlet coefficients and that the Hermite-Triebel-Lizorkin space is, in general, different from the respective classical spaces. In addition, Olafsson and Zheng [7] study the Triebel-Lizorkin space associated with the Peetre type maximal function for Hermite operator H , which is defined by

$$\varphi_{j,s}^* f(x) = \sup_{y \in \mathbb{R}^d} \frac{|\varphi_j(H)f(y)|}{(1 + 2^{j/2}|x-y|)^s}, \quad j \in \mathbb{Z}, \quad s > 0. \quad (11)$$

They gave the maximal characterization of this space as follows.

Theorem GS. Let $\varphi_j \in \{\varphi_j(x)\}_{-\infty}^\infty$, $0 < p < \infty$, $0 < q \leq \infty$, and $\alpha \in \mathbb{R}$. If $s > d/\min(p, q)$, then

$$\|f\|_{F_p^{\alpha,q}} \sim \left\| \left\{ 2^{j\alpha} \varphi_{j,s}^*(H)f \right\} \right\|_{L^p(\ell^q)}, \quad (12)$$

where $\varphi_{j,s}^*(H)f$ is the Peetre type maximal function for Hermite operator.

In [3], Thangavelu introduced Riesz transforms associated with Hermite expansions and proved that they are bounded operators on $L^p(\mathbb{R}^d)$, $1 < p < \infty$. In [8], Harboure et al. gave a different proof to show that the L^p -norms of these operators are bounded by a constant not depending on the dimension d . Moreover, they also defined the Riesz transforms of higher order and obtained the free dimensional estimates of the L^p -bounds of these operators. After that, Stempak and Torrea [9] obtained the weighted L^p -inequalities for the gradient square function associated with the Poisson semigroup. They get the result proposed in [3] by using a slightly different proof and they also get the analogous result for the g -function associated with the Poisson semigroup.

Although many results have been obtained about the Triebel-Lizorkin spaces associated with the Hermite operators, all of them only discuss the case of the discrete quasi-norm of the Triebel-Lizorkin spaces associated with the Hermite operators. Up to now, as we know, there are few papers related to the continuous case due to the fact that the Hermite expansions do not satisfy the general convolution property. This makes the study of the continuous case a meaningful problem. We first present a general characterization of the Triebel-Lizorkin spaces associated with the multidimensional Hermite expansions satisfied continuous quasi-norm on $\mathbb{R}^d$. Furthermore, analogous to the classical Triebel-Lizorkin space, we construct the Littlewood-Paley theorem on these spaces. Finally, we obtain the endpoint estimate of these spaces and establish the boundedness of the Riesz transforms associated with Hermite functions expansions.

Throughout this paper, $|x|$ is the Euclidean norm of x for $x \in \mathbb{R}^d$; $A \sim B$ means $C_1 A \leq B \leq C_2 A$, where C_1, C_2 are positive constants; C denotes a positive constant that may vary at every occurrence. For simplicity, we abbreviate $\|f\|_{F_p^{\alpha,q}(H)}$ to $\|f\|_{F_p^{\alpha,q}}$.

2. The Characterizations of $F_p^{\alpha,q}$

First, we give the definition of the Triebel-Lizorkin spaces associated with Hermite operators. Let $\varphi_j \in \{\varphi_j(x)\}_{-\infty}^\infty$, $\varphi_j(x) = \varphi(2^{-j}x)$; the Triebel-Lizorkin spaces associated with Hermite operator H , denoted by $F_p^{\alpha,q}$, $\alpha \in \mathbb{R}$, $0 < p$, and $q < \infty$, are defined by the quasi-norm

$$\|f\|_{F_p^{\alpha,q}} = \left\| \left(\sum_{j=-\infty}^{\infty} (2^{j\alpha} |\varphi_j(H)f(x)|)^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^d)}, \quad (13)$$

where

$$\varphi_j(H)f(x) = \sum_{k=0}^{\infty} \varphi(2^{-j}(2k+d)) P_k f(x). \quad (14)$$

The results of Dziubański [5] show that the definition of $F_p^{\alpha,q}$ does not depend on the particular choice of φ . Next, we will

give the general characterization of Triebel-Lizorkin spaces associated with the Hermite expansions.

Theorem 1. Let $\alpha \in \mathbb{R}$, $1 < p, q < \infty$, and $\varphi \in S(\mathbb{R}^d)$; let $\varphi_j(x) = \varphi(2^{-j}x)$. If $x \in \mathbb{R}^d \setminus \{0\}$ and $j \in \mathbb{Z}$, then

$$\left\| \left(\int_0^\infty |t^{-\alpha} \varphi(tH) f(\cdot)|^q \frac{dt}{t} \right)^{1/q} \right\|_{L^p(\mathbb{R}^d)} \quad (15)$$

(with usual modification when $q = \infty$) is an equivalent quasi-norm in $F_p^{\alpha,q}$.

This theorem shows that any “discrete” quasi-norm of the Triebel-Lizorkin spaces associated with the Hermite expansions of type (13) has a “continuous” counterpart (15).

In order to prove our theorem, we need some necessary lemmas.

Lemma 2. Let $\{\varepsilon_k\}$, $\{v_k\}$ be two sequences of complex numbers. Then, for any l , we have

$$\sum_k \varepsilon_k v_k = \sum_k \left(\Delta^l \varepsilon_k \right) A_k^{l-1} S_k^{l-1}, \quad (16)$$

where

$$S_k^{l-1} = \left(A_k^{l-1} \right)^{-1} \sum_{i=1}^k A_{k-i}^{l-1} v_i \quad (17)$$

is the $(l-1)$ th linear mean of first k terms of $\{v_k\}$ and Δ^m is the m th difference operator.

Since the proof of Lemma 2 is easy, we omit the proof for the sake of compactness of the paper.

In [3], a basic estimate for the kernel of Riesz means for the Hermite expansions has been obtained. We describe the result in the following lemma.

