Research Article

Two-Weight Extrapolation on Lorentz Spaces

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We give several two-weight extrapolation theorems on Lorentz spaces which extend the results of Cruz-Uribe and Pérez (2000) and some applications for the singular integral operators, the potential type operators, and commutators.

1. Introduction and Main Results

The classical extrapolation theorem is due to Rubio de Francia [1], who showed that if sublinear operator T is bounded on $L^{p_0}(w)$ for some $p_0, 1 \leq p_0 < \infty$, and every $w \in A_{p_0}$, then T is bounded on $L^p(w)$ for every $p, 1 < p < \infty$ and every $w \in A_p$. Cruz-Uribe et al. [2] and Curbera et al. [3] generalized the extrapolation theorem of Rubio de Francia and got a lot of new extrapolation theorems associated with A_∞ weights. These theorems have proved to be the key to solving many problems in harmonic analysis.

Cruz-Uribe and Pérez [4] gave several other extrapolation theorems for pairs of weights of the forms $(w, M^k w)$ and $(w, (Mw/w)^r w)$, where w is any nonnegative function, $r > 1$, and M^k is the k th iteration of the Hardy-Littlewood maximal operator. The purpose of this paper is to extend the extrapolation theorems in [4] and derive some extrapolation results on Lorentz spaces.

Let $w(x), x \in \mathbb{R}^n$, be a nonnegative, locally integrable weight function. For a measurable function f ,

$$\lambda_f^w(t) = w \{x \in \mathbb{R}^n : |f(x)| > t\}, \quad t \geq 0, \quad (1)$$

is the distribution function of f with respect to the measure $w(x)dx$. For $0 < p, q \leq \infty$, the weighted Lorentz spaces

$L^{p,q}(w)$ are the collection of all functions f , such that $\|f\|_{L^{p,q}(w)} < \infty$, where

$$\|f\|_{L^{p,q}(w)} = \begin{cases} \left(q \int_0^\infty \lambda_f^w(t)^{q/p} t^{q-1} dt \right)^{1/q}, & 0 < q < \infty, \\ \sup_{t>0} t \lambda_f^w(t)^{1/p}, & q = \infty. \end{cases} \quad (2)$$

For $1 \leq p < \infty$, p' denote the conjugate exponent of p : $1/p + 1/p' = 1$.

In the case when either $1 < p < \infty$ and $1 \leq q \leq \infty$, or $p = q = 1$, or $p = q = \infty$, $L^{p,q}(w)$ is a Banach space. Furthermore, we have the relationship

$$\begin{aligned} C_1 \|f\|_{L^{p,q}(w)} &\leq \sup_{\|g\|_{L^{p',q'}(w)} \leq 1} \left| \int_{\mathbb{R}^n} f(x) g(x) w(x) dx \right| \\ &\leq C_2 \|f\|_{L^{p,q}(w)}. \end{aligned} \quad (3)$$

Our main results are the following.

Theorem 1. Let S and T be operators (not necessarily linear), and let f be a function in a suitable test class for both S and T . Suppose that there exist positive constants p_0 and C_0 and a positive integer k such that for all weights w

$$\|Tf\|_{L^{p_0}(w)} \leq C_0 \|Sf\|_{L^{p_0}(M^k w)}. \quad (4)$$

Then, for all $p, p_0 < p < \infty$, $p_0 \leq q \leq \infty$, there exists a constant C_p depending only on C_0, p, p_0, k , and n , such that, for all weights w ,

$$\|Tf\|_{L^{p,q}(w)} \leq C_p \|Sf\|_{L^{p,q}(M^{[kp/p_0]+1}w)}, \quad (5)$$

where $[kp/p_0]$ is the largest integer less than or equal to kp/p_0 .

