Research Article

Herz-Morrey-Hardy Spaces with Variable Exponents and Their Applications

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The authors introduce Herz-Morrey-Hardy spaces with variable exponents and establish the characterization of these spaces in terms of atom. Applying the characterization, the authors obtain the boundedness of some singular integral operators on these spaces.

1. Introduction

The Herz spaces go back to Beurling and Herz; see [1, 2]. Firstly, they attracted a lot of authors' attention because they could be used to characterize Fourier multipliers for Hardy spaces; see [3]. Then, in 1989 Chen and Lau in [4] and García-Cuerva in [5] introduced now called nonhomogeneous Herz type Hardy spaces. They found that these Herz type Hardy spaces have a decomposition via central atoms. After that, Lu et al. considered homogeneous Herz type Hardy spaces and also obtained a central atomic decomposition for them. Since then Herz type spaces have been studied extensively; see monograph [6] for details. Meanwhile, in the last three decades, the interest of the study for variable exponent spaces has been increasing year by year. Variable exponent spaces have many applications: in electrorheological fluid [7], in differential equations [8] and references therein, and in image restoration [9–11], for instance. Indeed, many spaces with variable exponents appeared, such as: Lebesgue spaces, Sobolev spaces and Bessel potential spaces with variable exponent, Besov and Triebel-Lizorkin spaces with variable exponents, Morrey spaces with variable exponents, Campanato spaces with variable exponent, and Hardy spaces with variable exponent; see [12–23] and references therein. Moreover, the atomic, molecular, and wavelet decompositions of variable exponent Besov and Triebel-Lizorkin spaces were given in [13, 14, 20, 21, 24]. The duality and reflexivity of spaces $B_{p(\cdot),q}^s$ and $F_{p(\cdot),q}^s$ were discussed in [25]. The atomic

and molecular decompositions of Hardy spaces with variable exponent and their applications for the boundedness of singular integral operators were obtained in [22, 26].

Recently, as a generalization of Lebesgue spaces with variable exponent, Herz spaces with variable exponents are introduced. In fact, in 2010 Izuki proved the boundedness of sublinear operators on Herz space with variable exponents $\dot{K}_{p(\cdot)}^{\alpha,q}$ and $K_{p(\cdot)}^{\alpha,q}$ in [27]. In 2012, Almeida and Drihem obtained boundedness results for a wide class of classical operators on Herz spaces $\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}$ and $K_{p(\cdot)}^{\alpha(\cdot),q}$ in [28]. Shi and the first author in [29] considered Herz type Besov and Triebel-Lizorkin spaces with one variable exponent. Then Dong and first author in [30] established the boundedness of vector-valued Hardy-Littlewood maximal operator in spaces $\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}$ and $K_{p(\cdot)}^{\alpha(\cdot),q}$ and gave characterizations of Herz type Besov and Triebel-Lizorkin spaces with variable exponents by maximal functions. In [31], Wang and Liu introduced a certain Herz type Hardy spaces with variable exponent. In 2013, Samko introduced Herz spaces with three variable exponents and obtained the boundedness of Hardy-Littlewood maximal operator on them. In [32–34], the boundedness of singular integrals and their commutators of BMO functions are discussed in Herz Morrey spaces with variable exponents. The Herz-Morrey spaces with constants were considered in [35, 36]; however, there is no theory of Herz-Morrey type Hardy spaces. In this paper we fill the gap and introduce Herz-Morrey-Hardy spaces with variable exponents.

The outline of the paper is as follows. In the rest of the section we will recall some definitions and notions. In Section 2, we will define the Herz-Morrey-Hardy spaces with variable exponents $HM\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}$ and $HM\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}$ and give their atomic characterization. In Section 3, we obtain that certain singular integral operators are bounded from Herz-Morrey-Hardy spaces with variable exponents into Herz-Morrey spaces with variable exponents as an application of the atomic characterization.

Throughout this paper $|E|$ denotes the Lebesgue measure and χ_E the characteristic function for a measurable set $E \subset \mathbb{R}^n$. For a multi-index $\beta = (\beta_1, \beta_2, \dots, \beta_n)$, we denote $|\beta| = \beta_1 + \beta_2 + \dots + \beta_n$. We also use the notation $a \leq b$ if there exists a constant $c > 0$ such that $a \leq cb$. If $a \leq b$ and $b \leq a$ we will write $a \approx b$. Finally we claim that C is always a positive constant but it may change from line to line.

Definition 1. Let E be a measurable set in $\mathbb{R}^n$ with $|E| > 0$. Let $p(\cdot) : E \rightarrow [1, \infty)$ be a measurable function. Denote

$$L^{p(\cdot)}(E) := \left\{ f \text{ is measurable on } E : \rho_{p(\cdot)}\left(\frac{f}{\lambda}\right) < \infty \text{ for some constant } \lambda > 0 \right\}, \quad (1)$$

where $\rho_{p(\cdot)}(f) := \int_E |f(x)|^{p(x)} dx$, and

$$\|f\|_{L^{p(\cdot)}(E)} := \inf \left\{ \lambda > 0 : \rho_{p(\cdot)}\left(\frac{f}{\lambda}\right) \leq 1 \right\}. \quad (2)$$

Then $L^{p(\cdot)}(E)$ is a Banach space with the norm $\|\cdot\|_{L^{p(\cdot)}(E)}$.

Let $L_{\text{loc}}^1(\mathbb{R}^n)$ be the collection of all locally integrable functions on $\mathbb{R}^n$. Given a function $f \in L_{\text{loc}}^1(\mathbb{R}^n)$, the Hardy-Littlewood maximal operator $\mathcal{M}$ is defined by

$$\mathcal{M}f(x) := \sup_{r>0} r^{-n} \int_{B(x,r)} |f(y)| dy, \quad \forall x \in \mathbb{R}^n, \quad (3)$$

where and what follows $B(x, r) := \{y \in \mathbb{R}^n : |x - y| < r\}$. We also use the following notation: $p_- := \text{ess inf}\{p(x) : x \in \mathbb{R}^n\}$ and $p_+ := \text{ess sup}\{p(x) : x \in \mathbb{R}^n\}$. The set $\mathcal{P}(\mathbb{R}^n)$ consists of all $p(\cdot)$ satisfying $p_- > 1$ and $p_+ < \infty$. $\mathcal{B}(\mathbb{R}^n)$ is the set of $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfying the condition that $\mathcal{M}$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. It is well known that if $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfies the following global log-Hölder continuous then $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$; see [37–42].

Definition 2. Let $\alpha(\cdot)$ be a real-valued function on $\mathbb{R}^n$. If there exists $C > 0$ such that, for all $x, y \in \mathbb{R}^n$, $|x - y| < 1/2$,

$$|\alpha(x) - \alpha(y)| \leq \frac{C}{-\log(|x - y|)}, \quad (4)$$

then $\alpha(\cdot)$ is said local log-Hölder continuous on $\mathbb{R}^n$.

If there exists $C > 0$, such that for all $x \in \mathbb{R}^n$,

$$|\alpha(x) - \alpha(0)| \leq \frac{C}{\log(e + 1/|x|)}, \quad (5)$$

then $\alpha(\cdot)$ is said log-Hölder continuous at origin.

If there exist $\alpha_\infty \in \mathbb{R}$ and a constant $C > 0$ such that for all $x \in \mathbb{R}^n$

$$|\alpha(x) - \alpha_\infty| \leq \frac{C}{\log(e + |x|)}, \quad (6)$$

then $\alpha(\cdot)$ is said log-Hölder continuous at infinity.

If $\alpha(\cdot)$ is both local log-Hölder continuous and log-Hölder continuous at infinity, then $\alpha(\cdot)$ is said global log-Hölder continuous.

The sets of log-Hölder continuous functions, log-Hölder continuous functions at origin, log-Hölder continuous functions at infinity, global log-Hölder continuous are denoted by $\mathcal{P}_{\text{loc}}^{\log}(\mathbb{R}^n)$, $\mathcal{P}_0^{\log}(\mathbb{R}^n)$, $\mathcal{P}_\infty^{\log}(\mathbb{R}^n)$, and $\mathcal{P}^{\log}(\mathbb{R}^n)$, respectively.

We denote $p'(\cdot)$ by the conjugate exponent to $p(\cdot)$, which means $p'(\cdot) = p(\cdot)/(p(\cdot) - 1)$. It is also well known that $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ is equivalent to $p'(\cdot) \in \mathcal{B}(\mathbb{R}^n)$; see [39].

For simplicity, we denote $L^{p(\cdot)}(\mathbb{R}^n)$ by $L^{p(\cdot)}$. We will use the following results.

Lemma 3 (see [43]). Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. If $f \in L^{p(\cdot)}$ and $g \in L^{p'(\cdot)}$, then fg is integrable on $\mathbb{R}^n$ and

$$\int_{\mathbb{R}^n} |f(x)g(x)| dx \leq r_p \|f\|_{L^{p(\cdot)}} \|g\|_{L^{p'(\cdot)}}, \quad (7)$$

where $r_p = 1 + 1/p^- - 1/p^+$.

Lemma 4 (see [27]). Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Then there exist $0 < \delta_1, \delta_2 < 1$, and a positive constant C depending only on $p(\cdot)$ and n such that for all balls B in $\mathbb{R}^n$ and all measurable subsets $S \subset B$,

$$\begin{aligned} \frac{\|\chi_B\|_{L^{p(\cdot)}}}{\|\chi_S\|_{L^{p(\cdot)}}} &\leq C \frac{|B|}{|S|}, & \frac{\|\chi_S\|_{L^{p(\cdot)}}}{\|\chi_B\|_{L^{p(\cdot)}}} &\leq C \left(\frac{|S|}{|B|} \right)^{\delta_1}, \\ \frac{\|\chi_S\|_{L^{p'(\cdot)}}}{\|\chi_B\|_{L^{p'(\cdot)}}} &\leq C \left(\frac{|S|}{|B|} \right)^{\delta_2}. \end{aligned} \quad (8)$$

Lemma 5 (see [27]). Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Then there exists a positive constant C such that, for any ball B in $\mathbb{R}^n$,

$$\|\chi_B\|_{L^{p(\cdot)}} \|\chi_B\|_{L^{p'(\cdot)}} \leq C |B|. \quad (9)$$

To give the definition of Herz-Morrey spaces with variable exponents, let us introduce the following notations. Let $k \in \mathbb{Z}$, $B_k := \{x \in \mathbb{R}^n : |x| \leq 2^k\}$, $L \in \mathbb{Z}$, $D_k := B_k \setminus B_{k-1}$, and $\chi_k := \chi_{D_k}$. The symbol $\mathbb{N}_0$ denotes the set of all nonnegative integers. For $m \in \mathbb{N}_0$, we denote $\tilde{\chi}_m := \chi_{D_m}$ if $m \geq 1$ and $\tilde{\chi}_0 := \chi_{B_0}$.

Definition 6. Let $0 < q \leq \infty$, $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, and $0 \leq \lambda < \infty$. Let $\alpha(\cdot)$ be a bounded real-valued measurable function on $\mathbb{R}^n$. The homogeneous Herz-Morrey space $M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}$ and

nonhomogeneous Herz-Morrey space $M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}$ are defined, respectively, by

$$\begin{aligned} M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q} &:= \left\{ f \in L_{\text{loc}}^{p(\cdot)}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}} < \infty \right\}, \\ MK_{p(\cdot),\lambda}^{\alpha(\cdot),q} &:= \left\{ f \in L_{\text{loc}}^{p(\cdot)}(\mathbb{R}^n) : \|f\|_{MK_{p(\cdot),\lambda}^{\alpha(\cdot),q}} < \infty \right\}, \end{aligned} \quad (10)$$

where

$$\begin{aligned} \|f\|_{M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}} &:= \sup_{L \in \mathbb{Z}} 2^{-L\lambda} \left(\sum_{k=-\infty}^L \|2^{\alpha(\cdot)k} f \chi_k\|_{L^{p(\cdot)}}^q \right)^{1/q}, \\ \|f\|_{MK_{p(\cdot),\lambda}^{\alpha(\cdot),q}} &:= \sup_{L \in \mathbb{N}_0} 2^{-L\lambda} \left(\sum_{k=0}^L \|2^{\alpha(\cdot)k} f \tilde{\chi}_k\|_{L^{p(\cdot)}}^q \right)^{1/q}. \end{aligned} \quad (11)$$

Here there is the usual modification when $q = \infty$.

Proposition 7. Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $q \in (0, \infty]$, and $\lambda \in [0, \infty)$. If $\alpha(\cdot) \in L^\infty(\mathbb{R}^n) \cap \mathcal{P}_0^{\log}(\mathbb{R}^n) \cap \mathcal{P}_\infty^{\log}(\mathbb{R}^n)$, then

$$\begin{aligned} \|f\|_{M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}} &\approx \max \left\{ \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda} \left(\sum_{k=-\infty}^L 2^{k\alpha(0)q} \|f \chi_k\|_{L^{p(\cdot)}}^q \right)^{1/q}, \right. \\ &\quad \left. \sup_{L > 0, L \in \mathbb{Z}} \left[2^{-L\lambda} \left(\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q} \|f \chi_k\|_{L^{p(\cdot)}}^q \right)^{1/q} \right. \right. \\ &\quad \left. \left. + 2^{-L\lambda} \left(\sum_{k=0}^L 2^{k\alpha(\infty)q} \|f \chi_k\|_{L^{p(\cdot)}}^q \right)^{1/q} \right] \right\}. \end{aligned} \quad (12)$$

Proposition 7 is the generalization of Herz spaces with variable exponents in [28], and it was used in [33, 34].

Lemma 8. Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, $0 < q < \infty$, and $\lambda \in [0, \infty)$. Let $\alpha(\cdot)$ be bounded and log-Hölder continuous both at the origin and at infinity such that $-n\delta_1 < \alpha(0) \leq \alpha_\infty < n\delta_2$, where $0 < \delta_1, \delta_2 < 1$ are constants in Lemma 4. Suppose that T is a sublinear and bounded operator on $L^{p(\cdot)}$ satisfying size condition

$$|Tf(x)| \leq C \int_{\mathbb{R}^n} |x-y|^{-n} |f(y)| dy \quad (13)$$

for all $f \in L_{\text{loc}}^1(\mathbb{R}^n)$ with compact support and a.e. $x \notin \text{supp } f$. Then there exists a positive constant C such that

$$\|Tf\|_{M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}} \leq C \|f\|_{M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}}, \quad \|Tf\|_{MK_{p(\cdot),\lambda}^{\alpha(\cdot),q}} \leq C \|f\|_{MK_{p(\cdot),\lambda}^{\alpha(\cdot),q}} \quad (14)$$

for any function f belongs to $M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}$ and $MK_{p(\cdot),\lambda}^{\alpha(\cdot),q}$, respectively.

Lemma 8 is the generalization of Herz spaces with variable exponents in [28]. For a proof, see [33].