Lemma 3. For $\alpha > (3d-2)/6$, the following estimate is valid:

$$\begin{aligned} |S_R^\alpha(x, y)| &\leq CR^{d/2} \left\{ \left(1 + R^{1/2} |x - y| \right)^{-\alpha + (3d-2)/6-1} \right. \\ &\quad \left. + \left(1 + R^{1/2} |x + y| \right)^{-\alpha + (3d-2)/6-1} \right\}, \end{aligned} \quad (18)$$

where $S_R^\alpha(x, y)$ is the kernel of the Riesz means $S_R^\alpha f$ for the Hermite expansions, which is defined by

$$S_R^\alpha f = \sum_{k=0}^\infty \left(1 - \frac{2k+d}{R} \right)_+^\alpha P_k f. \quad (19)$$

Lemma 4. Assume that $f \in L^p(\mathbb{R}^d)$, $1 \leq p < \infty$, and $\alpha > (9d-8)/6$. We have the pointwise inequality

$$S_*^\alpha f(x) \leq CM_p f(x), \quad (20)$$

where

$$S_*^\alpha f(x) = \sup_{R>0} |S_R^\alpha f(x)| \quad (21)$$

is the maximal operator and

$$M_p f(x) = \left(\sup_{r>0} \frac{1}{B_r(x)} \int_{B_r(x)} |f(y)|^p dy \right)^{1/p} \quad (22)$$

is the Hardy-Littlewood maximal function of $|f|^p$ raised to the power $1/p$ and $B_r(x)$ is the ball of radius r centered at x .

Proof. The proof is based on the estimate of Lemma 3:

$$\begin{aligned} |S_R^\alpha f(x)| &= \left| \int_{\mathbb{R}^n} S_R^\alpha(x, y) f(y) dy \right| \\ &\leq \left(\int_{\mathbb{R}^d} |S_R^\alpha(x, y)|^q \right)^{1/q} (|f(y)|^p)^{1/p} \\ &\leq C \left(\int_{\mathbb{R}^d} \left| R^{d/2} (1 + R^{1/2} |x - y|)^{-\alpha + (3d-2)/6-1} \right|^q dy \right)^{1/q} \\ &\quad \cdot \|f\|_{L^p} \\ &\quad + C \left(\int_{\mathbb{R}^d} \left| R^{d/2} (1 + R^{1/2} |x + y|)^{-\alpha + (3d-2)/6-1} \right|^q dy \right)^{1/q} \\ &\quad \cdot \|f\|_{L^p} = I + II, \end{aligned} \quad (23)$$

where $1/p + 1/q = 1$. Consider a partition of $\mathbb{R}^d$ into dyadic annuli $B_j = \{y : 2^j \leq |x - y| \leq 2^{j+1}\}$. For a given function f , we set $f_j(y) = f(y)$ for $2^j \leq |x - y| \leq 2^{j+1}$ and $f_j(y) = 0$ otherwise. Then

$$\begin{aligned} I &\leq C \sum_{j=-\infty}^\infty \left(\int_{B_j} \left| R^{d/2} (1 + R^{1/2} |x - y|)^{-\alpha + (3d-2)/6-1} \right|^q dy \right)^{1/q} \\ &\quad \cdot \|f_j\|_{L^p} \\ &\leq C \sum_{j=-\infty}^\infty R^{d/2} (1 + R^{1/2} 2^j)^{-\alpha + (3d-2)/6-1} |B_j|^{1/q+1/p} M_p f(x) \\ &\leq C \sum_{j=-\infty}^\infty (R^{1/2} 2^j)^d (1 + R^{1/2} 2^j)^{-\alpha + (3d-2)/6-1} M_p f(x) \\ &= G(R^{1/2}), \end{aligned} \quad (24)$$

where

$$G(t) = \sum_{j=-\infty}^\infty (t 2^j)^d (1 + t 2^j)^{-\alpha + (3d-2)/6-1}. \quad (25)$$

It is obvious that $G(t)$ is locally bounded. Since $G(2^i t) = G(t)$, it is bounded on $(0, \infty)$ (see [3]). The boundedness of II can also be obtained by replacing y in the proof of the boundedness of I by $-y$ and following from the same estimate tactics as above. Hence, we have $S_*^\alpha f(x) \leq CM_p f(x)$. $\square$

We give the proof of Theorem 1 in what follows.

Proof. Consider the following.

Step 1. Let $f \in F_p^{\alpha,q}$. We will prove that $\|f\|_{F_p^{\alpha,q}}$ in (13) can be estimated from above by the quasi-norm in (15). Let

$\rho_j \in \{\varphi_j(x)\}_{-\infty}^{\infty}$ and $l-1 > (9d-8)/6$. It follows from Lemma 4 that

$$\begin{aligned} |\rho_j(H)f(x)| &= \left| \frac{\rho_j(H)}{\varphi(\lambda H)} \varphi(\lambda H)f(x) \right| \\ &= \left| \sum_k \Delta^l \left(\frac{\rho_j(2k+d)}{\varphi(\lambda(2k+d))} \right) A_k^{l-1} S_k^{l-1}(\varphi_j(\lambda H)f) \right. \\ &\quad \cdot (x) \left| \leq \sum_k \left| \Delta^l \left(\frac{\rho_j(2k+d)}{\varphi(\lambda(2k+d))} \right) A_k^{l-1} \right| \right. \\ &\quad \cdot |M(\varphi_j(\lambda H)f)(x)|. \end{aligned} \quad (26)$$

For the first term of the right-hand side of (26), taking the supremum with respect to λ on $1 < \lambda < 2$ yields

$$\begin{aligned} \sup_{1 \leq \lambda \leq 2} \sum_k \left| \Delta^l \left(\frac{\rho_j(2k+d)}{\varphi(\lambda(2k+d))} \right) A_k^{l-1} \right| \\ \leq C \sum_k \left| 2^{(-j+1)l} \sum_{i=0}^l C_l^i \rho^{(l-i)}(2^{-j}(2k+d)) \right. \\ \cdot \left(\frac{1}{\varphi(2^{-j}(2k+d))} \right)^{(i)} \left| k^{l-1} \leq C \sum_k 2^{2(-j+1)l} \right. \\ \cdot \frac{(1 + |2^{-j}(2k+d)|)^b}{(1 + |2^{-j}(2k+d)|)^a} k^{l-1} \leq C \sum_k 2^{2(-j+1)l} 2^{(a-b)j} \\ \cdot \frac{1}{(1 + |2k+d|)^{a-b}} k^{l-1}. \end{aligned} \quad (27)$$

We choose $a-b = 2l > l-1 > 0$; then we have

$$\sup_{1 \leq \lambda \leq 2} \sum_k \left| \Delta^l \left(\frac{\rho_j(2k+d)}{\varphi(\lambda(2k+d))} \right) A_k^{l-1} \right| \leq C. \quad (28)$$

Afterwards, integrating the modified inequality with respect to λ that appears now only in the maximal function yields

$$|\rho_j(H)f(x)| \leq C \left| \int_1^2 M(\varphi_j(\lambda H)f)(x) d\lambda \right|, \quad (29)$$

where $M(f)$ is the Hardy-Littlewood maximal function.