Theorem 2. Let S and T be operators (not necessarily linear), and let f be a function in a suitable test class for both S and T . Suppose that there exist positive constants p_0 and C_0 and a positive integer k such that for all weights w

$$\|Tf\|_{L^{p_0}(w)} \leq C_0 \|Sf\|_{L^{p_0}(Mw)}. \quad (6)$$

Then, for all $p, p_0 < p < \infty$, $p_0 \leq q \leq \infty$, there exists a constant C_p depending only on C_0, p, p_0, k , and n , such that, for all weights w ,

$$\|Tf\|_{L^{p,q}(w)} \leq C_p \|Sf\|_{L^{p,q}((Mw/w)^{[p/p_0]}Mw)}. \quad (7)$$

2. Applications

Let M be the Hardy-Littlewood maximal operator; Fefferman and Stein [5] showed that, for every $p, 1 < p < \infty$, every nonnegative function w , and every function f ,

$$\int_{\mathbb{R}^n} |Mf(x)|^p w(x) dx \leq C \int_{\mathbb{R}^n} |f(x)|^p Mw(x) dx. \quad (8)$$

Using Theorem 1, we can get the following result.

Proposition 3. For $p, 1 < p < \infty$, $1 \leq q \leq \infty$, there exists a constant $C > 0$, such that, for all weights w and function f ,

$$\|Mf\|_{L^{p,q}(w)} \leq C \|f\|_{L^{p,q}(Mw)}. \quad (9)$$

Let T be a Calderón-Zygmund operator. Wilson [6] and Pérez [7] showed that if $1 < p < \infty$, then, for any weight w and every f ,

$$\int_{\mathbb{R}^n} |Tf(x)|^p w(x) dx \leq C \int_{\mathbb{R}^n} |f(x)|^p M^{[p]+1}w(x) dx, \quad (10)$$

where the exponent $[p] + 1$ is sharp. Using Theorem 1 we can get the following result.

Proposition 4. Let T be a Calderón-Zygmund operator, $1 < p < \infty$, $1 \leq q \leq \infty$; then, for any weight w and every f ,

$$\|Tf\|_{L^{p,q}(w)} \leq C \|f\|_{L^{p,q}(M^{[p]+1}w)}. \quad (11)$$

For the commutators of singular integral operators with a BMO function $[T, b]$, Pérez [8] showed the analogous weighted inequalities as in (10). By Theorem 1, we can give similar results as in Proposition 4; details are omitted.

Lerner [9] establishes

$$\int_{\mathbb{R}^n} |Tf(x)| w(x) dx \leq C \int_{\mathbb{R}^n} Mf(x) Mw(x) dx, \quad (12)$$

for any Calderón-Zygmund operator T and any arbitrary weight w . Using Theorems 1 and 2, we can get the following results.

Proposition 5. Let T be a Calderón-Zygmund operator, $1 < p < \infty$, $1 \leq q \leq \infty$; then, for any weight w and all f ,

$$\|Tf\|_{L^{p,q}(w)} \leq C \|Mf\|_{L^{p,q}(M^{[p]}w)}, \quad (13)$$

$$\|Tf\|_{L^{p,q}(w)} \leq C \|Mf\|_{L^{p,q}((Mw/w)^{[p]}Mw)}.$$

Now, we consider the potential type operators and commutators. For a nonnegative, locally integrable function Φ on $\mathbb{R}^n$, assume that Φ satisfies the following weak growth condition: there are constants $\delta, c > 0$, $0 \leq \varepsilon < 1$ with the property that, for all $k \in \mathbb{Z}$,

$$\sup_{2^k < |x| \leq 2^{k+1}} \Phi(x) \leq \frac{c}{2^{kn}} \int_{\delta(1-\varepsilon)2^k < |y| \leq 2\delta(1+\varepsilon)2^k} \Phi(y) dy. \quad (14)$$

The basic examples are provided by the Riesz potential of order α , I_α , defined by the kernel $\Phi(x) = |x|^{\alpha-n}$, $0 < \alpha < n$.

Define the potential type operator T_Φ and the maximal operator $M_{\widetilde{\Phi}}$ by

$$T_\Phi f(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y) dy, \quad (15)$$

$$M_{\widetilde{\Phi}} f(x) = \sup_{Q \ni x} \frac{\widetilde{\Phi}(l(Q))}{|Q|} \int_Q |f(y)| dy,$$

where

$$\widetilde{\Phi}(t) = \int_{|z| < t} \Phi(z) dz, \quad t \geq 0. \quad (16)$$

For the Riesz potential of order α , $\widetilde{\Phi}(t) = t^\alpha$. Pérez [10] studies the two-weight strong type (p, q) inequalities for T_Φ , $1 < p \leq q < \infty$.