2. The Atomic Characterization

In this section, we will introduce Herz-Morrey-Hardy spaces with variable exponents $HM\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}$ and $HMK_{p(\cdot),\lambda}^{\alpha(\cdot),q}$. To do this, we need to recall some notations. $\mathcal{S}(\mathbb{R}^n)$ denotes the Schwartz space of all rapidly decreasing infinitely differentiable functions on $\mathbb{R}^n$, and $\mathcal{S}'(\mathbb{R}^n)$ denotes the dual space of $\mathcal{S}(\mathbb{R}^n)$. Let $G_N f$ be the grand maximal function of f defined by

$$G_N f(x) := \sup_{\phi \in \mathcal{A}_N} |\phi_\nabla^*(f)(x)|, \quad x \in \mathbb{R}^n, \quad (15)$$

where $\mathcal{A}_N := \{\phi \in \mathcal{S}(\mathbb{R}^n) : \sup_{|\alpha|, |\beta| \leq N, \forall x \in \mathbb{R}^n} |x^\alpha D^\beta \phi(x)| \leq 1\}$ and $N > n+1$ and ϕ_∇^* is the nontangential maximal operator defined by

$$\phi_\nabla^*(f)(x) := \sup_{|y-x| < t} |\phi_t * f(y)|, \quad (16)$$

$$\forall x \in \mathbb{R}^n \quad \text{with } \phi_t(\cdot) = t^{-n} \phi\left(\frac{\cdot}{t}\right).$$

The grand maximal operator G_N was firstly introduced by Fefferman and Stein in [44] to study classical Hardy spaces. For classical Hardy spaces, one can also see [45–47]. Nakai and Sawano generalized them to variable exponent case in [22].

Definition 9. Let $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$, $0 < q \leq \infty$, $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $0 \leq \lambda < \infty$, and $N > n+1$. The homogeneous Herz-Morrey-Hardy space with variable exponents $HM\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}$ and nonhomogeneous Herz-Morrey-Hardy space with variable exponents $HMK_{p(\cdot),\lambda}^{\alpha(\cdot),q}$ are defined, respectively, by

$$\begin{aligned} HM\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q} &:= \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{HM\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}} := \|G_N f\|_{M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}} < \infty \right\}, \\ HMK_{p(\cdot),\lambda}^{\alpha(\cdot),q} &:= \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{HMK_{p(\cdot),\lambda}^{\alpha(\cdot),q}} := \|G_N f\|_{MK_{p(\cdot),\lambda}^{\alpha(\cdot),q}} < \infty \right\}. \end{aligned} \quad (17)$$

Remark 10. If $\alpha(\cdot) \equiv \alpha$ and $\lambda = 0$, these spaces were considered by Wang and Liu in [31]. If $p(\cdot)$ and $\alpha(\cdot)$ are constant and $\lambda = 0$, these are the classical Herz type Hardy spaces; see [6].

Let $\psi(r) = 1$ for $r \in [0, 1]$ and $\psi(r) = r^{-N}$ for $r \in (1, +\infty)$. Then there exists $C > 0$ such that $\phi(x) \leq C\psi(|x|)$ for all $\phi \in \mathcal{A}_N$. Therefore, by [46, Proposition in Page 57], there exists $C > 0$ such that $G_N f(x) \leq CMf(x)$ for all $x \in \mathbb{R}^n$. This means that $G_N f$ satisfies the size condition in Lemma 8. By Lemma 8, if $-n\delta_1 < \alpha(0) \leq \alpha_\infty < n\delta_2$ and $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, then

$$\begin{aligned} HM\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q} \cap L_{\text{loc}}^{p(\cdot)}(\mathbb{R}^n \setminus \{0\}) &= M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}, \\ HMK_{p(\cdot),\lambda}^{\alpha(\cdot),q} \cap L_{\text{loc}}^{p(\cdot)}(\mathbb{R}^n) &= MK_{p(\cdot),\lambda}^{\alpha(\cdot),q}. \end{aligned} \quad (18)$$

Thus we are interested in the case $n\delta_2 \leq \alpha(0)$, $\alpha_\infty < \infty$. In this case, we will establish a characterization of the spaces $HM\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}$ and $HM\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}$ in terms of central atom. For $u \in \mathbb{R}$ we denote by $[u]$ the largest integer less than or equal to u .

Definition 11. Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$ be log-Hölder continuous both at the origin and infinity, and nonnegative integer $s \geq [\alpha_r - n\delta_2]$; here $\alpha_r = \alpha(0)$, if $r < 1$, and $\alpha_r = \alpha_\infty$, if $r \geq 1$, $n\delta_2 \leq \alpha_r < \infty$ and δ_2 as in Lemma 4.

- (i) A function a on $\mathbb{R}^n$ is called a central $(\alpha(\cdot), p(\cdot))$ -atom, if it satisfies (1) $\text{supp } a \subset B(0, r)$; (2) $\|a\|_{L^{p(\cdot)}} \leq |B(0, r)|^{-\alpha_r/n}$; (3) $\int_{\mathbb{R}^n} a(x)x^\beta dx = 0$, $|\beta| \leq s$.
- (ii) A function a on $\mathbb{R}^n$ is called a central $(\alpha(\cdot), p(\cdot))$ -atom of restricted type, if it satisfies (1) $\text{supp } a \subset B(0, r)$, $r \geq 1$; (2) $\|a\|_{L^{p(\cdot)}} \leq |B(0, r)|^{-\alpha_r/n}$; (3) $\int_{\mathbb{R}^n} a(x)x^\beta dx = 0$, $|\beta| \leq s$.

Remark 12. If $p(\cdot) \equiv p$ and $\alpha(\cdot) \equiv \alpha$ are constant, then taking $\delta_2 = 1 - 1/p$ we recover the classical case in [6].

Theorem 13. Let $0 < q < \infty$, $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, $0 \leq \lambda < \infty$, and $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$ be log-Hölder continuous both at the origin and infinity, $2\lambda \leq \alpha(\cdot)$, $n\delta_2 \leq \alpha(0)$, $\alpha_\infty < \infty$, and δ_2 as in Lemma 4.

- (i) $f \in HM\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}$ if and only if $f = \sum_{k=-\infty}^{\infty} \lambda_k a_k$ in the sense of $\mathcal{S}'(\mathbb{R}^n)$, where each a_k is a central $(\alpha(\cdot), p(\cdot))$ -atom with support contained in B_k and $\sup_{L \in \mathbb{Z}} 2^{-L\lambda} \sum_{k=-\infty}^L |\lambda_k|^q < \infty$. Moreover,

$$\|f\|_{HM\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}} \approx \inf \sup_{L \in \mathbb{Z}} 2^{-L\lambda} \left(\sum_{k=-\infty}^L |\lambda_k|^q \right)^{1/q}, \quad (19)$$

where the infimum is taken over all above decompositions of f .

- (ii) $f \in HM\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}$ if and only if $f = \sum_{k=0}^{\infty} \lambda_k a_k$ in the sense of $\mathcal{S}'(\mathbb{R}^n)$, where each a_k is a central $(\alpha(\cdot), p(\cdot))$ -atom of restricted type with support contained in B_k and $\sup_{L \in \mathbb{Z}} 2^{-L\lambda} \sum_{k=0}^L |\lambda_k|^q < \infty$. Moreover

$$\|f\|_{HM\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}} \approx \inf \sup_{L \in \mathbb{Z}} 2^{-L\lambda} \left(\sum_{k=0}^L |\lambda_k|^q \right)^{1/q}, \quad (20)$$

where the infimum is taken over all above decompositions of f .

Proof. We only prove (i). The proof of (ii) is similar. We use the ideas in [6]. To prove the necessity, we choose $\phi \in C_0^\infty(\mathbb{R}^n)$ such that $\phi \geq 0$, $\int_{\mathbb{R}^n} \phi(x) dx = 1$, and $\text{supp } \phi \subset \{x : |x| \leq 1\}$. For $j \in \mathbb{N}_0$, let $\phi_{(j)}(x) := 2^{jn} \phi(2^j x)$, $\forall x \in \mathbb{R}^n$. For each $f \in \mathcal{S}'(\mathbb{R}^n)$, set $f^{(j)}(x) = f * \phi_{(j)}(x)$, $\forall x \in \mathbb{R}^n$. It is obvious that $f^{(j)} \in C^\infty(\mathbb{R}^n)$ and $\lim_{j \rightarrow \infty} f^{(j)} = f$ in $\mathcal{S}'(\mathbb{R}^n)$. Let ψ be a radial smooth function such that $\text{supp } \psi \subset \{x :$

$1/2 - \varepsilon \leq |x| \leq 1 + \varepsilon\}$ with $0 < \varepsilon < 1/4$, $\psi(x) = 1$ for $1/2 \leq |x| \leq 1$. Let $\psi_k(x) := \psi(2^{-k}x)$ for $k \in \mathbb{Z}$ and

$$\widetilde{A}_{k,\varepsilon} := \{x : 2^{k-1} - 2^k \varepsilon \leq |x| \leq 2^k + 2^k \varepsilon\}. \quad (21)$$

Observe that $\text{supp } \psi_k \subset \widetilde{A}_{k,\varepsilon}$ and $\psi_k(x) = 1$ for $x \in A_k := \{x : 2^{k-1} \leq |x| \leq 2^k\}$. Obviously, $1 \leq \sum_{k=-\infty}^{\infty} \psi_k(x) \leq 2$, $x \neq 0$. Let

$$\Phi_k(x) := \begin{cases} \frac{\psi_k(x)}{\sum_{l=-\infty}^{\infty} \psi_l(x)}, & x \neq 0 \\ 0, & x = 0. \end{cases} \quad (22)$$

Then $\sum_{k=-\infty}^{\infty} \Phi_k(x) = 1$ for $x \neq 0$. For each $m \in \mathbb{N}$, we denote by $\mathcal{P}_m$ the class of all the real polynomials with the degree less than m . Let $P_k^{(j)}(x) := P_{\widetilde{A}_{k,\varepsilon}}^{(j)}(f^{(j)}\Phi_k)(x)\chi_{\widetilde{A}_{k,\varepsilon}} \in \mathcal{P}_m(\mathbb{R}^n)$ be the unique polynomial satisfying

$$\begin{aligned} \int_{\widetilde{A}_{k,\varepsilon}} (f^{(j)}(x)\Phi_k(x) - P_k^{(j)}(x))x^\beta dx &= 0, \\ |\beta| \leq m &= \max\{[\alpha(0) - n\delta_2], [\alpha_\infty - n\delta_2]\}. \end{aligned} \quad (23)$$

Write

$$\begin{aligned} f^{(j)}(x) &= \sum_{k=-\infty}^{\infty} (f^{(j)}(x)\Phi_k(x) - P_k^{(j)}(x)) + \sum_{k=-\infty}^{\infty} P_k^{(j)}(x) \\ &:= I_{(j)} + II_{(j)}. \end{aligned} \quad (24)$$

For the term $I_{(j)}$, let $g_k^{(j)}(x) := f^{(j)}(x)\Phi_k(x) - P_k^{(j)}(x)$ and $a_k^{(j)}(x) := g_k^{(j)}(x)/\lambda_k$, where $\lambda_k := b|B_{k+1}|^{\alpha_{k+1}/n} \sum_{l=k-1}^{k+1} \|(G_N f)\chi_l\|_{L^{p(\cdot)}}$ and b is a constant which will be chosen later. Note that $\text{supp } a_k^{(j)} \subset B_{k+1}$, $I_{(j)} = \sum_{k=-\infty}^{\infty} \lambda_k a_k^{(j)}(x)$.

Now we estimate $\|g_k^{(j)}\|_{L^{p(\cdot)}}$. To do this, let $\{\phi_\gamma^k : |\gamma| \leq m\}$ be the orthogonal polynomials restricted to $\widetilde{A}_{k,\varepsilon}$ with respect to the weight $1/|\widetilde{A}_{k,\varepsilon}|$, which are obtained from $\{x^\beta : |\beta| \leq m\}$ by the Gram-Schmidt method, which means

$$\langle \phi_\nu^k, \phi_\mu^k \rangle = \frac{1}{|\widetilde{A}_{k,\varepsilon}|} \int_{\widetilde{A}_{k,\varepsilon}} \phi_\nu^k(x) \phi_\mu^k(x) dx = \delta_{\nu\mu}, \quad (25)$$

where $\delta_{\nu\mu} = 1$ for $\nu = \mu$, otherwise 0.

It is easy to see that $P_k^{(j)}(x) = \sum_{|\gamma| \leq m} \langle f^{(j)}\Phi_k, \phi_\gamma^k \rangle \phi_\gamma^k(x)$ for $x \in \widetilde{A}_{k,\varepsilon}$. On the other hand, from $(1/|\widetilde{A}_{k,\varepsilon}|) \int_{\widetilde{A}_{k,\varepsilon}} \phi_\nu^k(x) \phi_\mu^k(x) dx = \delta_{\nu\mu}$ we infer that

$$\frac{1}{|\widetilde{A}_{1,\varepsilon}|} \int_{\widetilde{A}_{1,\varepsilon}} \phi_\nu^k(2^{k-1}y) \phi_\mu^k(2^{k-1}y) dy = \delta_{\nu\mu}. \quad (26)$$

Thus, we deduce $\phi_\nu^k(2^{k-1}y) = \phi_\nu^1(y)$ a.e. That is, $\phi_\nu^k(x) = \phi_\nu^1(2^{1-k}x)$ almost everywhere for $x \in \widetilde{A}_{k,\varepsilon}$. Therefore

$|\phi_y^k(x)| \leq C$ for $x \in \bar{A}_{k,\varepsilon}$. By the generalized Hölder inequality we have

$$\begin{aligned} |P_k^{(j)}(x)| &\leq \frac{1}{|\bar{A}_{k,\varepsilon}|} \int_{\bar{A}_{k,\varepsilon}} |f^{(j)}(x) \Phi_k(x)| dx \\ &\leq \frac{1}{|\bar{A}_{k,\varepsilon}|} \|f^{(j)} \Phi_k\|_{L^{p(\cdot)}} \|\chi_{\bar{A}_{k,\varepsilon}}\|_{L^{p'(\cdot)}}. \end{aligned} \quad (27)$$

By Lemma 5 we have

$$\begin{aligned} \|g_k^{(j)}\|_{L^{p(\cdot)}} &\leq \|f^{(j)} \Phi_k\|_{L^{p(\cdot)}} + \|P_k^{(j)}\|_{L^{p(\cdot)}} \\ &\leq \|f^{(j)} \Phi_k\|_{L^{p(\cdot)}} + \frac{1}{|\bar{A}_{k,\varepsilon}|} \|f^{(j)} \Phi_k\|_{L^{p(\cdot)}} \\ &\quad \times \|\chi_{\bar{A}_{k,\varepsilon}}\|_{L^{p'(\cdot)}} \|\chi_{\bar{A}_{k,\varepsilon}}\|_{L^{p(\cdot)}} \\ &\leq \|f^{(j)} \Phi_k\|_{L^{p(\cdot)}} + \|f^{(j)} \Phi_k\|_{L^{p(\cdot)}} \\ &\leq \|(f * \phi_{(j)}) \Phi_k\|_{L^{p(\cdot)}} \\ &\leq C \sum_{l=k-1}^{k+1} \|(G_N f) \chi_l\|_{L^{p(\cdot)}}. \end{aligned} \quad (28)$$

Choose $b = C$; then $\|a_k^{(j)}\|_{L^{p(\cdot)}} \leq |B_{k+1}|^{-\alpha_{k+1}/n}$ and each $a_k^{(j)}$ is a central $(\alpha(\cdot), p(\cdot))$ -atom with support contained in B_{k+1} . Here and below we abuse $\alpha_k := \alpha_{2k}$ and it is well defined in Definition 11. Thus,

$$\begin{aligned} &\sup_{L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^L |\lambda_k|^q \\ &\leq \sup_{L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^L |B_{k+1}|^{q\alpha_{k+1}/n} \left(\sum_{l=k-1}^{k+1} \|(G_N f) \chi_l\|_{L^{p(\cdot)}} \right)^q \\ &\leq \sup_{L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^L |B_{k+1}|^{q\alpha_{k+1}/n} \|(G_N f) \chi_k\|_{L^{p(\cdot)}}^q \\ &:= A. \end{aligned} \quad (29)$$

Now we estimate A . By the condition of $\alpha(\cdot)$ and Proposition 7 we consider it in two cases.