For $1 < p < \infty$ and $1 < q < \infty$, we first multiply (29) by $2^{j\alpha}$; and then we apply the l_q -norm with respect to

j ; afterwards, we apply the L^p -norm with respect to x . That gives

$$\begin{aligned} &\left\| \left(\sum_{j=-\infty}^{\infty} (2^{j\alpha} |\rho_j(H)f(x)|)^q \right)^{1/q} \right\|_{L^p} \\ &\leq C \left\| \left(\sum_{j=-\infty}^{\infty} \left| \int_1^2 M(2^{\alpha j} \varphi_j(\lambda H)f)(x) d\lambda \right|^q \right)^{1/q} \right\|_{L^p} \\ &\leq C \left\| \left(\sum_{j=-\infty}^{\infty} 2^{\alpha q j} \left| \int_1^2 \sum_k \varphi(2^{-j}\lambda(2k+d)) P_k f(x) d\lambda \right|^q \right)^{1/q} \right\|_{L^p} \\ &\leq C \left\| \left(\sum_{j=-\infty}^{\infty} \int_{2^{-j}}^{2^{-j+1}} \left| t^{-\alpha} \sum_k \varphi(t(2k+d)) P_k f(x) \right|^q \frac{dt}{t} \right)^{1/q} \right\|_{L^p} \\ &\leq C \left\| \left(\int_0^{\infty} \left| t^{-\alpha} \sum_k \varphi(t(2k+d)) P_k f(x) \right|^q \frac{dt}{t} \right)^{1/q} \right\|_{L^p} \\ &\leq C \left\| \left(\int_0^{\infty} |t^{-\alpha} \varphi(tH)f(x)|^q \frac{dt}{t} \right)^{1/q} \right\|_{L^p}. \end{aligned} \quad (30)$$

Step 2. We will show that the quasi-norm in (15) can be deduced from above by $C\|f\|_{F_p^{\alpha,q}}$. Let $\rho_j \in \{\varphi_j(x)\}_{-\infty}^{\infty}$. Then we have (13) with ρ instead of φ . Let $\rho_m(x) = 0$, $-m \in \mathbb{N}$. For $j \in \mathbb{Z}$, we get

$$\begin{aligned} &2^{\alpha j} (\varphi_j(H)f)(x) \\ &= \sum_{m=-\infty}^{\infty} 2^{\alpha j} (\varphi_j(H)\rho_{m+j}(H)f)(x) \\ &= \sum_{m=-\infty}^K \cdots + \sum_{m=K+1}^{\infty} \cdots. \end{aligned} \quad (31)$$

We give the estimate of the first sum in (31). Let

$$\tilde{\rho}_j(x) = |2^{-j}x|^{\alpha_1} \rho_j(x) \quad j \in \mathbb{Z}, \quad \alpha_1 > \alpha. \quad (32)$$

Then we have

$$\begin{aligned} &\sum_{m=-\infty}^K 2^{\alpha j} (\varphi_j(H)\rho_{m+j}(H)f)(x) = \sum_{m=-\infty}^K 2^{\alpha j} \\ &\quad \cdot \sum_k \varphi(2^{-j}(2k+d)) \rho_{m+j}(2k+d) P_k f(x) \\ &= \sum_{m=-\infty}^K 2^{\alpha j} \sum_k \frac{\varphi(2^{-j}(2k+d))}{|2^{-(m+j)}(2k+d)|^{\alpha_1}} \tilde{\rho}_{m+j}(2k+d) \\ &\quad \cdot P_k f(x) = \sum_{m=-\infty}^K 2^{\alpha(j+m)} 2^{(\alpha_1-\alpha)m} \\ &\quad \cdot \sum_k \left(\Delta^l \frac{\varphi(2^{-j}(2k+d))}{|2^{-j}(2k+d)|^{\alpha_1}} \right) A_k^{l-1} S_k^{l-1}(\tilde{\rho}_{m+j}(H)f) \\ &\quad \cdot (x). \end{aligned} \quad (33)$$

Thus

$$\begin{aligned}
 & \left| \sum_{m=-\infty}^K 2^{\alpha j} (\varphi_j(H) \rho_{m+j}(H) f)(x) \right| \\
 & \leq \sum_{m=-\infty}^K 2^{\alpha(j+m)} 2^{(\alpha_1-\alpha)m} \sum_k \left| \Delta^l \frac{\varphi(2^{-j}(2k+d))}{|2^{-j}(2k+d)|^{\alpha_1}} \right| \\
 & \cdot |A_k^{l-1}| |S_k^{l-1}(\tilde{\rho}_{m+j}(H) f)(x)| \\
 & \leq C \sum_{m=-\infty}^K 2^{\alpha(j+m)} 2^{(\alpha_1-\alpha)m} M(\tilde{\rho}_{m+j}(H) f)(x),
 \end{aligned} \quad (34)$$

where $j \in \mathbb{Z}$ and $l-1 > (9d-8)/6$. In addition, following the same estimate procedure of (27), we have

$$\sum_k \left| \Delta^l \frac{\varphi(2^{-j}(2k+d))}{|2^{-j}(2k+d)|^{\alpha_1}} \right| |A_k^{l-1}| \leq C. \quad (35)$$

We use $\varphi(tx)$, $2^{-j} \leq t \leq 2^{-j+1}$, instead of $\varphi_j(x) = \varphi(2^{-j}x)$ in (34); this gives

$$\begin{aligned}
 & \sup_{2^{-j} \leq t \leq 2^{-j+1}} \left| \sum_{m=-\infty}^K 2^{\alpha j} (\varphi(tH) \rho_{m+j}(H) f)(x) \right| \\
 & \leq C \sum_{m=-\infty}^K 2^{\alpha(j+m)} 2^{(\alpha_1-\alpha)m} M(\tilde{\rho}_{m+j}(H) f)(x).
 \end{aligned} \quad (36)$$