Lemma 6 (see [10]). Let T_Φ be the potential type operator with Φ satisfying (14), let f and g be bounded functions with compact support, and let $a > 2^n$; then, there exist a family of cubes $Q_{k,j}$ and a family of pairwise disjoint subsets $E_{k,j}$, $E_{k,j} \subset Q_{k,j}$, with $|Q_{k,j}| < (1-2^n/a)^{-1} |E_{k,j}|$ for all k, j , such that

$$\begin{aligned} & \int_{\mathbb{R}^n} |T_\Phi f(x) g(x)| dx \\ & \leq C \sum_{k,j} \frac{1}{|3Q_{k,j}|} \int_{3Q_{k,j}} f(x) dx \frac{\widetilde{\Phi}(l(\rho Q_{k,j}))}{|\rho Q_{k,j}|} \\ & \quad \times \int_{\rho Q_{k,j}} g(x) dx |E_{k,j}|, \end{aligned} \quad (17)$$

where $\rho = \delta(1+\varepsilon)$.

By Lemma 6, we can easily prove the following result.

Lemma 7. Let T_Φ be the potential type operator with Φ satisfying (14); then, there is a constant $C > 0$ such that, for any weight w and all f ,

$$\int_{\mathbb{R}^n} |T_\Phi f(x)| w(x) dx \leq C \int_{\mathbb{R}^n} M_{\widetilde{\Phi}} f(x) Mw(x) dx. \quad (18)$$

Using (18) and Theorems 1 and 2, we can get the following results.

Proposition 8. *Let T_Φ be the potential type operator with Φ satisfying (14), $1 < p < \infty$, $1 \leq q \leq \infty$; then, for any weight w and every f ,*

$$\begin{aligned} \|T_\Phi f\|_{L^{p,q}(w)} &\leq C \|M_\Phi f\|_{L^{p,q}(M^{[p]}w)}, \\ \|T_\Phi f\|_{L^{p,q}(w)} &\leq C \|M_\Phi f\|_{L^{p,q}((Mw/w)^{[p]}Mw)}. \end{aligned} \quad (19)$$

For $k \geq 1$ and $b \in \text{BMO}$, the commutators of potential type integral operator with a BMO function are defined by

$$T_{\Phi,b}^k f(x) = \int_{\mathbb{R}^n} (b(x) - b(y))^k \Phi(x - y) f(y) dy. \quad (20)$$

Li [11] gave the two-weight strong type (p, q) inequalities for $T_{\Phi,b}^k$, $1 < p \leq q < \infty$. We can easily get similar results as in Lemmas 6 and 7 from [11]; by Theorem 1, we can obtain some weighted inequalities for the commutators of potential type integral operators; details are omitted.

3. Proofs of Theorems

We need some notations and facts. Let $B(t) : [0, \infty) \rightarrow [0, \infty)$ be a Young function, that is, a continuous, convex, and increasing function with $B(0) = 0$ such that $B(t) \rightarrow \infty$ as $t \rightarrow \infty$. Given a Young function B , we define the B -average of a function f over a cube Q by

$$\|f\|_{B,Q} = \inf \left\{ \lambda > 0 : \frac{1}{|Q|} \int_Q B\left(\frac{|f(x)|}{\lambda}\right) dx \leq 1 \right\}. \quad (21)$$

In particular, for the Young function $B_k(t) = t(1 + \log^+ t)^k$, $k = 1, 2, \dots$, the B_k -average of a function f over a cube Q is denoted by $\|f\|_{L(\log L)^k, Q}$. Define the maximal function associated with the Young function B_k as

$$M_{L(\log L)^k} f(x) = \sup_{Q \ni x} \|f\|_{L(\log L)^k, Q}. \quad (22)$$

Pérez [8] obtained that $M_{L(\log L)^k} f(y) \approx M^{k+1} f(y)$.