Case 1 ($L \leq 0$). Consider

$$\begin{aligned} &2^{-L\lambda q} \sum_{k=-\infty}^L |B_{k+1}|^{q\alpha_{k+1}/n} \|(G_N f) \chi_k\|_{L^{p(\cdot)}}^q \\ &\leq \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^L 2^{kq\alpha(0)} \|(G_N f) \chi_k\|_{L^{p(\cdot)}}^q \\ &\leq \|G_N f\|_{MK_{p(\cdot),\lambda}^{\alpha(\cdot),q}}^q. \end{aligned} \quad (30)$$

Case 2 ($L > 0$). Consider

$$\begin{aligned} &2^{-L\lambda q} \sum_{k=-\infty}^L |B_{k+1}|^{q\alpha_{k+1}/n} \|(G_N f) \chi_k\|_{L^{p(\cdot)}}^q \\ &= 2^{-L\lambda q} \sum_{k=-\infty}^{-2} 2^{(k+1)q\alpha(0)} \|(G_N f) \chi_k\|_{L^{p(\cdot)}}^q \end{aligned}$$

$$\begin{aligned} &+ 2^{-L\lambda q} \sum_{k=-1}^L 2^{(k+1)q\alpha(\infty)} \|(G_N f) \chi_k\|_{L^{p(\cdot)}}^q \\ &\leq 2^{-L\lambda q} \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \|(G_N f) \chi_k\|_{L^{p(\cdot)}}^q \\ &\quad + 2^{-L\lambda q} \sum_{k=0}^L 2^{kq\alpha(\infty)} \|(G_N f) \chi_k\|_{L^{p(\cdot)}}^q \\ &\leq \|G_N f\|_{MK_{p(\cdot),\lambda}^{\alpha(\cdot),q}}^q. \end{aligned} \quad (31)$$

Hence,

$$\sup_{L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^L |\lambda_k|^q \leq \|G_N f\|_{MK_{p(\cdot),\lambda}^{\alpha(\cdot),q}}^p. \quad (32)$$

It remains to estimate $II_{(j)}$. Let $\{\psi_d^k : |\gamma| \leq m\}$ be the dual basis of $\{x^\beta : |\beta| \leq m\}$ with respect to the weight $1/|\bar{A}_{k,\varepsilon}|$ on $\bar{A}_{k,\varepsilon}$, that is,

$$\langle \psi_\gamma^k, x^\beta \rangle = \frac{1}{|\bar{A}_{k,\varepsilon}|} \int_{\bar{A}_{k,\varepsilon}} x^\beta \psi_\gamma^k(x) dx = \delta_{\beta\gamma}. \quad (33)$$

Similar to the method of [48], let

$$\begin{aligned} h_{k,\gamma}^{(j)}(x) &:= \sum_{l=-\infty}^k \left(\frac{\psi_\gamma^k(x) \chi_{\bar{A}_{k,\varepsilon}}(x)}{|\bar{A}_{k,\varepsilon}|} - \frac{\psi_\gamma^{k+1}(x) \chi_{\bar{A}_{k+1,\varepsilon}}(x)}{|\bar{A}_{k+1,\varepsilon}|} \right) \\ &\quad \times \int_{\mathbb{R}^n} f^{(j)}(y) \Phi_l(y) y^\gamma dy. \end{aligned} \quad (34)$$

We write

$$\begin{aligned} II_{(j)} &= \sum_{k=-\infty}^{\infty} \sum_{|\gamma| \leq m} \langle f^{(j)} \Phi_k, x^\gamma \rangle \psi_\gamma^k(x) \chi_{\bar{A}_{k,\varepsilon}}(x) \\ &= \sum_{|\gamma| \leq m} \sum_{k=-\infty}^{\infty} \left(\int_{\mathbb{R}^n} f^{(j)} \Phi_k x^\gamma dx \right) \frac{\psi_\gamma^k(x) \chi_{\bar{A}_{k,\varepsilon}}(x)}{|\bar{A}_{k,\varepsilon}|} \\ &= \sum_{|\gamma| \leq m} \sum_{k=-\infty}^{\infty} \left(\sum_{l=-\infty}^k \int_{\mathbb{R}^n} f^{(j)}(x) \Phi_l(x) x^\gamma dx \right) \\ &\quad \times \left(\frac{\psi_\gamma^k(x) \chi_{\bar{A}_{k,\varepsilon}}(x)}{|\bar{A}_{k,\varepsilon}|} - \frac{\psi_\gamma^{k+1}(x) \chi_{\bar{A}_{k+1,\varepsilon}}(x)}{|\bar{A}_{k+1,\varepsilon}|} \right) \\ &=: \sum_{|\gamma| \leq m} \sum_{k=-\infty}^{\infty} \frac{\alpha_{k,\gamma} h_{k,\gamma}^{(j)}(x)}{\alpha_{k,\gamma}} \\ &=: \sum_{|\gamma| \leq m} \sum_{k=-\infty}^{\infty} \alpha_{k,\gamma} a_{k,\gamma}^{(j)}(x), \end{aligned} \quad (35)$$

where

$$\alpha_{k,\gamma} := \tilde{b} \sum_{l=k-1}^{k+1} \|(G_N f) \chi_l\|_{L^{p(\cdot)}} |B_{k+2}|^{\alpha_{k+2}/n} \quad (36)$$

and $\tilde{b}$ is a constant which will be chosen later. Note that

$$\begin{aligned} \int_{\mathbb{R}^n} \sum_{l=-\infty}^k |\Phi_l(x) x^\gamma| dx &= \sum_{l=-\infty}^k \int_{\tilde{A}_{k,\varepsilon}} |\Phi_l(x) x^\gamma| dx \\ &\leq 2^{k(n+|\gamma|)}. \end{aligned} \quad (37)$$

By a computation we have

$$\begin{aligned} \left| \int_{\mathbb{R}^n} f^{(j)}(y) \sum_{l=-\infty}^k \Phi_l(y) y^\gamma dy \right| &\leq 2^{k(n+|\gamma|)} G_N f(x), \\ x &\in B_{k+2}. \end{aligned} \quad (38)$$

Since

$$\left| \frac{\psi_\gamma^k(x) \chi_{\tilde{A}_{k,\varepsilon}}(x)}{|\tilde{A}_{k,\varepsilon}|} - \frac{\psi_\gamma^{k+1}(x) \chi_{\tilde{A}_{k+1,\varepsilon}}(x)}{|\tilde{A}_{k+1,\varepsilon}|} \right| \leq 2^{-k(n+|\gamma|)} \sum_{l=k-1}^{k+1} \chi_l(x), \quad (39)$$

it follows that

$$\|h_{k,\gamma}^{(j)}\|_{L^{p(\cdot)}} \leq C \sum_{l=k-1}^{k+1} \|(G_N f) \chi_l\|_{L^{p(\cdot)}}. \quad (40)$$

Take $\tilde{b} = C$. It is easy to show that each $a_{k,\gamma}^{(j)}$ is a central $(\alpha(\cdot), p(\cdot))$ -atom with support contained in $\tilde{A}_{k,\varepsilon} \cup \tilde{A}_{k+1,\varepsilon} \subset B_{k+2}$, and

$$\alpha_{k,\gamma} = C \sum_{l=k-1}^{k+1} \|(G_N f) \chi_l\|_{L^{p(\cdot)}} |B_{k+2}|^{\alpha_{k+2}/n}, \quad (41)$$

where C is a constant independent of j, f, k , and γ . Moreover, we have

$$\begin{aligned} &\sup_{L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^L \sum_{|\gamma| \leq m} |\alpha_{k,\gamma}|^q \\ &\leq \sup_{L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^L |B_{k+2}|^{q\alpha_{k+2}/n} \left(\sum_{l=k-1}^{k+1} \|(G_N f) \chi_l\|_{L^{p(\cdot)}} \right)^q \\ &\leq \sup_{L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^L |B_{k+2}|^{q\alpha_{k+2}/n} \|(G_N f) \chi_k\|_{L^{p(\cdot)}}^q \\ &:= B. \end{aligned} \quad (42)$$

Using the same argument as before for A , we obtain

$$B \leq \|G_N f\|_{MK_{p(\cdot),\lambda}^{\alpha(\cdot),q}}^q. \quad (43)$$

Therefore,

$$\sup_{L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^L \left| \sum_{|\gamma| \leq m} \alpha_{k,\gamma} \right|^q \leq \|G_N f\|_{MK_{p(\cdot),\lambda}^{\alpha(\cdot),q}}^p. \quad (44)$$

Thus, we obtain that

$$f^{(j)}(x) = \sum_{d=-\infty}^{\infty} \lambda_d a_d^{(j)}(x), \quad (45)$$

where each $a_d^{(j)}$ is a central $(\alpha(\cdot), p(\cdot))$ -atom with support contained in $\tilde{A}_{d,\varepsilon} \cup \tilde{A}_{d+1,\varepsilon} \subset B_{d+2}$, λ_d is independent of j and

$$\sup_{L \in \mathbb{Z}} 2^{-L\lambda} \left(\sum_{d=-\infty}^L |\lambda_d|^q \right)^{1/q} \leq C \|G_N f\|_{MK_{p(\cdot),\lambda}^{\alpha(\cdot),q}} < \infty, \quad (46)$$

where C is independent of j and f .

Since

$$\sup_{j \in \mathbb{N}_0} \|a_0^{(j)}\|_{L^{p(\cdot)}} \leq |B_2|^{-\alpha_2/n}, \quad (47)$$

by the Banach-Alaoglu theorem we obtain a subsequence $\{a_0^{(j_{n_0})}\}$ of $\{a_0^{(j)}\}$ converging in the weak $*$ topology of $L^{p(\cdot)}$ to some $a_0 \in L^{p(\cdot)}$. It is easy to verify that a_0 is a central $(\alpha(\cdot), p(\cdot))$ -atom supported on B_2 . Next, since

$$\sup_{j_{n_0} \in \mathbb{N}_0} \|a_1^{(j_{n_0})}\|_{L^{p(\cdot)}} \leq |B_3|^{-\alpha_3/n}, \quad (48)$$

another application of the Banach-Alaoglu theorem yields a subsequence $\{a_1^{(j_{n_1})}\}$ of $\{a_1^{(j_{n_0})}\}$ which converges weak $*$ in $L^{p(\cdot)}$ to a central $(\alpha(\cdot), p(\cdot))$ -atom a_1 with support in B_3 . Furthermore,

$$\sup_{j_{n_1} \in \mathbb{N}_0} \|a_{-1}^{(j_{n_1})}\|_{L^{p(\cdot)}} \leq |B_1|^{-\alpha_1/n}. \quad (49)$$

Similarly, there exists a subsequence $\{a_{-1}^{(j_{n_1})}\}$ of $\{a_{-1}^{(j_{n_0})}\}$ which converges weak $*$ in $L^{p(\cdot)}$ to some $a_{-1} \in L^{p(\cdot)}$, and a_{-1} is a central $(\alpha(\cdot), p(\cdot))$ -atom supported on B_1 . Repeating the above procedure for each $d \in \mathbb{Z}$, we can find a subsequence $\{a_d^{(j_{n_d})}\}$ of $\{a_d^{(j)}\}$ converging weak $*$ in $L^{p(\cdot)}$ to some $a_d \in L^{p(\cdot)}$ which is a central $(\alpha(\cdot), p(\cdot))$ -atom supported on B_{d+2} . By using the diagonal method we obtain a subsequence $\{j_\nu\}$ of $\mathbb{N}_0$ such that, for each $d \in \mathbb{Z}$, $\lim_{\nu \rightarrow \infty} a_d^{(j_\nu)} = a_d$ in the weak $*$ topology of $L^{p(\cdot)}$ and therefore in $\mathcal{S}'(\mathbb{R}^n)$.