For the above inequality, we first apply the l_q -quasi-norm with respect to j and then apply L^p -quasi-norm with respect to x . Note that $\alpha_1 > \alpha$. Then, we have

$$\begin{aligned}
 & \left\| \left(\sum_{j=-\infty}^{\infty} \left| \sup_{2^{-j} \leq t \leq 2^{-j+1}} \left| \sum_{m=-\infty}^K 2^{\alpha j} (\varphi(tH) \rho_{m+j}(H) f)(x) \right|^q \right)^{1/q} \right\|_{L^p} \\
 & \leq C \left\| \left(\sum_{j=-\infty}^{\infty} \left| \sum_{m=-\infty}^K 2^{\alpha(j+m)} 2^{(\alpha_1-\alpha)m} M(\tilde{\rho}_{m+j}(H) f)(x) \right|^q \right)^{1/q} \right\|_{L^p} \\
 & \leq C \left\| \left(\sum_{i=-\infty}^{\infty} |2^{i\alpha} M(\tilde{\rho}_i(H) f)(x)|^q \right)^{1/q} \right\|_{L^p} \\
 & \leq C \left\| \left(\sum_{i=-\infty}^{\infty} |2^{i\alpha} \tilde{\rho}_i(H) f(x)|^q \right)^{1/q} \right\|_{L^p} \\
 & \leq C \left\| \left(\sum_{i=-\infty}^{\infty} \left| 2^{i\alpha} \sum_k |2^{-i}(2k+d)|^{\alpha_1} \rho_i(2k+d) P_k f(x) \right|^q \right)^{1/q} \right\|_{L^p}.
 \end{aligned} \quad (37)$$

Let $\psi \in S$, $\text{supp } \psi \subset \text{supp } \rho$, and $\psi(x) = 1$ if $x \in \text{supp } \rho$. Then we have

$$\begin{aligned}
 & \left\| \left(\sum_{i=-\infty}^{\infty} \left| 2^{i\alpha} \sum_k |2^{-i}(2k+d)|^{\alpha_1} \rho_i(2k+d) P_k f(x) \right|^q \right)^{1/q} \right\|_{L^p} \\
 & = \left\| \left(\sum_{i=-\infty}^{\infty} \left| 2^{i\alpha} \sum_k |2^{-i}(2k+d)|^{\alpha_1} \psi(2^{-i}(2k+d)) \rho_i(2k+d) P_k f(x) \right|^q \right)^{1/q} \right\|_{L^p} \\
 & = \left\| \left(\sum_{i=-\infty}^{\infty} \left| 2^{i\alpha} \sum_k \Delta^l |2^{-i}(2k+d)|^{\alpha_1} \psi(2^{-i}(2k+d)) A_k^{l-1} S_k^{l-1}(\rho_i(H) f)(x) \right|^q \right)^{1/q} \right\|_{L^p} \\
 & \leq C \left\| \left(\sum_{i=-\infty}^{\infty} 2^{i\alpha q} |\rho_i(H) f(x)|^q \right)^{1/q} \right\|_{L^p} = C \|f\|_{F_p^{\alpha,q}}.
 \end{aligned} \quad (38)$$

In the proof of the above inequality, we use the following fact:

$$\begin{aligned}
 & \sum_k |\Delta^l |2^{-i}(2k+d)|^{\alpha_1} \psi(2^{-i}(2k+d)) A_k^{l-1}| \\
 & \leq C \sum_k \left| 2^{(-i+1)l} \sum_{s=0}^l C_l^s (|2^{-i}(2k+d)|^{\alpha_1})^{(l-s)} \right. \\
 & \cdot \psi^{(s)}(2^{-i}(2k+d)) \left. \right| k^{l-1} \leq C \sum_k 2^{2(-i+1)l}
 \end{aligned}$$

$$\begin{aligned}
 & \cdot \frac{|2^{-i}(2k+d)|^b}{(1+|2^{-i}(2k+d)|)^a} k^{l-1} \leq C \sum_k 2^{2(-i+1)l} 2^{(a-b)j} \\
 & \cdot \frac{1}{(2^i+|2k+d|)^{a-b}} k^{l-1}.
 \end{aligned} \quad (39)$$

Taking $a-b = 2l > l-1 > 0$ yields

$$\sum_k |\Delta^l |2^{-i}(2k+d)|^{\alpha_1} \psi(2^{-i}(2k+d)) A_k^{l-1}| \leq C. \quad (40)$$

Let us estimate the second sum in (31) now. Similar to the estimate of the first sum, we introduce

$$\rho'_j(x) = |2^{-j}x|^{\alpha_0} \rho_j(x) \quad j \in \mathbb{Z}, \quad \alpha_0 < \alpha. \quad (41)$$

Then we have

$$\begin{aligned} & \left| \sum_{m=K+1}^{\infty} 2^{\alpha j} (\varphi_j(H) \rho_{m+j}(H) f)(x) \right| \\ &= \left| \sum_{m=K+1}^{\infty} 2^{\alpha(j+m)} 2^{(\alpha_0-\alpha)m} \right. \\ & \quad \cdot \frac{\varphi(2^{-j}(2k+d))}{|2^{-j}(2k+d)|^{\alpha_0}} \rho_{m+j}(2k+d) P_k f(x) \left. \right| \\ &\leq \sum_{m=K+1}^{\infty} 2^{\alpha(j+m)} 2^{(\alpha_0-\alpha)m} \sum_k \left| \Delta^l \frac{\varphi(2^{-j}(2k+d))}{|2^{-j}(2k+d)|^{\alpha_0}} \right| \\ & \quad \cdot |A_k^{l-1}| |S_k^{l-1}(\rho'_{m+j}(H) f)(x)| \\ &\leq C \sum_{m=K+1}^{\infty} 2^{\alpha(j+m)} 2^{(\alpha_0-\alpha)m} M(\rho'_{m+j}(H) f)(x), \end{aligned} \quad (42)$$

where $j \in \mathbb{Z}$. Following the same estimate procedure of (27) gives

$$\sum_k \left| \Delta^l \frac{\varphi(2^{-j}(2k+d))}{|2^{-j}(2k+d)|^{\alpha_0}} \right| |A_k^{l-1}| \leq C. \quad (43)$$

Similarly, we use $\varphi(tx)$, $2^{-j} \leq t \leq 2^{-j+1}$, instead of $\varphi_j(x) = \varphi(2^{-j}x)$ in (42). Then, we have

$$\begin{aligned} & \sup_{2^{-j} \leq t \leq 2^{-j+1}} \left| \sum_{m=K+1}^{\infty} 2^{\alpha j} (\varphi(tH) \rho_{m+j}(H) f)(x) \right| \\ &\leq C \sum_{m=K+1}^{\infty} 2^{\alpha(j+m)} 2^{(\alpha_0-\alpha)m} M(\tilde{\rho}_{m+j}(H) f)(x). \end{aligned} \quad (44)$$