Proof of Theorem 1. Fix $p, p_0 < p < \infty$, $p_0 \leq q \leq \infty$, and let $r = p/p_0$, $s = q/p_0$. Then, by duality,

$$\begin{aligned} \|Tf\|_{L^{p,q}(w)}^{p_0} &= \|(Tf)^{p_0}\|_{L^{r,s}(w)} \\ &\leq \sup_{g \in L^{r',s'}(w), \|g\|_{L^{r',s'}(w)} \leq 1} \int_{\mathbb{R}^n} Tf(x)^{p_0} g(x) w(x) dx. \end{aligned} \quad (23)$$

The last inequality follows since $L^{r',s'}(w)$ and $L^{r,s}(w)$ are associate spaces. Fixing one of these g 's, we use (4) to continue with

$$\begin{aligned} &\int_{\mathbb{R}^n} (Tf(x))^{p_0} g(x) w(x) dx \\ &\leq C \int_{\mathbb{R}^n} Sf(x)^{p_0} M^k(gw)(x) dx \end{aligned}$$

$$\begin{aligned} &\leq C \int_{\mathbb{R}^n} Sf(x)^{p_0} \frac{M_{L(\log L)^{k-1}}(gw)(x)}{\widetilde{M}(w)(x)} \widetilde{M}(w)(x) dx \\ &\leq C \|(Sf)^{p_0}\|_{L^{r,s}(\widetilde{M}(w))} \left\| \frac{M_{L(\log L)^{k-1}}(gw)(x)}{\widetilde{M}(w)(x)} \right\|_{L^{r',s'}(\widetilde{M}(w))} \\ &= C \|Sf\|_{L^{p,q}(\widetilde{M}(w))}^{p_0} \left\| \frac{M_{L(\log L)^{k-1}}(gw)(x)}{\widetilde{M}(w)(x)} \right\|_{L^{r',s'}(\widetilde{M}(w))}, \end{aligned} \quad (24)$$

where $\widetilde{M}$ is an appropriate maximal operator to be chosen soon. To conclude, we just need to show that

$$\left\| \frac{M_{L(\log L)^{k-1}}(gw)(x)}{\widetilde{M}(w)(x)} \right\|_{L^{r',s'}(\widetilde{M}(w))} \leq C \|g\|_{L^{r',s'}(w)}, \quad (25)$$

or equivalently

$$F : L^{r',s'}(w) \longrightarrow L^{r',s'}(\widetilde{M}(w)), \quad (26)$$

where

$$Ff = \frac{M_{L(\log L)^{k-1}}(fw)}{\widetilde{M}(w)}. \quad (27)$$

To do this, we choose $\widetilde{M}$ pointwise bigger than $M_{L(\log L)^{k-1}}$; then, we trivially have

$$F : L^\infty(w) \longrightarrow L^\infty(\widetilde{M}(w)). \quad (28)$$

Therefore, by Marcinkiewicz's interpolation theorem for Lorentz spaces in [12], it will be enough to show that for some $\varepsilon > 0$

$$F : L^{(r+\varepsilon)'}(w) \longrightarrow L^{(r+\varepsilon)'}(\widetilde{M}(w)), \quad (29)$$

which amounts to proving

$$\begin{aligned} &\int_{\mathbb{R}^n} M_{L(\log L)^{k-1}}(fw)(x)^{(r+\varepsilon)'} (\widetilde{M}w(x))^{1-(r+\varepsilon)'} dx \\ &\leq C \int_{\mathbb{R}^n} f(x)^{(r+\varepsilon)'} w(x) dx. \end{aligned} \quad (30)$$

But this result follows from [13]: it is shown there that, for $\nu > 1$ and $\eta > 0$,

$$\begin{aligned} &\int_{\mathbb{R}^n} M_{L(\log L)^{k-1}}(f)(x)^{\nu'} (M_{L(\log L)^{k\nu-1+\eta}}w(x))^{1-\nu'} dx \\ &\leq C \int_{\mathbb{R}^n} f(x)^{\nu'} w(x)^{1-\nu'} dx. \end{aligned} \quad (31)$$

We finally choose the appropriate parameters and weight. Let $\nu = r + \varepsilon$, $\eta = \varepsilon = ([kr] + 1 - kr)/(k + 1) > 0$, and pick the weight

$$\widetilde{M}w = M_{L(\log L)^{[kr]}}w. \quad (32)$$

This shows that

$$\|Tf\|_{L^{p,q}(w)} \leq C \|Sf\|_{L^{p,q}(M_{L(\log L)^{[kp/p_0]}}(w))}. \quad (33)$$