Now we only need to prove that $f = \sum_{d=-\infty}^{\infty} \lambda_d a_d$ in the sense of $\mathcal{S}'(\mathbb{R}^n)$. For each $\varphi \in \mathcal{S}(\mathbb{R}^n)$, note that $\text{supp } a_d^{(j_\nu)} \subset (\tilde{A}_{d,\varepsilon} \cup \tilde{A}_{d+1,\varepsilon}) \subset (A_{d-1} \cup A_d \cup A_{d+1} \cup A_{d+2})$. Using the same argument in [48], we have

$$\langle f, \varphi \rangle = \lim_{\nu \rightarrow \infty} \sum_{d=-\infty}^{\infty} \lambda_d \int_{\mathbb{R}^n} a_d^{(j_\nu)}(x) \varphi(x) dx. \quad (50)$$

Recall that $m = \max\{\alpha(0) - n\delta_2, [\alpha_\infty - n\delta_2]\}$. If $d \leq 0$, then by Lemmas 3 and 4 we have

$$\begin{aligned}
& \left| \int_{\mathbb{R}^n} a_d^{(j_\nu)}(x) \varphi(x) dx \right| \\
&= \left| \int_{\mathbb{R}^n} a_d^{(j_\nu)}(x) \left(\varphi(x) - \sum_{|\beta| \leq m} \frac{D^\beta \varphi(0)}{\beta!} x^\beta \right) dx \right| \\
&\leq \int_{\mathbb{R}^n} |a_d^{(j_\nu)}(x)| \cdot |x|^{m+1} dx \\
&\leq 2^{d(m+1)} \int_{\mathbb{R}^n} |a_d^{(j_\nu)}(x)| dx \\
&\leq 2^{d(m+1-\alpha_{d+2})} \|\chi_{B_{d+2}}\|_{L^{p'(\cdot)}} \\
&\leq 2^{d(m+1-\alpha_{d+2})} \left(\frac{|B_{d+2}|}{|B_2|} \right)^{\delta_2} \|\chi_{B_2}\|_{L^{p'(\cdot)}} \\
&\leq 2^{d(m+1-\alpha_{d+2}+n\delta_2)} \frac{|B_2|}{|B_0|} \|\chi_{B_0}\|_{L^{p'(\cdot)}} \\
&\leq 2^{d(m+1-\alpha_{d+2}+n\delta_2)}.
\end{aligned} \tag{51}$$

If $d > 0$, let $k_0 \in \mathbb{N}_0$ such that $\min\{k_0 + \alpha(0) - n, k_0 + \alpha_\infty - n\} > 0$; then by Lemmas 4 and 3 again we have

$$\begin{aligned}
& \left| \int_{\mathbb{R}^n} a_d^{(j_\nu)}(x) \varphi(x) dx \right| \leq \int_{\mathbb{R}^n} |a_d^{(j_\nu)}(x)| |x|^{-k_0} dx \\
&\leq 2^{-d(k_0+\alpha_{d+2})} \|\chi_{B_{d+2}}\|_{L^{p'(\cdot)}} \\
&\leq 2^{-d(k_0+\alpha_{d+2})} \frac{|B_{d+2}|}{|B_0|} \|\chi_{B_0}\|_{L^{p'(\cdot)}} \\
&\leq 2^{-d(k_0+\alpha_{d+2}-n)}.
\end{aligned} \tag{52}$$

Let

$$\mu_d = \begin{cases} |\lambda_d| 2^{d(m+1-\alpha_{d+2}+n\delta_2)}, & d \leq 0 \\ |\lambda_d| 2^{-d(k_0+\alpha_{d+2}-n)}, & d > 0. \end{cases} \tag{53}$$

Then

$$\begin{aligned}
& \sup_{L \in \mathbb{Z}} 2^{-L\lambda} \sum_{d=-\infty}^L |\mu_d| \leq \left(\sup_{L \in \mathbb{Z}} 2^{-L\lambda} \sum_{d=-\infty}^L |\lambda_d|^q \right)^{1/q} \\
&\leq \|G_N f\|_{\dot{MK}_{p(\cdot),\lambda}^{\alpha(\cdot),q}} < \infty,
\end{aligned} \tag{54}$$

$$|\lambda_d| \left| \int_{\mathbb{R}^n} a_d^{(j_\nu)}(x) \varphi(x) dx \right| \leq |\mu_d|,$$

which implies that

$$\begin{aligned}
\langle f, \varphi \rangle &= \sum_{d=-\infty}^{\infty} \lim_{\nu \rightarrow \infty} \lambda_d \int_{\mathbb{R}^n} a_d^{(j_\nu)}(x) \varphi(x) dx \\
&= \sum_{d=-\infty}^{\infty} \lambda_d \int_{\mathbb{R}^n} a_d(x) \varphi(x) dx.
\end{aligned} \tag{55}$$

This establishes the identity we wanted.

To prove the sufficiency, for convenience, we denote $\sup_{L \in \mathbb{Z}} 2^{-L\lambda} \sum_{k=-\infty}^L |\lambda_k|^q = \Lambda$. Firstly we have

$$\begin{aligned}
& \|G_N f\|_{\dot{MK}_{p(\cdot),\lambda}^{\alpha(\cdot),q}}^q \\
&\approx \max \left\{ \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda} \sum_{k=-\infty}^L 2^{kq\alpha(0)} \|(G_N f) \chi_k\|_{L^{p(\cdot)}}^q, \right. \\
&\quad \left. \sup_{L > 0, L \in \mathbb{Z}} 2^{-L\lambda} \left(\sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \|(G_N f) \chi_k\|_{L^{p(\cdot)}}^q \right. \right. \\
&\quad \left. \left. + \sum_{k=0}^L 2^{kq\alpha(\infty)} \|(G_N f) \chi_k\|_{L^{p(\cdot)}}^q \right) \right\} \\
&\leq \max \{I, II + III\},
\end{aligned} \tag{56}$$

where

$$\begin{aligned}
I &:= \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda} \sum_{k=-\infty}^L 2^{kq\alpha(0)} \|(G_N f) \chi_k\|_{L^{p(\cdot)}}^q \\
II &:= \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \|(G_N f) \chi_k\|_{L^{p(\cdot)}}^q \\
III &:= \sup_{L > 0, L \in \mathbb{Z}} 2^{-L\lambda} \sum_{k=0}^L 2^{kq\alpha(\infty)} \|(G_N f) \chi_k\|_{L^{p(\cdot)}}^q.
\end{aligned} \tag{57}$$

Now we have

$$\begin{aligned}
I &= \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda} \sum_{k=-\infty}^L 2^{kq\alpha(0)} \|(G_N f) \chi_k\|_{L^{p(\cdot)}}^q \\
&\leq \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda} \sum_{k=-\infty}^L 2^{kq\alpha(0)} \left(\sum_{l=k}^{\infty} |\lambda_l| \|a_l\|_{L^{p(\cdot)}} \right)^q \\
&\quad + \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda} \sum_{k=-\infty}^L 2^{kq\alpha(0)} \left(\sum_{l=-\infty}^{k-1} |\lambda_l| \|(G_N a_l) \chi_k\|_{L^{p(\cdot)}} \right)^q \\
&:= I_1 + I_2.
\end{aligned}$$

$$\begin{aligned}
II &= \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \|(G_N f) \chi_k\|_{L^{p(\cdot)}}^q \\
&\leq \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \left(\sum_{l=k}^{\infty} |\lambda_l| \|a_l\|_{L^{p(\cdot)}} \right)^q \\
&\quad + \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \left(\sum_{l=-\infty}^{k-1} |\lambda_l| \|(G_N a_l) \chi_k\|_{L^{p(\cdot)}} \right)^q \\
&:= II_1 + II_2
\end{aligned}$$

$$\begin{aligned}
III &= \sup_{L > 0, L \in \mathbb{Z}} 2^{-L\lambda} \sum_{k=0}^L 2^{kq\alpha(\infty)} \|(G_N f) \chi_k\|_{L^{p(\cdot)}}^q \\
&\leq \sup_{L > 0, L \in \mathbb{Z}} 2^{-L\lambda} \sum_{k=0}^L 2^{kq\alpha(\infty)} \left(\sum_{l=k}^L |\lambda_l| \|a_l\|_{L^{p(\cdot)}} \right)^q
\end{aligned}$$

$$\begin{aligned}
& + \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^L 2^{kq\alpha_\infty} \left(\sum_{l=-\infty}^{k-1} |\lambda_l| \| (G_N a_l) \chi_k \|_{L^{p(\cdot)}} \right)^q \\
& := III_1 + III_2.
\end{aligned} \tag{58}$$

To estimate I , II , and III we need a pointwise estimate for $G_N a_l(x)$ on D_k , where $k \geq l+2$. Let $\phi \in \mathcal{A}_N$, $m \in \mathbb{N}$ such that $\alpha_k - n\delta_2 < m+1$. Denote by P_m the m th order Taylor series expansion of ϕ at y/t . If $|x-y| < t$, then from the vanishing moment condition of a_l we have

$$\begin{aligned}
& |a_l * \phi_t(y)| \\
& = t^{-n} \left| \int_{\mathbb{R}^n} a_l(z) \left(\phi\left(\frac{y-z}{t}\right) - P_m\left(-\frac{z}{t}\right) \right) dz \right| \\
& \leq t^{-n} \int_{\mathbb{R}^n} |a_l(z)| \left| \frac{z}{t} \right|^{m+1} \left(1 + \left| \frac{(y-\theta z)}{t} \right| \right)^{-(n+m+1)} dz \\
& \leq \int_{\mathbb{R}^n} |a_l(z)| |z|^{m+1} (t + |y-\theta z|)^{-(n+m+1)} dz,
\end{aligned} \tag{59}$$

where $0 < \theta < 1$. Since $x \in D_k$ for $k \in \mathbb{Z}$, we have $|x| \geq 2^{k-1}$. From $|x-y| < t$ and $|z| < 2^l$, we have

$$t + |y - \theta z| \geq |x - y| + |y - \theta z| \geq |x| - |z| \geq \frac{|x|}{2}. \tag{60}$$

Thus,

$$\begin{aligned}
& |a_l * \phi_t(y)| \\
& \leq \int_{\mathbb{R}^n} |a_l(z)| |z|^{m+1} (|x-y| + |y-\theta z|)^{-(n+m+1)} dz \\
& \leq 2^{l(m+1)} |x|^{-(n+m+1)} \int_{\mathbb{R}^n} |a_l(z)| dz \\
& \leq 2^{l(m+1)} 2^{-k(n+m+1)} |B_l|^{-\alpha_l/n} \|\chi_{B_l}\|_{L^{p'(\cdot)}}.
\end{aligned} \tag{61}$$

Therefore, we have

$$\begin{aligned}
G_N a_l(x) & \leq 2^{l(m+1)} 2^{-k(n+m+1)} |B_l|^{-\alpha_l/n} \|\chi_{B_l}\|_{L^{p'(\cdot)}}, \\
& x \in D_k, \quad k \geq l+2.
\end{aligned} \tag{62}$$

To proceed, we consider them into two cases $0 < q \leq 1$ and $1 < q < \infty$.

If $0 < q \leq 1$,

$$\begin{aligned}
I_1 & = \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^L 2^{\alpha(0)kq} \left(\sum_{l=k}^{\infty} |\lambda_l| \| (Tb_l) \chi_k \|_{L^{p(\cdot)}} \right)^q \\
& \leq \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^L 2^{\alpha(0)kq} \left(\sum_{l=k}^{\infty} |\lambda_l| 2^{-\alpha_l l} \right)^q \\
& \leq \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^L 2^{\alpha(0)kq} \\
& \quad \times \left(\sum_{l=k}^{-1} |\lambda_l|^q 2^{-\alpha(0)lq} + \sum_{l=0}^{\infty} |\lambda_l|^q 2^{-\alpha_\infty lq} \right)
\end{aligned}$$

$$\begin{aligned}
& \leq \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^L \sum_{l=k}^{-1} |\lambda_l|^q 2^{\alpha(0)(k-l)q} \\
& \quad + \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^L 2^{\alpha(0)kq} \sum_{l=0}^{\infty} |\lambda_l|^q 2^{-\alpha_\infty lq} \\
& \leq \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{l=-\infty}^{-1} |\lambda_l|^q \sum_{k=-\infty}^l 2^{\alpha(0)(k-l)q} \\
& \quad + \sup_{L \leq 0, L \in \mathbb{Z}} \sum_{l=0}^{\infty} 2^{-l\lambda q} |\lambda_l|^q 2^{(\lambda-\alpha_\infty)lq} 2^{-L\lambda q} \sum_{k=-\infty}^L 2^{\alpha(0)kq} \\
& \leq \Lambda + \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{l=L}^{-1} |\lambda_l|^q \sum_{k=-\infty}^l 2^{\alpha(0)(k-l)q} \\
& \quad + \Lambda \sup_{L \leq 0, L \in \mathbb{Z}} \sum_{l=0}^{\infty} 2^{(\lambda-\alpha_\infty)lq} \sum_{k=-\infty}^L 2^{\alpha(0)kq-L\lambda q} \\
& \leq \Lambda + \sup_{L \leq 0, L \in \mathbb{Z}} \sum_{l=L}^{-1} 2^{-l\lambda q} |\lambda_l|^q 2^{(l-L)\lambda q} \sum_{k=-\infty}^l 2^{\alpha(0)(k-l)q} + \Lambda \\
& \leq \Lambda + \Lambda \sup_{L \leq 0, L \in \mathbb{Z}} \sum_{l=L}^{-1} 2^{(l-L)\lambda q} \sum_{k=-\infty}^l 2^{\alpha(0)(k-l)q} \\
& \leq \Lambda. \\
I_2 & = \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^L 2^{kq\alpha(0)} \left(\sum_{l=-\infty}^{k-1} |\lambda_l| \| (G_N a_l) \chi_k \|_{L^{p(\cdot)}} \right)^q \\
& \leq \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \\
& \quad \times \sum_{k=-\infty}^L 2^{kq\alpha(0)} \left(\sum_{l=-\infty}^{k-1} |\lambda_l| 2^{l(m+1)-k(n+m+1)} \right. \\
& \quad \left. \times |B_l|^{-\alpha_l/n} \|\chi_{B_l}\|_{L^{p'(\cdot)}} \|\chi_{B_k}\|_{L^{p(\cdot)}} \right)^q \\
& \leq \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^L \left(\sum_{l=-\infty}^{k-1} |\lambda_l| 2^{(l-k)(m+1+n\delta_2)-(l-k)\alpha(0)} \right)^q \\
& \leq \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^L \left(\sum_{l=-\infty}^{k-1} |\lambda_l| 2^{(l-k)(m+1+n\delta_2-\alpha(0))} \right)^q \\
& = \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{l=-\infty}^{L-1} \sum_{k=l+1}^L |\lambda_l|^q 2^{(l-k)(m+1+n\delta_2-\alpha(0))q} \\
& \leq \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{l=-\infty}^L |\lambda_l|^q \\
& \leq \Lambda.
\end{aligned} \tag{63}$$

Then we turn to estimate *II*:

$$\begin{aligned}
II_1 &= \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \left(\sum_{l=k}^{\infty} |\lambda_l| \|a_l\|_{L^{p(\cdot)}} \right)^q \\
&\leq \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \left(\sum_{l=k}^{\infty} |\lambda_l| |B_l|^{-\alpha_l/n} \right)^q \\
&\leq \sum_{k=-\infty}^{-1} 2^{\alpha(0)kq} \left(\sum_{l=k}^{-1} |\lambda_l|^q 2^{-\alpha(0)lq} + \sum_{l=0}^{\infty} |\lambda_l|^q 2^{-\alpha_{\infty}lq} \right) \\
&\leq \sum_{k=-\infty}^{-1} \sum_{l=k}^{-1} |\lambda_l|^q 2^{\alpha(0)(k-l)q} \\
&\quad + \sum_{k=-\infty}^{-1} 2^{\alpha(0)kq} \sum_{l=0}^{\infty} |\lambda_l|^q 2^{-\alpha_{\infty}lq} \\
&\leq \sum_{l=-\infty}^{-1} |\lambda_l|^q \sum_{k=-\infty}^l 2^{\alpha(0)(k-l)q} \\
&\quad + \sum_{l=0}^{\infty} |\lambda_l|^q 2^{-\alpha_{\infty}lq} \sum_{k=-\infty}^{-1} 2^{\alpha(0)kq} \\
&\leq \sum_{l=-\infty}^{-1} |\lambda_l|^q + \sum_{l=0}^{\infty} 2^{-l\lambda q} |\lambda_l|^q 2^{-\alpha_{\infty}lq} \sum_{k=-\infty}^{-1} 2^{\alpha(0)kq} \\
&\leq \Lambda + \Lambda \sum_{i=-\infty}^l |\lambda_i|^q \sum_{l=0}^{\infty} 2^{(\lambda-\alpha_{\infty})lq} \sum_{k=-\infty}^l 2^{\alpha(0)kq} \\
&\leq \Lambda. \\
II_2 &= \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \left(\sum_{l=-\infty}^{k-1} |\lambda_l| \|(G_N a_l) \chi_k\|_{L^{p(\cdot)}} \right)^q \\
&\leq \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \left(\sum_{l=-\infty}^{k-1} |\lambda_l| 2^{l(m+1)-k(n+m+1)} \right. \\
&\quad \left. \times |B_l|^{-\alpha_l/n} \|\chi_{B_l}\|_{L^{p'(\cdot)}} \|\chi_{B_k}\|_{L^{p(\cdot)}} \right)^q \\
&\leq \sum_{k=-\infty}^{-1} \left(\sum_{l=-\infty}^{k-1} |\lambda_l| 2^{(l-k)(m+1+n\delta_2)-(l-k)\alpha(0)} \right)^q \\
&\leq \sum_{k=-\infty}^{-1} \left(\sum_{l=-\infty}^{k-1} |\lambda_l| 2^{(l-k)(m+1+n\delta_2-\alpha(0))} \right)^q \\
&\leq \sum_{k=-\infty}^{-1} \left(\sum_{l=-\infty}^{k-1} |\lambda_l|^q 2^{(l-k)(m+1+n\delta_2-\alpha(0))q} \right) \\
&= \sum_{l=-\infty}^{-2} \sum_{k=l+1}^{-1} |\lambda_l|^q 2^{(l-k)(m+1+n\delta_2-\alpha(0))q} \\
&\leq \sum_{l=-\infty}^{-1} |\lambda_l|^q \leq \Lambda.
\end{aligned} \tag{64}$$