Since $\alpha_0 < \alpha$, similar to the estimate of the first sum and the estimate of (38), we arrive at

$$\begin{aligned} & \left\| \left(\sum_{m=K+1}^{\infty} \left| \sup_{2^{-j} \leq t \leq 2^{-j+1}} \left| \sum_{m=K+1}^{\infty} 2^{\alpha j} (\varphi(tH) \rho_{m+j}(H) f)(x) \right|^q \right)^{1/q} \right\|_{L^p} \\ &\leq C \left\| \left(\sum_{i=-\infty}^{\infty} \left| 2^{i\alpha} \sum_k |2^{-i}(2k+d)|^{\alpha_0} \rho_i(2k+d) P_k f(x) \right|^q \right)^{1/q} \right\|_{L^p} \\ &\leq C \left\| \left(\sum_{i=-\infty}^{\infty} 2^{i\alpha q} |\rho_i(H) f(x)|^q \right)^{1/q} \right\|_{L^p} = C \|f\|_{F_p^{\alpha,q}}, \end{aligned} \quad (45)$$

which is even stronger than the desired estimate.

The proof is complete. $\square$

The Poisson semigroup $\{P_t\}_{t>0}$, associated with H , is given by

$$P_t f = e^{-tH^{1/2}} f = \sum_{k=0}^{\infty} e^{-t(2k+d)^{1/2}} P_k f, \quad f \in L^2. \quad (46)$$

We refer the reader to Thangavelu's monograph [3] for a detailed description of the Poisson semigroup $\{P_t\}_{t>0}$ associated with H . In [9], the authors studied the g -function associated with the Poisson semigroup:

$$g(f)(x) = \left(\int_0^{\infty} |\partial_t f(t, x)|^2 t dt \right)^{1/2}, \quad (47)$$

where $f \in L^p$, $1 \leq p < \infty$, and $f(t, x)$ is the Poisson integral of f that is given by the convergent series

$$f(t, x) = \sum_{k=0}^{\infty} e^{-t(2k+d)^{1/2}} \sum_{|\alpha|=k} \langle f, h_{\alpha} \rangle h_{\alpha}(x). \quad (48)$$

They obtained that, for $1 < p < \infty$ and $\omega \in A_p$,

$$\|g(f)\|_{L_{\omega}^p} \leq C \|f\|_{L_{\omega}^p}, \quad f \in L_{\omega}^p, \quad (49)$$

which implies

$$C_p^{-1} \|f\|_{L^p} \leq \|g(f)\|_{L^p} \leq C_p \|f\|_{L^p}, \quad 1 < p < \infty. \quad (50)$$

That is a consequent of a general result for symmetric contraction semigroups. This general result is a refinement done by Coifman et al. [10], of the Littlewood-Paley theory for symmetric contraction semigroups satisfying, also, positivity and conservation properties which was developed by Stein [11]; see also [12]. Inspired by these results, we give the harmonic and thermic characterization of the Triebel-Lizorkin spaces associated with the Hermite expansions by the Poisson semigroup related to the Hermite expansion, which shows $F_p^{0,2} = L^p$, for $1 < p < \infty$.

Theorem 5. Let $\alpha \in \mathbb{R}$, $1 < p < \infty$, and $1 < q \leq \infty$. If m is a sufficiently large natural number, then

$$\left\| \left(\int_0^{\infty} t^{(m-\alpha)q} \left| \frac{\partial^m}{\partial t^m} P_t f(x) \right|^q \frac{dt}{t} \right)^{1/q} \right\|_{L^p(\mathbb{R}^d)} \quad (51)$$

(with usual modification when $q = \infty$) is equivalent quasi-norm in $F_p^{\alpha,q}$.

Proof. We use Theorem 1 to prove that (51) is an equivalent quasi-norm in $F_p^{\alpha,q}$. Let $\varphi(x) = -|x|^{m/2} e^{-|x|^{1/2}}$. Obviously, $\varphi \in \mathcal{S}$; we get

$$\begin{aligned} \varphi(t^2 H) f(x) &= -t^m H^{m/2} e^{-tH^{1/2}} f(x) \\ &= t^m \frac{\partial^m}{\partial t^m} P_t f(x). \end{aligned} \quad (52)$$

It means that (51) is an equivalent quasi-norm in $F_p^{\alpha,q}$. Here, we substituted t by t^2 . $\square$

3. Endpoint Estimates

From Theorem 5, we know that $F_p^{0,2} = L^p$, for $1 < p < \infty$. However, this conclusion does not hold when $p = 1$. So, the purpose of this section is to discuss the characterization of the space $F_1^{0,2}$. We first present the definition of the local Hardy space below.

Definition 6. Let $\varphi \in S(\mathbb{R}^d)$, $\varphi(0) = 1$, and $\varphi_t(x) = \varphi(tx)$. The local Hardy space associated with the Hermite operator is defined by the quasi-norm

$$\|f\|_{h_n^1} = \left\| \sup_{0 < t < 2^{-n}} |\varphi(tH) f(\cdot)| \right\|_{L^1(\mathbb{R}^d)}. \quad (53)$$

We now give some properties about the local Hardy space.

Proposition 7. Let $m(x)$ be an infinitely differentiable complex-valued function in $\mathbb{R}^d$ such that

$$\sup_{x \in \mathbb{R}^d} (1 + |x|)^{|\alpha|} |D^\alpha m(x)| < \infty \quad (54)$$

for every multi-index α . If $p = 1$, then there exists a constant C such that

$$\|m(H) f(\cdot)\|_{h_n^1(\mathbb{R}^d)} \leq C \|f\|_{h_n^1(\mathbb{R}^d)} \quad (55)$$

holds for all $f \in h_n^1(\mathbb{R}^d)$.