We conclude the proof of (5). $\square$

Proof of Theorem 2. The proof of this theorem proceeds exactly as that of (5) with minor changes. At inequality (23), fixing one of these g 's, we use (6) to continue with

$$\begin{aligned} & \int_{\mathbb{R}^n} (Tf(x))^{p_0} g(x) w(x) dx \\ & \leq C \int_{\mathbb{R}^n} Sf(x)^{p_0} M(gw)(x) dx. \end{aligned} \quad (34)$$

For g and w , we have $M(gw) \leq CM_c(gw)$, where C is a constant depending only on the dimension n and M_c is the unweighted centered Hardy-Littlewood maximal operator:

$$M_c(f)(x) = \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(x)| dx, \quad (35)$$

in which $B_r(x)$ is the ball of radius r centered at x . Furthermore,

$$\begin{aligned} & M_c(gw)(x) \\ & = \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} g(y) w(y) dy \\ & = \sup_{r>0} \frac{w(B_r(x))}{|B_r(x)|} \frac{1}{w(B_r(x))} \int_{B_r(x)} g(y) w(y) dy \\ & \leq M_{w,c} g(x) Mw(x), \end{aligned} \quad (36)$$

where $M_{w,c}$ is the weighted centered maximal operator. Then, for $\delta > 0$,

$$\begin{aligned} & \int_{\mathbb{R}^n} (Sf)^{p_0} M(gw) dx \\ & \leq C \int_{\mathbb{R}^n} |Sf|^{p_0} M_{w,c} g \left(\frac{w}{Mw} \right)^{r+\delta-1} \left(\frac{Mw}{w} \right)^{r+\delta} w dx \\ & \leq C \| (Sf)^{p_0} \|_{L^{r',s'}((Mw/w)^{r+\delta} w)} \\ & \quad \times \left\| M_{w,c} g \left(\frac{w}{Mw} \right)^{r+\delta-1} \right\|_{L^{r',s'}((Mw/w)^{r+\delta} w)}. \end{aligned} \quad (37)$$

To conclude, we just need to show that

$$\left\| M_{w,c} g \left(\frac{w}{Mw} \right)^{r+\delta-1} \right\|_{L^{r',s'}((Mw/w)^{r+\delta} w)} \leq C \|g\|_{L^{r',s'}(w)}, \quad (38)$$

or equivalently

$$F : L^{r',s'}(w) \longrightarrow L^{r',s'} \left(\left(\frac{Mw}{w} \right)^{r+\delta} w \right), \quad (39)$$

where

$$Ff = M_{w,c} f \left(\frac{w}{Mw} \right)^{r+\delta-1}. \quad (40)$$

We notice that $Mw \leq (Mw/w)^{r+\delta} w$ for each w . With this, we trivially have

$$F : L^\infty(w) \longrightarrow L^\infty \left(\left(\frac{Mw}{w} \right)^{r+\delta} w \right). \quad (41)$$

Therefore, by Marcinkiewicz's interpolation theorem for Lorentz spaces, it will be enough to show that for some $\varepsilon > 0$

$$F : L^{(r+\varepsilon)'}(w) \longrightarrow L^{(r+\varepsilon)'} \left(\left(\frac{Mw}{w} \right)^{r+\delta} w \right), \quad (42)$$

which amounts to proving

$$\begin{aligned} & \int_{\mathbb{R}^n} \left(M_{w,c} g \left(\frac{w}{Mw} \right)^{r+\delta-1} \right)^{(r+\varepsilon)'} \left(\frac{Mw}{w} \right)^{r+\delta} w dx \\ & \leq C \int_{\mathbb{R}^n} g^{(r+\varepsilon)'} w dx. \end{aligned} \quad (43)$$

Taking $\varepsilon = \delta$ and using the well-known fact that $M_{w,c}$ is bounded on $L^p(w)$, $1 < p < \infty$, with a constant that depends only on p and n , we get (43).

This shows that

$$\|Tf\|_{L^{p,q}(w)}^{p_0} \leq C \|Sf\|_{L^{p,q}((Mw/w)^{p/p_0+\delta} w)}^{p_0}. \quad (44)$$

Taking $\delta = [p/p_0] + 1 - p/p_0 > 0$, we get (7). This ends the proof of Theorem 2. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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