Third, we estimate *III*:

$$\begin{aligned}
III_1 &= \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^L 2^{kq\alpha_{\infty}} |B_k|^{\alpha_{\infty}q/n} \left(\sum_{l=k}^{\infty} |\lambda_l| \|a_l\|_{L^{p(\cdot)}} \right)^q \\
&\leq \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^L 2^{kq\alpha_{\infty}} |B_k|^{\alpha_{\infty}q/n} \left(\sum_{l=k}^{\infty} |\lambda_l| |B_l|^{-\alpha_l/n} \right)^q \\
&\leq \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^L \sum_{l=k}^{\infty} |B_k|^{\alpha_{\infty}q/n} |\lambda_l|^q |B_l|^{-\alpha_{\infty}q/n} \\
&\leq \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^L \sum_{l=k}^{\infty} |\lambda_l|^q 2^{(k-l)\alpha_{\infty}q} \\
&= \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \\
&\quad \times \left[\sum_{l=0}^L |\lambda_l|^q \sum_{k=0}^l 2^{(k-l)\alpha_{\infty}q} + \sum_{l=L}^{\infty} |\lambda_l|^q \sum_{k=0}^L 2^{(k-l)\alpha_{\infty}q} \right] \\
&\leq \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{l=0}^L |\lambda_l|^q \\
&\quad + \sup_{L>0, L \in \mathbb{Z}} \sum_{l=L}^{\infty} 2^{(l\lambda q - L\lambda q)} 2^{-l\lambda q} \sum_{i=-\infty}^l |\lambda_i|^q \sum_{k=0}^L 2^{(k-l)\alpha_{\infty}q/2} \\
&\leq \Lambda + \Lambda \sup_{L>0, L \in \mathbb{Z}} \sum_{l=L}^{\infty} 2^{(l-L)\lambda q} 2^{(L-l)\alpha_{\infty}q/2} \\
&\leq \Lambda + \Lambda \sup_{L>0, L \in \mathbb{Z}} \sum_{l=L}^{\infty} 2^{(l-L)q(\lambda-\alpha_{\infty}/2)} \\
&\leq \Lambda. \\
III_2 &= \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^L 2^{kq\alpha_{\infty}} \left(\sum_{l=-\infty}^{k-1} |\lambda_l| \|(G_N a_l) \chi_k\|_{L^{p(\cdot)}} \right)^q \\
&\leq \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \\
&\quad \times \sum_{k=0}^L 2^{kq\alpha_{\infty}} \left(\sum_{l=-\infty}^{k-1} |\lambda_l| 2^{l(m+1)-k(n+m+1)} \right. \\
&\quad \left. \times |B_l|^{-\alpha_l/n} \|\chi_{B_l}\|_{L^{p'(\cdot)}} \|\chi_{B_k}\|_{L^{p(\cdot)}} \right)^q \\
&\leq \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^L \left(\sum_{l=-\infty}^{k-1} |\lambda_l| 2^{(l-k)(m+1+n\delta_2)-l\alpha_l+k\alpha_{\infty}} \right)^q \\
&\leq \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^L \left(\sum_{l=-\infty}^{-1} |\lambda_l| 2^{(l-k)(m+1+n\delta_2)-l\alpha(0)+k\alpha_{\infty}} \right)^q \\
&\quad + \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^L \left(\sum_{l=0}^{k-1} |\lambda_l| 2^{(l-k)(m+1+n\delta_2-\alpha_{\infty})} \right)^q
\end{aligned}$$

$$\begin{aligned}
&\leq \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^L 2^{qk[\alpha_{\infty} - (m+1+n\delta_2)]} \\
&\quad \times \left(\sum_{l=-\infty}^{-1} |\lambda_l| 2^{l(m+1+n\delta_2-\alpha(0))} \right)^q \\
&\quad + \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^L \sum_{l=0}^{k-1} |\lambda_l|^q 2^{(l-k)(m+1+n\delta_2-\alpha_{\infty})q} \\
&\leq \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{l=-\infty}^{-1} |\lambda_l|^q 2^{l(m+1+n\delta_2-\alpha(0))q} \\
&\quad + \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{l=0}^{L-1} \sum_{k=l+1}^L |\lambda_l|^q 2^{(l-k)(m+1+n\delta_2-\alpha_{\infty})q} \\
&\leq \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{l=-\infty}^{-1} |\lambda_l|^q + \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{l=0}^L |\lambda_l|^q \\
&= \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{l=-\infty}^L |\lambda_l|^q.
\end{aligned} \tag{66}$$

If $1 < q < \infty$, we have

$$\begin{aligned}
I_1 &= \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^L 2^{kq\alpha(0)} \left(\sum_{l=k}^{\infty} |\lambda_l| \|a_l\|_{L^{p(\cdot)}} \right)^q \\
&\leq \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^L 2^{kq\alpha(0)} \left(\sum_{l=k}^{\infty} |\lambda_l| |B_l|^{-\alpha_l/n} \right)^q \\
&\leq \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^L \left(\sum_{l=k}^{-1} |\lambda_l| 2^{\alpha(0)(k-l)} \right)^q \\
&\quad + \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^L 2^{\alpha(0)kq} \left(\sum_{l=0}^{\infty} |\lambda_l| 2^{-\alpha_{\infty}l} \right)^q \\
&\leq \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^L \left(\sum_{l=k}^{-1} |\lambda_l|^q 2^{\alpha(0)(k-l)q/2} \right) \\
&\quad \times \left(\sum_{l=k}^{-1} 2^{\alpha(0)(k-l)q'/2} \right)^{q/q'} \\
&\quad + \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^L 2^{\alpha(0)kq} \left(\sum_{l=0}^{\infty} |\lambda_l|^q 2^{-\alpha_{\infty}jq/2} \right) \\
&\quad \times \left(\sum_{l=0}^{\infty} 2^{-\alpha_{\infty}jq'/2} \right)^{q/q'} \\
&\leq \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^L \sum_{l=k}^{-1} |\lambda_l|^q 2^{\alpha(0)(k-l)q/2} \\
&\quad + \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^L 2^{\alpha(0)kq} \sum_{l=0}^{\infty} |\lambda_l|^q 2^{-\alpha_{\infty}jq/2}
\end{aligned}$$

$$\begin{aligned}
&\leq \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{l=-\infty}^{-1} |\lambda_l|^q \sum_{k=-\infty}^l 2^{\alpha(0)(k-l)q/2} \\
&\quad + \sup_{L \leq 0, L \in \mathbb{Z}} \sum_{l=0}^{\infty} 2^{-l\lambda q} |\lambda_l|^q 2^{(\lambda-\alpha_{\infty}/2)lq} 2^{-L\lambda q} \sum_{k=-\infty}^L 2^{\alpha(0)kq} \\
&\leq \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{l=-\infty}^L |\lambda_l|^q \\
&\quad + \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{l=L}^{-1} |\lambda_l|^q \sum_{k=-\infty}^l 2^{\alpha(0)(k-l)q/2} \\
&\quad + \Lambda \sup_{L \leq 0, L \in \mathbb{Z}} \sum_{l=0}^{\infty} 2^{(\lambda-\alpha_{\infty}/2)lq} \sum_{k=-\infty}^L 2^{\alpha(0)kq-L\lambda q} \\
&\leq \Lambda + \sup_{L \leq 0, L \in \mathbb{Z}} \sum_{l=L}^{-1} 2^{-l\lambda q} |\lambda_l|^q 2^{(l-L)\lambda q} \\
&\quad \times \sum_{k=-\infty}^l 2^{\alpha(0)(k-l)q/2} + \Lambda \\
&\leq \Lambda + \Lambda \sup_{L \leq 0, L \in \mathbb{Z}} \sum_{l=L}^{-1} 2^{(l-L)\lambda q} \sum_{k=-\infty}^j 2^{\alpha(0)(k-l)q/2} \\
&\leq \Lambda. \\
I_2 &= \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^L 2^{kq\alpha(0)} \left(\sum_{l=-\infty}^{k-1} |\lambda_l| \|(G_N a_l) \chi_k\|_{L^{p(\cdot)}} \right)^q \\
&\leq \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \\
&\quad \times \sum_{k=-\infty}^L 2^{kq\alpha(0)} \left(\sum_{l=-\infty}^{k-1} |\lambda_l| 2^{l(m+1)-k(n+m+1)} \right. \\
&\quad \left. \times |B_l|^{-\alpha_l/n} \|\chi_{B_l}\|_{L^{p'(\cdot)}} \|\chi_{B_k}\|_{L^{p(\cdot)}} \right)^q \\
&\leq \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^L \left(\sum_{l=-\infty}^{k-1} |\lambda_l| 2^{(l-k)(m+1+n\delta_2)-(l-k)\alpha(0)} \right)^q \\
&\leq \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^L \left(\sum_{l=-\infty}^{k-1} |\lambda_l| 2^{(l-k)(m+1+n\delta_2-\alpha(0))} \right)^q \\
&\leq \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^L \left(\sum_{l=-\infty}^{k-1} |\lambda_l|^q 2^{(l-k)(m+1+n\delta_2-\alpha(0))q/2} \right) \\
&\quad \times \left(\sum_{l=-\infty}^{k-1} 2^{(l-k)(m+1+n\delta_2-\alpha(0))q'/2} \right)^{q/q'} \\
&\leq \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^L \left(\sum_{l=-\infty}^{k-1} |\lambda_l|^q 2^{(l-k)(m+1+n\delta_2-\alpha(0))q/2} \right) \\
&= \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{l=-\infty}^{L-1} \sum_{k=l+1}^L |\lambda_l|^q 2^{(l-k)(m+1+n\delta_2-\alpha(0))q/2} \\
&\leq \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{l=-\infty}^L |\lambda_l|^q \\
&\leq \Lambda.
\end{aligned} \tag{67}$$

Second, we estimate II . As the same argument before, we obtain that

$$\begin{aligned}
II_1 &= \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \left(\sum_{l=k}^{\infty} |\lambda_l| \|a_l\|_{L^{p(\cdot)}} \right)^q \\
&\leq \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \left(\sum_{l=k}^{\infty} |\lambda_l| |B_l|^{-\alpha_l/n} \right)^q \\
&\leq \sum_{k=-\infty}^{-1} \left(\sum_{j=k}^{-1} |\lambda_j| 2^{\alpha(0)(k-j)} \right)^q \\
&\quad + \sum_{k=-\infty}^{-1} 2^{\alpha(0)kq} \left(\sum_{j=0}^{\infty} |\lambda_j| 2^{-\alpha_{\infty}j} \right)^q \\
&\leq \sum_{k=-\infty}^{-1} \left(\sum_{l=k}^{-1} |\lambda_l|^q 2^{\alpha(0)(k-l)q/2} \right) \left(\sum_{l=k}^{-1} 2^{\alpha(0)(k-l)q'/2} \right)^{q/q'} \\
&\quad + \sum_{k=-\infty}^{-1} 2^{\alpha(0)kq} \left(\sum_{l=0}^{\infty} |\lambda_l|^q 2^{-\alpha_{\infty}lq/2} \right) \left(\sum_{l=0}^{\infty} 2^{-\alpha_{\infty}lq'/2} \right)^{q/q'} \\
&\leq \sum_{k=-\infty}^{-1} |\lambda_l|^q \sum_{k=-\infty}^l 2^{\alpha(0)(k-l)q/2} \\
&\quad + \sum_{l=0}^{\infty} |\lambda_l|^q 2^{-\alpha_{\infty}lq/2} \sum_{k=-\infty}^{-1} 2^{\alpha(0)kq} \\
&\leq \sum_{l=-\infty}^{-1} |\lambda_l|^q \\
&\quad + \sum_{l=0}^{\infty} 2^{(\lambda-\alpha_{\infty}/2)lq} 2^{-l\lambda q} \sum_{i=-\infty}^l |\lambda_i|^q \sum_{k=-\infty}^{-1} 2^{\alpha(0)kq} \\
&\leq \Lambda + \Lambda \sum_{l=0}^{\infty} 2^{(\lambda-\alpha_{\infty}/2)lq} \sum_{k=-\infty}^{-1} 2^{\alpha(0)kq} \\
&\leq \Lambda. \\
II_2 &= \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \left(\sum_{l=-\infty}^{k-1} |\lambda_l| \|(G_N a_l) \chi_k\|_{L^{p(\cdot)}} \right)^q \\
&\leq \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \left(\sum_{l=-\infty}^{k-1} |\lambda_l| 2^{l(m+1)-(n+m+1)k} \right. \\
&\quad \left. \times |B_l|^{-\alpha_l/n} \|\chi_{B_l}\|_{L^{p'(\cdot)}} \|\chi_{B_k}\|_{L^{p(\cdot)}} \right)^q \\
&\leq \sum_{k=-\infty}^{-1} \left(\sum_{l=-\infty}^{k-1} |\lambda_l| 2^{(l-k)(m+1+n\delta_2)-(l-k)\alpha(0)} \right)^q \\
&\leq \sum_{k=-\infty}^{-1} \left(\sum_{l=-\infty}^{k-1} |\lambda_l| 2^{(l-k)(m+1+n\delta_2-\alpha(0))} \right)^q
\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{k=-\infty}^{-1} \left(\sum_{l=-\infty}^{k-1} |\lambda_l|^q 2^{(l-k)(m+1+n\delta_2-\alpha(0))q/2} \right) \\
&\quad \times \left(\sum_{l=-\infty}^{k-1} 2^{(l-k)(m+1+n\delta_2-\alpha(0))q'/2} \right)^{q/q'} \\
&\leq \sum_{k=-\infty}^{-1} \left(\sum_{l=-\infty}^{k-1} |\lambda_l|^q 2^{(l-k)(m+1+n\delta_2-\alpha(0))q/2} \right) \\
&= \sum_{l=-\infty}^{-2} \sum_{k=l+1}^{-1} |\lambda_l|^q 2^{(l-k)(m+1+n\delta_2-\alpha(0))q/2} \\
&\leq \sum_{l=-\infty}^{-1} |\lambda_l|^q \\
&\leq \Lambda. \tag{68}
\end{aligned}$$