In order to prove Proposition 7, we recall some results from the theory of local Hardy spaces [13]; compare also [14]. A function a is an atom for the local Hardy space h_n^1 if there is a ball $B(x_0, r)$, $r \leq 2^{1-n/2}$, such that

$$\begin{aligned} \text{supp } a &\subset B(x_0, r), \\ \|a\|_{L^\infty} &\leq |B(x_0, r)|^{-1}, \\ \text{if } r &\leq 2^{-1-n/2}, \quad \text{then } \int a(x) dx = 0. \end{aligned} \quad (56)$$

The atomic norm in h_n^1 is defined by

$$\|f\|_{h_{a,n}^1} = \inf \left(\sum_j |c_j| \right), \quad (57)$$

where the infimum is taken over all decompositions $f = \sum c_j a_j$; a_j are h_n^1 atoms.

Theorem GO. The norms $\|\cdot\|_{h_n^1}$ and $\|\cdot\|_{h_{a,n}^1}$ are equivalent with constants independent of $n \in \mathbb{Z}$.

Moreover, if $f \in h_n^1$, $\text{supp } f \subset B(x, 2^{1-n/2})$, then there are h_n^1 atoms a_j such that $\text{supp } a_j \subset B(x, 2^{2-n/2})$ and

$$f = \sum_j c_j a_j, \quad \sum_j |c_j| \leq \|f\|_{h_n^1} \quad (58)$$

with a constant C independent of n .

Proof of Proposition 7. We only need to show that there exists a constant $C > 0$ such that

$$\int_{\mathbb{R}^d} \sup_{0 < t \leq 2^{-n}} |\varphi_t(H) m(H) a(x)| dx \leq C, \quad (59)$$

where a is any h_n^1 atom associated with a ball $B(x_0, r)$. Let $B^* = B^*(x_0, 2r)$. Clearly, this ball is concentric with B . Then

$$\begin{aligned} \int_{B^*} \sup_{0 < t \leq 2^{-n}} |\varphi_t(H) m(H) a(x)| dx &\leq C |B^*|^{1/2} \left(\int_{B^*} \sup_{0 < t \leq 2^{-n}} \left| \sum_k \varphi(t(2k+d)) m(2k+d) P_k a(x) \right|^2 dx \right)^{1/2} \\ &\leq C |B^*|^{1/2} \left(\int_{B^*} \sup_{0 < t \leq 2^{-n}} \left| \sum_k \Delta^l(\varphi(t(2k+d)) m(2k+d)) A_k^{l-1} S_k^{l-1}(a)(x) \right|^2 dx \right)^{1/2} \\ &\leq C |B^*|^{1/2} \left(\int_{B^*} |M(a)(x)|^2 dx \right)^{1/2} \leq C |B^*|^{1/2} \left(\int_B |a(x)|^2 dx \right)^{1/2} \\ &\leq C |B^*|^{1/2} |B|^{1/2} |B|^{-1} \leq C. \end{aligned} \quad (60)$$

In the above inequality, we use the fact that

$$\sup_{0 < t \leq 2^{-n}} \left| \sum_k \Delta^l(\varphi(t(2k+d)) m(2k+d)) A_k^{l-1} \right| \leq C. \quad (61)$$

If $x \notin B^*$, we use the representation

$$\varphi_t(H) m(H) a(x) = \int K_t(x, y) a(y) dy, \quad (62)$$

where

$$\begin{aligned} K_t(x, y) &= \sum_k \varphi(t(2k+d)) m(2k+d) \Phi_k(x, y) \\ &= \sum_k \Delta^l(\varphi(t(2k+d)) m(2k+d)) A_k^{l-1} S_k^{l-1}(x, y), \end{aligned} \quad (63)$$

and $\Phi_k(x, y)$ is the kernel of the projection operator P_k ; see [3] for detailed description. From Lemma 3, we conclude that

$$\begin{aligned}
 & |K_t(x, y)| \\
 & \leq C \sum_k \frac{1}{(1+t|2k+d|)^a} \frac{1}{(1+|2k+d|)^b} k^{l-1} k^{d/2} \\
 & \cdot \left\{ \frac{1}{(1+|x-y|)^{l-(3d-2)/6}} \right. \\
 & \left. + \frac{1}{(1+|x+y|)^{l-(3d-2)/6}} \right\}. \tag{64}
 \end{aligned}$$

We choose $l-1 > (3d-2)/6$; then we have

$$\begin{aligned}
 & \int_{(B^*)^C} \sup_{0 < t \leq 2^{-n}} |\varphi_t(H) m(H) a(x)| dx \\
 & \leq C \int_{(B^*)^C} \sup_{0 < t \leq 2^{-n}} \int |K_t(x, y)| |a(y)| dy dx \\
 & \leq C \int_{(B^*)^C} \int \left\{ \frac{1}{(1+|x-y|)^{l-(3d-2)/6}} \right. \\
 & \left. + \frac{1}{(1+|x+y|)^{l-(3d-2)/6}} \right\} |a(y)| dy dx \\
 & \leq C \int \frac{1}{(1+|x|)^{l-(3d-2)/6}} dx \leq C. \tag{65}
 \end{aligned}$$

Then we complete our proof of Proposition 7. $\square$

If η has the same properties as the function φ (including $\eta_t(x) = \eta(tx)$), then the l^2 -valued counterpart of $\|f\|_{h_n^1}^\eta$ is given by

$$\begin{aligned}
 & \|f_j\|_{h_n^1(\mathbb{R}^d, l^2)}^\eta \\
 & = \left\| \sup_{0 < t \leq 2^{-n}} \left(\sum_{j=-\infty}^{\infty} |\eta_t(H) f_j(\cdot)|^2 \right)^{1/2} \right\|_{L^1(\mathbb{R}^d)}. \tag{66}
 \end{aligned}$$

In the sequel we omit the index η in (66). Then the l^2 -valued counterpart of (54), (55) reads as follows.

Proposition 8. *Let $\{m_{j,l}(x)\}_{j,l=-\infty}^\infty$ be a sequence of infinitely differentiable complex-valued functions in $\mathbb{R}^d$ such that*

$$\sup_{x \in \mathbb{R}^d} (1+|x|)^{|\alpha|} \left(\sum_{j,l=0}^{\infty} |D^\alpha m_{j,l}(x)|^2 \right)^{1/2} < \infty \tag{67}$$

for every multi-index α . Then there exists a constant C such that

$$\left\| \sum_{l=-\infty}^{\infty} m_{j,l}(H) f_l(\cdot) \right\|_{h_n^1(\mathbb{R}^d, l^2)} \leq C \|f_j\|_{h_n^1(\mathbb{R}^d, l^2)}. \tag{68}$$

After giving the above theoretical preparations, we now state our next result.