Third, we estimate III . We have

$$\begin{aligned}
III_1 &= \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^L 2^{kq\alpha_{\infty}} |B_k|^{\alpha_{\infty}q/n} \left(\sum_{l=k}^{\infty} |\lambda_l| \|a_l\|_{L^{p(\cdot)}} \right)^q \\
&\leq \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^L 2^{kq\alpha_{\infty}} |B_k|^{\alpha_{\infty}q/n} \left(\sum_{l=k}^{\infty} |\lambda_l| |B_l|^{-\alpha_l/n} \right)^q \\
&\leq \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^L 2^{kq\alpha_{\infty}} |B_k|^{\alpha_{\infty}q/n} \left(\sum_{l=k}^{\infty} |\lambda_l|^q |B_l|^{-\alpha_l q/(2n)} \right) \\
&\quad \times \left(\sum_{l=k}^{\infty} |B_l|^{-\alpha_l q'/(2n)} \right)^{q/q'} \\
&\leq \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^L \sum_{l=k}^{\infty} |B_k|^{\alpha_{\infty}q/(2n)} |\lambda_l|^q |B_l|^{-\alpha_l q/(2n)} \\
&\leq \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^L \sum_{l=k}^{\infty} |\lambda_l|^q 2^{(k-l)\alpha_{\infty}q/2} \\
&= \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \left[\sum_{l=0}^L |\lambda_l|^q \sum_{k=0}^l 2^{(k-l)\alpha_{\infty}q/2} \right. \\
&\quad \left. + \sum_{l=L}^{\infty} |\lambda_l|^q \sum_{k=0}^L 2^{(k-l)\alpha_{\infty}q/2} \right] \\
&\leq \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{l=0}^L |\lambda_l|^q \\
&\quad + \sup_{L>0, L \in \mathbb{Z}} \sum_{l=L}^{\infty} 2^{(j\lambda q - L\lambda q)} 2^{-l\lambda q} \sum_{i=-\infty}^l |\lambda_i|^q \sum_{k=0}^L 2^{(k-l)\alpha_{\infty}q/2} \\
&\leq \Lambda + \Lambda \sup_{L>0, L \in \mathbb{Z}} \sum_{l=L}^{\infty} 2^{(l-L)\lambda q} 2^{(L-l)\alpha_{\infty}q/2} \\
&\leq \Lambda + \Lambda \sup_{L>0, L \in \mathbb{Z}} \sum_{l=L}^{\infty} 2^{(l-L)q(\lambda-\alpha_{\infty}/2)} \\
&\leq \Lambda.
\end{aligned}$$

$$\begin{aligned}
III_2 &= \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^L 2^{kq\alpha_\infty} \left(\sum_{l=-\infty}^{k-1} |\lambda_l| \| (G_N a_l) \chi_k \|_{L^{p(\cdot)}} \right)^q \\
&\leq \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \\
&\quad \times \sum_{k=0}^L 2^{kq\alpha_\infty} \left(\sum_{l=-\infty}^{k-1} |\lambda_l| 2^{l(m+1-k)(n+m+1)} \right. \\
&\quad \left. \times |B_l|^{-\alpha_l/n} \| \chi_{B_l} \|_{L^{p'(\cdot)}} \| \chi_{B_k} \|_{L^{p(\cdot)}} \right)^q \\
&\leq \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^L \left(\sum_{l=-\infty}^{k-1} |\lambda_l| 2^{(l-k)(m+1+n\delta_2)-l\alpha_l+k\alpha_\infty} \right)^q \\
&\leq \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^L \left(\sum_{l=-\infty}^{-1} |\lambda_l| 2^{(l-k)(m+1+n\delta_2)-l\alpha(0)+k\alpha_\infty} \right)^q \\
&\quad + \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^L \left(\sum_{l=0}^{k-1} |\lambda_l| 2^{(l-k)(m+1+n\delta_2-\alpha_\infty)} \right)^q \\
&\leq \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^L 2^{qk[\alpha_\infty-(m+1+n\delta_2)]} \\
&\quad \times \left(\sum_{l=-\infty}^{-1} |\lambda_l| 2^{l(m+1+n\delta_2-\alpha(0))} \right)^q \\
&\quad + \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^L \left(\sum_{l=0}^{k-1} |\lambda_l|^q 2^{(l-k)(m+1+n\delta_2-\alpha_\infty)q/2} \right) \\
&\quad \times \left(\sum_{l=0}^{k-1} 2^{(l-k)(m+1+n\delta_2-\alpha_\infty)q'/2} \right)^{q/q'} \\
&\leq \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \left(\sum_{l=-\infty}^{-1} |\lambda_l|^q 2^{l(m+1+n\delta_2-\alpha(0))q/2} \right) \\
&\quad \times \left(\sum_{l=-\infty}^{-1} 2^{l(m+1+n\delta_2-\alpha(0))q'/2} \right)^{q/q'} \\
&\quad + \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^L \sum_{l=0}^{k-1} |\lambda_l|^q 2^{(l-k)(m+1+n\delta_2-\alpha_\infty)q/2} \\
&\leq \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{l=-\infty}^{-1} |\lambda_l|^q 2^{l(m+1+n\delta_2-\alpha(0))q/2} \\
&\quad + \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{l=0}^{L-1} \sum_{k=l+1}^L |\lambda_l|^q 2^{(l-k)(m+1+n\delta_2-\alpha_\infty)q/2} \\
&\leq \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{l=-\infty}^{-1} |\lambda_l|^q + \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{l=0}^L |\lambda_l|^q \\
&= \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{l=-\infty}^L |\lambda_l|^q \\
&\leq \Lambda.
\end{aligned} \tag{69}$$

Thus, we finish the proof of Theorem 13. $\square$

3. Applications

As an application of the atomic decompositions, we will prove the following result.

Theorem 14. Let $0 < q < \infty$, $0 \leq \lambda < \infty$, $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, and $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$ be log-Hölder continuous both at the origin and infinity, $2\lambda \leq \alpha(\cdot)$, $n\delta_2 \leq \alpha(0)$, $\alpha_\infty < \infty$, nonnegative integer and $s = \max\{[\alpha(0) - n\delta_2], [\alpha_\infty - n\delta_2]\}$, and δ_2 as in Lemma 4. If a sublinear operator T satisfies that

- (i) T is bounded on $L^{p(\cdot)}$;
- (ii) there exists a constant $\delta > 0$ such that $s + \delta > \max\{\alpha(0) - n\delta_2, \alpha_\infty - n\delta_2\}$, and for any compact support function f with

$$\int_{\mathbb{R}^n} f(x) x^\beta dx = 0, \quad |\beta| \leq s, \tag{70}$$

Tf satisfies the size condition

$$|Tf(x)| \leq C (\text{diam}(\text{supp } f))^{s+\delta} |x|^{-(n+s+\delta)} \|f\|_1, \tag{71}$$

when $\text{dist}(x, \text{supp } f) \geq \frac{|x|}{2}$.

Then there exists a constant C such that

$$\begin{aligned}
\|Tf\|_{\dot{MK}_{p(\cdot),\lambda}^{\alpha(\cdot),q}} &\leq C \|f\|_{\dot{HMK}_{p(\cdot),\lambda}^{\alpha(\cdot),q}}, \\
\|Tf\|_{\dot{MK}_{p(\cdot),\lambda}^{\alpha(\cdot),q}} &\leq C \|f\|_{\dot{HMK}_{p(\cdot),\lambda}^{\alpha(\cdot),q}}
\end{aligned} \tag{72}$$

for $f \in \dot{HMK}_{p(\cdot),\lambda}^{\alpha(\cdot),q}$ and $f \in \dot{HMK}_{p(\cdot),\lambda}^{\alpha(\cdot),q}$, respectively.

Proof. It suffices to prove the homogeneous case. Suppose $f \in \dot{HMK}_{p(\cdot),\lambda}^{\alpha(\cdot),q}$. By Theorem 13, $f = \sum_{j=-\infty}^\infty \lambda_j b_j$ converges in $\mathcal{S}'(\mathbb{R}^n)$, where each b_j is a central $(\alpha(\cdot), q(\cdot))$ -atom with support contained in B_j and

$$\|f\|_{\dot{HMK}_{p(\cdot),\lambda}^{\alpha(\cdot),q}} \approx \inf_{L \in \mathbb{Z}} \sup_{L \in \mathbb{Z}} 2^{-L\lambda} \left(\sum_{j=-\infty}^L |\lambda_j|^q \right)^{1/q}. \tag{73}$$

For simplicity, we denote $\Lambda = \sup_{L \in \mathbb{Z}} 2^{-L\lambda} \sum_{j=-\infty}^L |\lambda_j|^q$. By Proposition 7, we have

$$\begin{aligned}
&\|Tf\|_{\dot{MK}_{p(\cdot),\lambda}^{\alpha(\cdot),q}}^q \\
&\approx \max \left\{ \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^L 2^{kq\alpha(0)} \| (Tf) \chi_k \|_{L^{p(\cdot)}}^q, \right. \\
&\quad \left. \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \left(\sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \| (Tf) \chi_k \|_{L^{p(\cdot)}}^q \right. \right. \\
&\quad \left. \left. + \sum_{k=0}^L 2^{kq\alpha(\infty)} \| (Tf) \chi_k \|_{L^{p(\cdot)}}^q \right) \right\} \\
&\leq \max \{I, II + III\},
\end{aligned} \tag{74}$$

(74)

where

$$\begin{aligned} I &:= \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^L 2^{kq\alpha(0)} \|(Tf)\chi_k\|_{L^{p(\cdot)}}^q, \\ II &:= \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \|(Tf)\chi_k\|_{L^{p(\cdot)}}^q, \\ III &:= \sup_{L > 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^L 2^{kq\alpha(\infty)} \|(Tf)\chi_k\|_{L^{p(\cdot)}}^q. \end{aligned} \quad (75)$$

To complete our proof, we only need show that there exists a positive constant C such that $I, II, III \leq C\Lambda$.

First, we estimate I :

$$\begin{aligned} I &= \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^L 2^{kq\alpha(0)} \|(Tf)\chi_k\|_{L^{p(\cdot)}}^q \\ &\leq \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^L 2^{kq\alpha(0)} \left(\sum_{j=k}^{\infty} |\lambda_j| \|(Tb_j)\chi_k\|_{L^{p(\cdot)}} \right)^q \\ &\quad + \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^L 2^{kq\alpha(0)} \left(\sum_{j=-\infty}^{k-1} |\lambda_j| \|(Tb_j)\chi_k\|_{L^{p(\cdot)}} \right)^q \\ &:= I_1 + I_2. \end{aligned} \quad (76)$$

By the boundedness of T in $L^{p(\cdot)}$, we have

$$\|(Tb_j)\chi_k\|_{L^{p(\cdot)}} \leq \|b_j\|_{L^{p(\cdot)}} \leq |B_j|^{-\alpha_j/n} = 2^{-\alpha_j j}. \quad (77)$$

Therefore, when $0 < q \leq 1$, we get

$$\begin{aligned} I_1 &= \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^L 2^{\alpha(0)kq} \left(\sum_{j=k}^{\infty} |\lambda_j| \|(Tb_j)\chi_k\|_{L^{p(\cdot)}} \right)^q \\ &\leq \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^L 2^{\alpha(0)kq} \left(\sum_{j=k}^{\infty} |\lambda_j| 2^{-\alpha_j j} \right)^q \\ &\leq \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \\ &\quad \times \sum_{k=-\infty}^L 2^{\alpha(0)kq} \left(\sum_{j=k}^{-1} |\lambda_j|^q 2^{-\alpha(0)jq} + \sum_{j=0}^{\infty} |\lambda_j|^q 2^{-\alpha_{\infty}jq} \right) \\ &\leq \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^L \sum_{j=k}^{-1} |\lambda_j|^q 2^{\alpha(0)(k-j)q} \\ &\quad + \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^L 2^{\alpha(0)kq} \sum_{j=0}^{\infty} |\lambda_j|^q 2^{-\alpha_{\infty}jq} \\ &\leq \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{j=-\infty}^{-1} |\lambda_j|^q \sum_{k=-\infty}^j 2^{\alpha(0)(k-j)q} \\ &\quad + \sup_{L \leq 0, L \in \mathbb{Z}} \sum_{j=0}^{\infty} 2^{-j\lambda q} |\lambda_j|^q 2^{(\lambda-\alpha_{\infty})jq} 2^{-L\lambda q} \sum_{k=-\infty}^L 2^{\alpha(0)kq} \end{aligned}$$

$$\begin{aligned} &\leq \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{j=-\infty}^L |\lambda_j|^q \\ &\quad + \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{j=L}^{-1} |\lambda_j|^q \sum_{k=-\infty}^j 2^{\alpha(0)(k-j)q} \\ &\quad + \Lambda \sup_{L \leq 0, L \in \mathbb{Z}} \sum_{j=0}^{\infty} 2^{(\lambda-\alpha_{\infty})jq} \sum_{k=-\infty}^L 2^{\alpha(0)kq-L\lambda q} \\ &\leq \Lambda + \sup_{L \leq 0, L \in \mathbb{Z}} \sum_{j=L}^{-1} 2^{-j\lambda q} |\lambda_j|^q 2^{(j-L)\lambda q} \sum_{k=-\infty}^j 2^{\alpha(0)(k-j)q} + \Lambda \\ &\leq \Lambda + \Lambda \sup_{L \leq 0, L \in \mathbb{Z}} \sum_{j=L}^{-1} 2^{(j-L)\lambda q} \sum_{k=-\infty}^j 2^{\alpha(0)(k-j)q} \\ &\leq \Lambda. \end{aligned} \quad (78)$$

When $1 < q < \infty$, let $1/q + 1/q' = 1$ and we obtain

$$\begin{aligned} I_1 &= \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^L 2^{\alpha(0)kq} \left(\sum_{j=k}^{\infty} |\lambda_j| \|(Tb_j)\chi_k\|_{L^{p(\cdot)}} \right)^q \\ &\leq \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^L 2^{\alpha(0)kq} \left(\sum_{j=k}^{\infty} |\lambda_j| 2^{-\alpha_j j} \right)^q \\ &\leq \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^L \left(\sum_{j=k}^{-1} |\lambda_j| 2^{\alpha(0)(k-j)} \right)^q \\ &\quad + \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^L 2^{\alpha(0)kq} \left(\sum_{j=0}^{\infty} |\lambda_j| 2^{-\alpha_{\infty}j} \right)^q \\ &\leq \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^L \left(\sum_{j=k}^{-1} |\lambda_j|^q 2^{\alpha(0)(k-j)q/2} \right) \\ &\quad \times \left(\sum_{j=k}^{-1} 2^{\alpha(0)(k-j)q'/2} \right)^{q/q'} \\ &\quad + \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^L 2^{\alpha(0)kq} \left(\sum_{j=0}^{\infty} |\lambda_j|^q 2^{-\alpha_{\infty}jq/2} \right) \\ &\quad \times \left(\sum_{j=0}^{\infty} 2^{-\alpha_{\infty}jq'/2} \right)^{q/q'} \\ &\leq \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^L \sum_{j=k}^{-1} |\lambda_j|^q 2^{\alpha(0)(k-j)q/2} \\ &\quad + \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^L 2^{\alpha(0)kq} \sum_{j=0}^{\infty} |\lambda_j|^q 2^{-\alpha_{\infty}jq/2} \\ &\leq \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{j=-\infty}^{-1} |\lambda_j|^q \sum_{k=-\infty}^j 2^{\alpha(0)(k-j)q/2} \end{aligned}$$

$$\begin{aligned}
& + \sup_{L \leq 0, L \in \mathbb{Z}} \sum_{j=0}^{\infty} 2^{-j\lambda q} |\lambda_j|^q 2^{(\lambda - \alpha_{\infty}/2)jq} 2^{-L\lambda q} \sum_{k=-\infty}^L 2^{\alpha(0)kq} \\
& \leq \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{j=-\infty}^L |\lambda_j|^q \\
& + \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{j=L}^{-1} |\lambda_j|^q \sum_{k=-\infty}^j 2^{\alpha(0)(k-j)q/2} \\
& + \Lambda \sup_{L \leq 0, L \in \mathbb{Z}} \sum_{j=0}^{\infty} 2^{(\lambda - \alpha_{\infty}/2)jq} \sum_{k=-\infty}^L 2^{\alpha(0)kq - L\lambda q} \\
& \leq \Lambda + \sup_{L \leq 0, L \in \mathbb{Z}} \sum_{j=L}^{-1} 2^{-j\lambda q} |\lambda_j|^q 2^{(j-L)\lambda q} \sum_{k=-\infty}^j 2^{\alpha(0)(k-j)q/2} + \Lambda \\
& \leq \Lambda + \Lambda \sup_{L \leq 0, L \in \mathbb{Z}} \sum_{j=L}^{-1} 2^{(j-L)\lambda q} \sum_{k=-\infty}^j 2^{\alpha(0)(k-j)q/2} \\
& \leq \Lambda.
\end{aligned} \tag{79}$$

So, we have $I_1 \leq \Lambda$.

Second, we estimate I_2 . By (71) and Lemma 3, we get

$$\begin{aligned}
|Tb_j(x)| & \leq |x|^{-(n+s+\delta)} 2^{j(s+\delta)} \int_{B_j} |b_j(y)| dy \\
& \leq 2^{-k(n+s+\delta)} 2^{j(s+\delta)} \|b_j\|_{L^{p(\cdot)}} \|\chi_{B_j}\|_{L^{p'(\cdot)}} \\
& \leq 2^{j(s+\delta-\alpha_j)-k(s+\delta+n)} \|\chi_{B_j}\|_{L^{p'(\cdot)}}.
\end{aligned} \tag{80}$$

So by Lemmas 3 and 4, we have

$$\begin{aligned}
& \|(Tb_j)\chi_k\|_{L^{p(\cdot)}} \\
& \leq 2^{j(s+\delta-\alpha_j)-k(s+\delta+n)} \|\chi_{B_j}\|_{L^{p'(\cdot)}} \|\chi_{B_k}\|_{L^{p(\cdot)}} \\
& \leq 2^{j(s+\delta-\alpha_j)-k(s+\delta)} 2^{-kn} (|B_k| \|\chi_{B_k}\|_{L^{p'(\cdot)}}^{-1}) \|\chi_{B_j}\|_{L^{p'(\cdot)}} \\
& \leq 2^{j(s+\delta-\alpha_j)-k(s+\delta)} \frac{\|\chi_{B_j}\|_{L^{p'(\cdot)}}}{\|\chi_{B_k}\|_{L^{p'(\cdot)}}} \\
& \leq 2^{(s+\delta+n\delta_2)(j-k)-j\alpha_j}.
\end{aligned} \tag{81}$$

Therefore, when $0 < q \leq 1$, by $n\delta_2 \leq \alpha(0) < s + \delta + n\delta_2$ we get

$$\begin{aligned}
I_2 & = \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^L 2^{\alpha(0)kq} \left(\sum_{j=-\infty}^{k-1} |\lambda_j| \|(Tb_j)\chi_k\|_{L^{p(\cdot)}} \right)^q \\
& \leq \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \\
& \quad \times \sum_{k=-\infty}^L 2^{\alpha(0)kq} \left(\sum_{j=-\infty}^{k-1} |\lambda_j|^q 2^{[(s+\delta+n\delta_2)(j-k)-j\alpha(0)]q} \right)
\end{aligned}$$

$$\begin{aligned}
& = \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{j=-\infty}^L |\lambda_j|^q \sum_{k=j+1}^{-1} 2^{(j-k)(s+\delta+n\delta_2-\alpha(0))q} \\
& \leq \Lambda.
\end{aligned} \tag{82}$$

When $1 < q < \infty$, let $1/q + 1/q' = 1$. Since $n\delta_2 \leq \alpha(0) < s + \delta + n\delta_2$, by Hölder's inequality, we have

$$\begin{aligned}
I_2 & = \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^L 2^{\alpha(0)kq} \left(\sum_{j=-\infty}^{k-1} |\lambda_j| \|(Tb_j)\chi_k\|_{L^{p(\cdot)}} \right)^q \\
& \leq \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^L 2^{\alpha(0)kq} \left(\sum_{j=-\infty}^{k-1} |\lambda_j| 2^{(s+\delta+n\delta_2)(j-k)-j\alpha(0)} \right)^q \\
& \leq \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \\
& \quad \times \sum_{k=-\infty}^L 2^{\alpha(0)kq} \left(\sum_{j=-\infty}^{k-1} |\lambda_j|^q 2^{(j-k)(s+\delta+n\delta_2-\alpha(0))q/2} \right) \\
& \quad \times \left(\sum_{j=-\infty}^{k-1} 2^{(j-k)(s+\delta+n\delta_2-\alpha(0))q'/2} \right)^{q/q'} \\
& \leq \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \\
& \quad \times \sum_{j=-\infty}^L 2^{\alpha(0)jq} \left(\sum_{k=j+1}^{-1} |\lambda_k|^q 2^{(j-k)(s+\delta+n\delta_2-\alpha(0))q/2} \right) \\
& = \sup_{L \leq 0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{j=-\infty}^L |\lambda_j|^q \sum_{k=j+1}^{-1} 2^{(j-k)(s+\delta+n\delta_2-\alpha(0))q/2} \\
& \leq \Lambda.
\end{aligned} \tag{83}$$

Hence, we have $I \leq \Lambda$.

Third, we estimate II . Consider

$$\begin{aligned}
II & = \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \|(Tf)\chi_k\|_{L^{p(\cdot)}}^q \\
& \leq \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \left(\sum_{j=k}^{\infty} |\lambda_j| \|(Tb_j)\chi_k\|_{L^{p(\cdot)}} \right)^q \\
& \quad + \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \left(\sum_{j=-\infty}^{k-1} |\lambda_j| \|(Tb_j)\chi_k\|_{L^{p(\cdot)}} \right)^q \\
& := II_1 + II_2.
\end{aligned} \tag{84}$$

When $0 < q \leq 1$, we get

$$\begin{aligned}
II_1 & = \sum_{k=-\infty}^{-1} 2^{\alpha(0)kq} \left(\sum_{j=k}^{\infty} |\lambda_j| \|(Tb_j)\chi_k\|_{L^{p(\cdot)}} \right)^q \\
& \leq \sum_{k=-\infty}^{-1} 2^{\alpha(0)kq} \left(\sum_{j=k}^{\infty} |\lambda_j| 2^{-\alpha_j} \right)^q
\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{k=-\infty}^{-1} 2^{\alpha(0)kq} \left(\sum_{j=k}^{-1} |\lambda_j|^q 2^{-\alpha(0)jq} + \sum_{j=0}^{\infty} |\lambda_j|^q 2^{-\alpha_{\infty}jq} \right) \\
&\leq \sum_{k=-\infty}^{-1} \sum_{j=k}^{-1} |\lambda_j|^q 2^{\alpha(0)(k-j)q} + \sum_{k=-\infty}^{-1} 2^{\alpha(0)kq} \sum_{j=0}^{\infty} |\lambda_j|^q 2^{-\alpha_{\infty}jq} \\
&\leq \sum_{j=-\infty}^{-1} |\lambda_j|^q \sum_{k=-\infty}^j 2^{\alpha(0)(k-j)q} \\
&\quad + \sum_{j=0}^{\infty} |\lambda_j|^q 2^{-\alpha_{\infty}jq} \sum_{k=-\infty}^{-1} 2^{\alpha(0)kq} \\
&\leq \sum_{j=-\infty}^{-1} |\lambda_j|^q + \sum_{j=0}^{\infty} 2^{-j\lambda q} |\lambda_j|^q 2^{-\alpha_{\infty}jq} \sum_{k=-\infty}^{-1} 2^{\alpha(0)kq} \\
&\leq \Lambda + \Lambda \sum_{i=-\infty}^j |\lambda_i|^q \sum_{j=0}^{\infty} 2^{(\lambda-\alpha_{\infty})jq} \sum_{k=-\infty}^j 2^{\alpha(0)kq} \\
&\leq \Lambda.
\end{aligned} \tag{85}$$

When $1 < q < \infty$, let $1/q + 1/q' = 1$ and we obtain

$$\begin{aligned}
II_1 &= \sum_{k=-\infty}^{-1} 2^{\alpha(0)kq} \left(\sum_{j=k}^{\infty} |\lambda_j| \|(Tb_j)\chi_k\|_{L^{p(\cdot)}} \right)^q \\
&\leq \sum_{k=-\infty}^{-1} 2^{\alpha(0)kq} \left(\sum_{j=k}^{\infty} |\lambda_j| 2^{-\alpha_j j} \right)^q \\
&\leq \sum_{k=-\infty}^{-1} \left(\sum_{j=k}^{-1} |\lambda_j| 2^{\alpha(0)(k-j)} \right)^q \\
&\quad + \sum_{k=-\infty}^{-1} 2^{\alpha(0)kq} \left(\sum_{j=0}^{\infty} |\lambda_j| 2^{-\alpha_{\infty}j} \right)^q \\
&\leq \sum_{k=-\infty}^{-1} \left(\sum_{j=k}^{-1} |\lambda_j|^q 2^{\alpha(0)(k-j)q/2} \right) \left(\sum_{j=k}^{-1} 2^{\alpha(0)(k-j)q'/2} \right)^{q/q'} \\
&\quad + \sum_{k=-\infty}^{-1} 2^{\alpha(0)kq} \left(\sum_{j=0}^{\infty} |\lambda_j|^q 2^{-\alpha_{\infty}jq/2} \right) \left(\sum_{j=0}^{\infty} 2^{-\alpha_{\infty}jq'/2} \right)^{q/q'} \\
&\leq \sum_{k=-\infty}^{-1} |\lambda_j|^q \sum_{k=-\infty}^j 2^{\alpha(0)(k-j)q/2} \\
&\quad + \sum_{j=0}^{\infty} |\lambda_j|^q 2^{-\alpha_{\infty}jq/2} \sum_{k=-\infty}^{-1} 2^{\alpha(0)kq} \\
&\leq \sum_{j=-\infty}^{-1} |\lambda_j|^q \\
&\quad + \sum_{j=0}^{\infty} 2^{(\lambda-\alpha_{\infty}/2)jq} 2^{-j\lambda q} \sum_{i=-\infty}^j |\lambda_i|^q \sum_{k=-\infty}^{-1} 2^{\alpha(0)kq} \\
&\leq \Lambda + \Lambda \sum_{j=0}^{\infty} 2^{(\lambda-\alpha_{\infty}/2)jq} \sum_{k=-\infty}^{-1} 2^{\alpha(0)kq} \\
&\leq \Lambda.
\end{aligned} \tag{86}$$

For II_2 , when $0 < q \leq 1$, by $n\delta_2 \leq \alpha(0) < s + \delta + n\delta_2$ we get

$$\begin{aligned}
II_2 &= \sum_{k=-\infty}^{-1} 2^{\alpha(0)kq} \left(\sum_{j=-\infty}^{k-1} |\lambda_j| \|(Tb_j)\chi_k\|_{L^{p(\cdot)}} \right)^q \\
&\leq \sum_{k=-\infty}^{-1} 2^{\alpha(0)kq} \left(\sum_{j=-\infty}^{k-1} |\lambda_j|^q 2^{[(s+\delta+n\delta_2)(j-k)-j\alpha(0)]q} \right) \\
&= \sum_{k=-\infty}^{-1} |\lambda_j|^q \sum_{k=j+1}^{-1} 2^{(j-k)(s+\delta+n\delta_2-\alpha(0))q} \\
&\leq \sum_{k=-\infty}^{-1} |\lambda_j|^q \\
&\leq \Lambda.
\end{aligned} \tag{87}$$

When $1 < q < \infty$, let $1/q + 1/q' = 1$. Since $n\delta_2 \leq \alpha(0) < s + \delta + n\delta_2$, by Hölder's inequality, we have

$$\begin{aligned}
II_2 &= \sum_{k=-\infty}^{-1} 2^{\alpha(0)kq} \left(\sum_{j=-\infty}^{k-1} |\lambda_j| \|(Tb_j)\chi_k\|_{L^{p(\cdot)}} \right)^q \\
&\leq \sum_{k=-\infty}^{-1} 2^{\alpha(0)kq} \left(\sum_{j=-\infty}^{k-1} |\lambda_j| 2^{(s+\delta+n\delta_2)(j-k)-j\alpha(0)} \right)^q \\
&\leq \sum_{k=-\infty}^{-1} 2^{\alpha(0)kq} \left(\sum_{j=-\infty}^{k-1} |\lambda_j|^q 2^{(j-k)(s+\delta+n\delta_2-\alpha(0))q/2} \right) \\
&\quad \times \left(\sum_{j=-\infty}^{k-1} 2^{(j-k)(s+\delta+n\delta_2-\alpha(0))q'/2} \right)^{q/q'} \\
&\leq \sum_{j=-\infty}^{-1} 2^{\alpha(0)kq} \left(\sum_{j=-\infty}^{k-1} |\lambda_j|^q 2^{(j-k)(s+\delta+n\delta_2-\alpha(0))q/2} \right) \\
&= \sum_{j=-\infty}^{-1} |\lambda_j|^q \sum_{k=j+1}^{-1} 2^{(j-k)(s+\delta+n\delta_2-\alpha(0))q/2} \\
&\leq \Lambda.
\end{aligned} \tag{88}$$

So, we have $II \leq \Lambda$.