Theorem 9. *Let $\alpha = 0$, $q = 2$, and $p = 1$. Then*

$$F_1^{0,2}(\mathbb{R}^d) = h_n^1(\mathbb{R}^d) \tag{69}$$

which are equivalent quasi-norms.

Proof. Consider the following.

Step 1. The functions $f(x) \in S(\mathbb{R}^d)$ with compact support are dense in $F_1^{0,2}(\mathbb{R}^d)$ and dense in $h_n^1(\mathbb{R}^d)$; compare [13], so we assume $f(x) \in S(\mathbb{R}^d)$. First, we prove that

$$\|f\|_{h_n^1(\mathbb{R}^d)} \sim \|\varphi_j(H) f\|_{h_n^1(\mathbb{R}^d, l^2)} \tag{70}$$

with $\varphi_j \in \{\varphi_j(x)\}_{j=-\infty}^\infty$ (equivalent quasi-norms).

Let $f \in L^p(\mathbb{R}^d)$, $\varphi_j \in \{\varphi_j(x)\}_{j=-\infty}^\infty$; and let $m_{j,0} = \varphi_j$ and $m_{j,l} = 0$ for $j, l \in \mathbb{Z}$, $l \neq 0$. Substituting $f_0 = 0$ and $f_l = 0$ ($l \in \mathbb{Z}$, $l \neq 0$) into (68) yields

$$\begin{aligned}
 & \|\varphi_j(H) f\|_{h_n^1(\mathbb{R}^d, l^2)} = \left\| \sum_{l=-\infty}^{\infty} m_{j,l}(H) f_l(\cdot) \right\|_{h_n^1(\mathbb{R}^d, l^2)} \\
 & \leq C \|f_j\|_{h_n^1(\mathbb{R}^d, l^2)} = \|f\|_{h_n^1(\mathbb{R}^d)}. \tag{71}
 \end{aligned}$$

Hence $f \in h_n^1(\mathbb{R}^d, l^2)$. In order to prove the converse assertion, we assume that $f \in h_n^1(\mathbb{R}^d, l^2)$. Let

$$\psi_j(x) = \sum_{r=-1}^1 \varphi_{j+r}(x) \quad (\text{with } \varphi_{-1} = 0). \tag{72}$$

If $x \in \text{supp } \varphi_j$, it is easy to get $\psi_j = 1$. Let $m_{0,l} = \psi_l$ and $m_{j,l} = 0$ if $j \in \mathbb{Z}$, $l \in \mathbb{Z}$, and $l \neq 0$. We apply (68) with $f_j = \varphi_j(H)f$. Then

$$\begin{aligned}
& \left\| \sum_{l=-\infty}^{\infty} m_{j,l}(H) f_l(\cdot) \right\|_{h_n^1(\mathbb{R}^d, l^2)} \\
&= \left\| \sup_{0 < t \leq 2^{-n}} \left(\sum_{j=-\infty}^{\infty} \left| \eta_t(H) \sum_{l=-\infty}^{\infty} m_{j,l}(H) f_l(\cdot) \right|^2 \right)^{1/2} \right\|_{L^1(\mathbb{R}^d)} \\
&= \left\| \sup_{0 < t \leq 2^{-n}} \left| \eta_t(H) \sum_{l=-\infty}^{\infty} \psi_l(H) f_l(\cdot) \right| \right\|_{L^1(\mathbb{R}^d)} \\
&= \left\| \sup_{0 < t \leq 2^{-n}} \left| \eta_t(H) \sum_{l=-\infty}^{\infty} \varphi_l(H) f(\cdot) \right| \right\|_{L^1(\mathbb{R}^d)} \\
&= \left\| \sup_{0 < t \leq 2^{-n}} |\eta_t(H) f(\cdot)| \right\|_{L^1(\mathbb{R}^d)} \leq C \|f_j\|_{h_n^1(\mathbb{R}^d, l^2)} \\
&= \|\varphi_j(H)f\|_{h_n^1(\mathbb{R}^d, l^2)}.
\end{aligned} \tag{73}$$

Therefore

$$\|f\|_{h_n^1(\mathbb{R}^d)} \leq C \|\varphi_j(H)f\|_{h_n^1(\mathbb{R}^d, l^2)}. \tag{74}$$

Thus, (70) holds.

Step 2. We recall that $\eta(0) = 1$. It follows from (66) with $f_j = \varphi_j(H)f$ and (70) that

$$\begin{aligned}
& \|f_j\|_{h_n^1(\mathbb{R}^d, l^2)} \\
&= \|\varphi_j(H)f\|_{h_n^1(\mathbb{R}^d, l^2)} \\
&= \left\| \sup_{0 < t \leq 2^{-n}} \left(\sum_{j=-\infty}^{\infty} |\eta_t(H) \varphi_j(H)f(\cdot)|^2 \right)^{1/2} \right\|_{L^1(\mathbb{R}^d)} \\
&\geq C \left\| \left(\sum_{j=-\infty}^{\infty} |\varphi_j(H)f(\cdot)|^2 \right)^{1/2} \right\|_{L^1(\mathbb{R}^d)} \\
&= \|f\|_{F_1^{0,2}(\mathbb{R}^d)}.
\end{aligned} \tag{75}$$

Step 3. We prove the converse assertion of (75). From (66) and (75) we have

$$\begin{aligned}
& \|f\|_{h_n^1(\mathbb{R}^d)} \\
&\leq C \left\| \sup_{0 < t \leq 2^{-n}} \left(\sum_{j=-\infty}^{\infty} |\eta_t(H) \varphi_j(H)f(\cdot)|^2 \right)^{1/2} \right\|_{L^1(\mathbb{R}^d)}.
\end{aligned} \tag{76}$$