Finally, we estimate III :

$$\begin{aligned}
III &= \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^L 2^{kq\alpha_{\infty}} \|(Tf)\chi_k\|_{L^{p(\cdot)}}^q \\
&\leq \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^L 2^{kq\alpha_{\infty}} \left(\sum_{j=k}^{\infty} |\lambda_j| \|(Tb_j)\chi_k\|_{L^{p(\cdot)}} \right)^q \\
&\quad + \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^L 2^{kq\alpha_{\infty}} \left(\sum_{j=-\infty}^{k-1} |\lambda_j| \|(Tb_j)\chi_k\|_{L^{p(\cdot)}} \right)^q \\
&:= III_1 + III_2.
\end{aligned} \tag{89}$$

When $0 < q \leq 1$, by the boundedness of T in $L^{p(\cdot)}$, we have

$$\begin{aligned}
 III_1 &= \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^L 2^{\alpha_{\infty} k q} \left(\sum_{j=k}^{\infty} |\lambda_j| \|(Tb_j)\chi_k\|_{L^{p(\cdot)}} \right)^q \\
 &\leq \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^L 2^{\alpha_{\infty} k q} \sum_{j=k}^{\infty} |\lambda_j|^q \|(Tb_j)\chi_k\|_{L^{p(\cdot)}}^{q/2} \\
 &\leq \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^L 2^{\alpha_{\infty} k q} \sum_{j=k}^{\infty} |\lambda_j|^q 2^{-\alpha_j j q} \\
 &\leq \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^L 2^{\alpha_{\infty} k q} \sum_{j=k}^{\infty} |\lambda_j|^q 2^{-\alpha_{\infty} j q} \\
 &= \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{j=0}^L |\lambda_j|^q \sum_{k=0}^j 2^{(k-j)\alpha_{\infty} q} \\
 &\quad + \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{j=L}^{\infty} |\lambda_j|^q \sum_{k=0}^L 2^{(k-j)\alpha_{\infty} q} \quad (90) \\
 &\leq \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{j=0}^L |\lambda_j|^q \\
 &\quad + \sup_{L>0, L \in \mathbb{Z}} \sum_{j=L}^{\infty} 2^{(j\lambda q - L\lambda q)} 2^{-j\lambda q} \sum_{i=-\infty}^j |\lambda_i|^q \sum_{k=0}^L 2^{(k-j)\alpha_{\infty} q} \\
 &\leq \Lambda + \Lambda \sup_{L>0, L \in \mathbb{Z}} \sum_{j=L}^{\infty} 2^{(j-L)\lambda q} 2^{(L-j)\alpha_{\infty} q} \\
 &\leq \Lambda + \Lambda \sup_{L>0, L \in \mathbb{Z}} \sum_{j=L}^{\infty} 2^{(j-L)q(\lambda - \alpha_{\infty})} \\
 &\leq \Lambda.
 \end{aligned}$$

When $1 < q < \infty$, by the boundedness of T in $L^{p(\cdot)}$ and Hölder's inequality, we have

$$\begin{aligned}
 III_1 &= \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^L 2^{\alpha_{\infty} k q} \left(\sum_{j=k}^{\infty} |\lambda_j| \|(Tb_j)\chi_k\|_{L^{p(\cdot)}} \right)^q \\
 &\leq \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^L 2^{\alpha_{\infty} k q} \left(\sum_{j=k}^{\infty} |\lambda_j|^q \|(Tb_j)\chi_k\|_{L^{p(\cdot)}}^{q/2} \right) \\
 &\quad \times \left(\sum_{j=k}^{\infty} \|(Tb_j)\chi_k\|_{L^{p(\cdot)}}^{q'/2} \right)^{q/q'} \\
 &\leq \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^L 2^{\alpha_{\infty} k q} \left(\sum_{j=k}^{\infty} |\lambda_j|^q \|b_j\|_{L^{p(\cdot)}}^{q/2} \right) \\
 &\quad \times \left(\sum_{j=k}^{\infty} \|b_j\|_{L^{p(\cdot)}}^{q'/2} \right)^{q/q'}
 \end{aligned}$$

$$\begin{aligned}
 &\leq \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^L 2^{\alpha_{\infty} k q} \left(\sum_{j=k}^{\infty} |\lambda_j|^q |B_j|^{-\alpha_j q/(2n)} \right) \\
 &\quad \times \left(\sum_{j=k}^{\infty} |B_j|^{-\alpha_j q'/(2n)} \right)^{q/q'} \\
 &\leq \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^L 2^{\alpha_{\infty} k q/2} \left(\sum_{j=k}^{\infty} |\lambda_j|^q |B_j|^{-\alpha_j q/(2n)} \right) \\
 &= \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{j=0}^L |\lambda_j|^q \sum_{k=0}^j 2^{(k-j)\alpha_{\infty} q/2} \\
 &\quad + \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{j=L}^{\infty} |\lambda_j|^q \sum_{k=0}^L 2^{(k-j)\alpha_{\infty} q/2} \\
 &\leq \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{j=0}^L |\lambda_j|^q \\
 &\quad + \sup_{L>0, L \in \mathbb{Z}} \sum_{j=L}^{\infty} 2^{(j\lambda q - L\lambda q)} 2^{-j\lambda q} \sum_{i=-\infty}^j |\lambda_i|^q \sum_{k=0}^L 2^{(k-j)\alpha_{\infty} q/2} \\
 &\leq \Lambda + \Lambda \sup_{L>0, L \in \mathbb{Z}} \sum_{j=L}^{\infty} 2^{(j-L)\lambda q} 2^{(L-j)\alpha_{\infty} q/2} \\
 &\leq \Lambda + \Lambda \sup_{L>0, L \in \mathbb{Z}} \sum_{j=L}^{\infty} 2^{(j-L)q(\lambda - \alpha_{\infty}/2)} \\
 &\leq \Lambda. \quad (91)
 \end{aligned}$$

When $0 < q \leq 1$, by $n\delta_2 \leq \alpha(0)$, $\alpha_{\infty} < s + \delta + n\delta_2$ we get

$$\begin{aligned}
 III_2 &= \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^L 2^{\alpha_{\infty} k q} \left(\sum_{j=-\infty}^{k-1} |\lambda_j| \|(Tb_j)\chi_k\|_{L^{p(\cdot)}} \right)^q \\
 &\leq \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \\
 &\quad \times \sum_{k=0}^L 2^{\alpha_{\infty} k q} \left(\sum_{j=-\infty}^{k-1} |\lambda_j|^q 2^{[(s+\delta+n\delta_2)(j-k)-j\alpha_j]q} \right) \\
 &= \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \\
 &\quad \times \sum_{k=0}^L 2^{\alpha_{\infty} k q} \left(\sum_{j=-\infty}^{-1} |\lambda_j|^q 2^{[(s+\delta+n\delta_2)(j-k)-j\alpha(0)]q} \right) \\
 &\quad + \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \\
 &\quad \times \sum_{k=0}^L 2^{\alpha_{\infty} k q} \left(\sum_{j=0}^{k-1} |\lambda_j|^q 2^{[(s+\delta+n\delta_2)(j-k)-j\alpha_{\infty}]q} \right)
 \end{aligned}$$

$$\begin{aligned}
&\leq \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^L 2^{[\alpha_{\infty} - (s+\delta+n\delta_2)]kq} \\
&\quad \times \sum_{j=-\infty}^{-1} |\lambda_j|^q 2^{(s+\delta+n\delta_2-\alpha(0))jq} \\
&\quad + \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^L |\lambda_j|^q \sum_{k=j+1}^{\infty} 2^{(j-k)(s+\delta+n\delta_2-\alpha_{\infty})q} \\
&\leq \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{j=-\infty}^{-1} |\lambda_j|^q + \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{j=0}^{L-1} |\lambda_j|^q \\
&\leq \Lambda.
\end{aligned} \tag{92}$$

When $1 < q < \infty$, let $1/q + 1/q' = 1$. Since $n\delta_2 \leq \alpha(0)$, $\alpha_{\infty} < s + \delta + n\delta_2$, by Hölder's inequality, we have

$$\begin{aligned}
III_2 &= \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^L 2^{\alpha_{\infty}kq} \left(\sum_{j=-\infty}^{k-1} |\lambda_j| \| (Tb_j)\chi_k \|_{L^{p(\cdot)}} \right)^q \\
&\leq \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^L 2^{\alpha_{\infty}kq} \left(\sum_{j=-\infty}^{k-1} |\lambda_j| 2^{[(s+\delta+n\delta_2)(j-k)-j\alpha_j]} \right)^q \\
&\leq \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^L 2^{\alpha_{\infty}kq} \left(\sum_{j=-\infty}^{-1} |\lambda_j| 2^{[(s+\delta+n\delta_2)(j-k)-j\alpha(0)]} \right)^q \\
&\quad + \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^L 2^{\alpha_{\infty}kq} \left(\sum_{j=0}^{k-1} |\lambda_j| 2^{[(s+\delta+n\delta_2)(j-k)-j\alpha_{\infty}]} \right)^q \\
&\leq \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^L 2^{[\alpha_{\infty} - (s+\delta+n\delta_2)]kq} \\
&\quad \times \left(\sum_{j=-\infty}^{-1} |\lambda_j| 2^{(s+\delta+n\delta_2-\alpha(0))j} \right)^q \\
&\quad + \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^L \left(\sum_{j=0}^{k-1} |\lambda_j| 2^{(j-k)(s+\delta+n\delta_2-\alpha_{\infty})} \right)^q \\
&\leq \left(\sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{j=-\infty}^{-1} |\lambda_j|^q 2^{(s+\delta+n\delta_2-\alpha(0))jq/2} \right) \\
&\quad \times \left(\sum_{j=-\infty}^{-1} 2^{(s+\delta+n\delta_2-\alpha(0))jq'/2} \right)^{q/q'} \\
&\quad + \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^L \left(\sum_{j=0}^{k-1} |\lambda_j|^q 2^{(j-k)(s+\delta+n\delta_2-\alpha_{\infty})q/2} \right) \\
&\quad \times \left(\sum_{j=0}^{k-1} 2^{(j-k)(s+\delta+n\delta_2-\alpha_{\infty})q'/2} \right)^{q/q'}
\end{aligned}$$

$$\begin{aligned}
&\leq \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{j=-\infty}^{-1} |\lambda_j|^q 2^{(s+\delta+n\delta_2-\alpha(0))jq/2} \\
&\quad + \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^L \sum_{j=0}^{k-1} |\lambda_j|^q 2^{(j-k)(s+\delta+n\delta_2-\alpha_{\infty})q/2} \\
&\leq \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{j=-\infty}^{-1} |\lambda_j|^q \\
&\quad + \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{j=0}^{L-1} |\lambda_j|^q \sum_{k=j+1}^L 2^{(j-k)(s+\delta+n\delta_2-\alpha_{\infty})q/2} \\
&\leq \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{j=-\infty}^{-1} |\lambda_j|^q + \sup_{L>0, L \in \mathbb{Z}} 2^{-L\lambda q} \sum_{j=0}^{L-1} |\lambda_j|^q \\
&\leq \Lambda.
\end{aligned} \tag{93}$$

Thus we have $III \leq \Lambda$. This finishes the proof of Theorem 14. $\square$

Definition 15. Let K be a locally integrable function on $\mathbb{R}^n \times \mathbb{R}^n \setminus \{x = y\}$. Then K is called a standard kernel if there exist $\delta \in (0, 1]$ and $C > 0$, such that

$$\begin{aligned}
|K(x, y)| &\leq \frac{C}{|x - y|^n}, \quad x \neq y; \\
|K(x, y) - K(x, y')| &\leq C \frac{|y - y'|^\delta}{|x - y|^{n+\delta}}, \\
|y - y'| &\leq \frac{1}{2}|x - y|; \\
|K(x, y) - K(x', y)| &\leq C \frac{|x - x'|^\delta}{|x - y|^{n+\delta}}, \\
|x - x'| &\leq \frac{1}{2}|x - y|.
\end{aligned} \tag{94}$$

A linear operator $T : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n)$ is called a Calderón-Zygmund operator associated to a standard kernel K if

- (i) T can be extended to a bounded operator on $L^2(\mathbb{R}^n)$;
- (ii) for any $f \in L^2$ with compact support and almost everywhere $x \notin \text{supp } f$,

$$Tf(x) = \int_{\mathbb{R}^n} K(x, y) f(y) dy. \tag{95}$$

It is well known that a Calderón-Zygmund operator is also bounded in $L^{p(\cdot)}$ for any $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$; for example, see [38].

Corollary 16. Let $\alpha(\cdot)$ be a bounded and log-Hölder continuous both at the origin and infinity such that $n\delta_2 \leq \alpha(0)$,

$\alpha_\infty < n\delta_2 + \delta$ with δ_2 as in Lemma 4. Suppose T is a Calderón-Zygmund operator associated to a standard kernel K . If $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, $0 < q < \infty$, and $0 \leq \lambda < \infty$, then there exists a constant C such that

$$\begin{aligned} \|Tf\|_{\dot{MK}_{p(\cdot),\lambda}^{\alpha(\cdot),q}} &\leq C \|f\|_{\dot{HMK}_{p(\cdot),\lambda}^{\alpha(\cdot),q}}, \\ \|Tf\|_{\dot{MK}_{p(\cdot),\lambda}^{\alpha(\cdot),q}} &\leq C \|f\|_{\dot{HMK}_{p(\cdot),\lambda}^{\alpha(\cdot),q}}, \end{aligned} \quad (96)$$

for $f \in \dot{HMK}_{p(\cdot),\lambda}^{\alpha(\cdot),q}$ and $f \in \dot{HMK}_{p(\cdot),\lambda}^{\alpha(\cdot),q}$, respectively.

Proof. It is easy to know that $s = \max\{[\alpha(0) - n\delta_2], [\alpha_\infty - n\delta_2]\} = 0$ when $n\delta_2 \leq \alpha(0)$ and $\alpha_\infty < n\delta_2 + \delta$. Then the result follows from Theorem 14. $\square$

Remark 17. For Hardy type spaces, there are some characterizations: maximal function, square function, atomic decomposition, and molecular decomposition. To discuss the boundedness of singular integrals in Hardy type spaces, we use the atomic decomposition for the domain Hardy space, while it is convenient to use another characterization of Hardy space for the target Hardy space; for example, see the proof of Theorem 6.7.4 in [49] and [50, 51]. In a future paper, we will give the molecular decomposition of spaces $\dot{HMK}_{p(\cdot),\lambda}^{\alpha(\cdot),q}$ and $\dot{HMK}_{p(\cdot),\lambda}^{\alpha(\cdot),q}$. Then, by the atomic and molecular decompositions, we will obtain the boundedness of T in Corollary 16 from $\dot{HMK}_{p(\cdot),\lambda}^{\alpha(\cdot),q}$ and $\dot{HMK}_{p(\cdot),\lambda}^{\alpha(\cdot),q}$ into themselves, respectively.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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