Since $\eta(x)$ has compact support, $\eta_t(2k+d)\varphi_j(2k+d) \neq 0$ implies $k \approx 2^j$. So, we have

$$\begin{aligned}
& \sup_{0 < t \leq 2^{-n}} \left(\sum_{j=-\infty}^{\infty} |\eta_t(H) \varphi_j(H)f(x)| \right) \\
&= \sup_{0 < t \leq 2^{-n}} \left(\sum_{j=-\infty}^{\infty} \left| \sum_k \eta(t(2k+d)) P_k \varphi_j(H)f(x) \right| \right) \\
&\leq C \sup_{0 < t \leq 2^{-n}} \int_{\mathbb{R}^d} |K_t(x, y) \varphi_j(H)f(y) dy|,
\end{aligned} \tag{77}$$

where

$$\begin{aligned}
|K_t(x, y)| &= \left| \sum_k \eta_t(2k+d) \Phi_k(x, y) \right| \\
&= \left| \sum_k \Delta^l(\eta_t(2k+d)) A_k^{l-1} S_k^{l-1}(x, y) \right| \\
&\leq C \sum_k \frac{1}{(1+t|2k+d|)^a} k^{l-1} k^{d/2} \\
&\cdot \left\{ \frac{1}{(1+2^{j/2}|x-y|)^{l-(3d-2)/6}} \right. \\
&\quad \left. + \frac{1}{(1+2^{j/2}|x+y|)^{l-(3d-2)/6}} \right\}.
\end{aligned} \tag{78}$$

If $\varphi_{j,s}^*(f)(x)$ is the maximal function in (11), where $s = l - (3d-2)/6 > d/\min(p, q)$, then by Theorem GS we get

$$\|f\|_{h_n^1(\mathbb{R}^d)} \leq C \left\| \left(\sum_{j=-\infty}^{\infty} |\varphi_{j,s}^*(f)(x)|^2 \right)^{1/2} \right\|_{L^1(\mathbb{R}^d)} \tag{79}$$

$$\leq C \|f\|_{F_1^{0,2}(\mathbb{R}^d)}. \tag{80}$$

In summary, (75) and (79) tell us that the quasi-norms $\|f\|_{h_n^1(\mathbb{R}^d)}$ and $\|f\|_{F_1^{0,2}(\mathbb{R}^d)}$ are equivalent on the functions $f \in S(\mathbb{R}^d)$ with compact support. Theorem 9 follows by completion. $\square$

4. The Boundedness of the Riesz

Transform on $F_p^{\alpha,q}$

This section presents the proof of the boundedness of the Riesz transforms associated with Hermite expansions on the space $F_p^{\alpha,q}$.

Since

$$L = -\frac{1}{2}[(\nabla + x) \cdot (\nabla - x) + (\nabla - x) \cdot (\nabla + x)], \quad (81)$$

it is reasonable to define the Riesz transforms $R_j^\pm$, $j = 1, 2, \dots, d$, by

$$R_j^\pm = \left(\frac{\partial}{\partial x_j} \pm x_j \right) L^{-1/2}. \quad (82)$$

This definition was suggested by Thangavelu [15]; compare also [3]. The definition fits a general framework and, in dimension one, matches the notion of conjugacy for the Hermite function expansions investigated by Gosselin and Stempak [16]. In [17], Stempak and Torrea have already proved that the kernels of the Riesz transforms satisfy the Calderón-Zygmund standard conditions. Their results are as below.

Theorem KJ. *The Riesz operators $R_j^\pm$, initially defined on L^2 , are Calderón-Zygmund operators associated with the (Calderón-Zygmund) kernels $R_j^\pm(x, y)$, which satisfy*

$$\begin{aligned} |R_j^\pm(x, y)| &\leq \frac{C}{|x - y|^d}, \\ |\nabla_x R_j^\pm(x, y)| + |\nabla_y R_j^\pm(x, y)| &\leq \frac{C}{|x - y|^{d+1}}. \end{aligned} \quad (83)$$

We now state the last result of this paper.

Theorem 10. *Let $\alpha \in \mathbb{R}$, $0 < p$, and $q < \infty$. Then the Riesz transform associated with H is a bounded operator on $\|f\|_{F_p^{\alpha,q}}$.*

Proof. Using the equivalent quasi-norm of $F_p^{\alpha,q}$ defined in (51), we yield that

$$TR_j^\pm = R_j^\pm T, \quad (84)$$

where T is the operator $f \rightarrow T(f)$ that is defined by

$$T(f)(x) = \left\{ \frac{\partial^m}{\partial t^m} P_t f(x) \right\}_{t>0}. \quad (85)$$

We obtain

$$\begin{aligned} \|R_j^\pm\|_{F_p^{\alpha,q}} &= \left\| \left(\int_0^\infty |t^{(m-\alpha)} TR_j^\pm f(x)|^q \frac{dt}{t} \right)^{1/q} \right\|_{L^p(\mathbb{R}^d)} \\ &= \left\| \left(\int_0^\infty |t^{(m-\alpha)} R_j^\pm T f(x)|^q \frac{dt}{t} \right)^{1/q} \right\|_{L^p(\mathbb{R}^d)}, \end{aligned} \quad (86)$$

where

$$\begin{aligned} &\left(\int_0^\infty |t^{(m-\alpha)} R_j^\pm T f(x)|^q \frac{dt}{t} \right)^{1/q} \\ &\leq \int_{\mathbb{R}^d} \left(\int_0^\infty |t^{(m-\alpha)} R_j^\pm(x, y) T f(y)|^q t dt \right)^{1/q} dy \\ &\leq C \int_{\mathbb{R}^d} \frac{1}{|x - y|^d} \left(\int_0^\infty |t^{(m-\alpha)} T f(y)|^q t dt \right)^{1/q} dy \\ &\leq C \left(\int_{\mathbb{R}^d} \frac{1}{|x - y|^{dp'}} dy \right)^{1/p'} \\ &\quad \cdot \left(\int_{\mathbb{R}^n} \left(\int_0^\infty |t^{(m-\alpha)} T f(y)|^q t dt \right)^{p/q} dy \right)^{1/p} \\ &\leq C \left(\int_{\mathbb{R}^d} \frac{1}{|x - y|^{dp'}} dy \right)^{1/p'} \|f\|_{F_p^{\alpha,q}}. \end{aligned} \quad (87)$$

Applying the L^p -norm with respect to x in (87) gives

$$\|R_j^\pm f\|_{F_p^{\alpha,q}} \leq C \|f\|_{F_p^{\alpha,q}}. \quad (88)$$

□

The results presented in Theorem 10 can be further enlarged to the boundedness of the g -functions of Littlewood-Paley defined with either the heat semigroup or the Poisson semigroup. We use a theorem to describe this bounded result.

Theorem 11. *If g is commutative with the operator T defined by (85) and if the kernel of g satisfies the Calderón-Zygmund size condition, then g is bounded on $F_p^{\alpha,q}$.*

Since the proof of Theorem 11 is very similar to the proof of Theorem 10, we omit the proof here in order to make the paper concise.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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