Research Article

The Generalized Green's Function for Boundary Value Problem of Second Order Difference Equation

Xiaoling Han and Juanjuan Huang

Department of Mathematics, Northwest Normal University, Lanzhou 730070, China

Correspondence should be addressed to Juanjuan Huang; hanxiaoling9@163.com

Received 30 June 2014; Accepted 27 July 2014

Academic Editor: Kishin Sadarangani

Copyright © 2015 X. Han and J. Huang. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Let $b > a + 2$ and $[a + 1, b + 1] = \{a + 1, a + 2, \dots, b + 1\}$. In this paper, by building the generalized Green's function for the problems, we study the solvability of the S-L problem $Lx = \Delta[p(t - 1)\Delta x(t - 1)] + [q(t) + \lambda r(t)]x(t) = -f(t)$, $U_1(x) = \alpha_1 x(a) + \alpha_2 \Delta x(a) = 0$, $U_2(x) = \beta_1 x(b + 1) + \beta_2 \Delta x(b + 1) = 0$, and the periodic S-L problem $Lx = \Delta[p(t - 1)\Delta x(t - 1)] + [q(t) + \lambda r(t)]x(t) = -f(t)$, $U_3(x) = x(a) - x(b + 1) = 0$, $U_4(x) = \Delta x(a) - \Delta x(b + 1) = 0$, where the parameter λ is an eigenvalue of the linear problem $Lx = 0$, $U_1(x) = 0$, $U_2(x) = 0$ or the problem $Lx = 0$, $U_3(x) = 0$, $U_4(x) = 0$, and $p : [a, b + 1] \rightarrow (0, +\infty)$, $r : [a + 1, b + 1] \rightarrow (0, +\infty)$, $q(t)$ is defined and real valued on $[a + 1, b + 1]$, $\alpha_1^2 + \alpha_2^2 \neq 0$, $\beta_1^2 + \beta_2^2 \neq 0$, and in the periodic S-L problem we have $p(a) = p(b + 1)$.

1. Introduction

There are many results about the solvability of S-L problem

$$Lx = \Delta[p(t - 1)\Delta x(t - 1)] + [q(t) + \lambda r(t)]x(t) = -f(t),$$

$$U_1(x) = \alpha_1 x(a) + \alpha_2 \Delta x(a) = 0,$$

$$U_2(x) = \beta_1 x(b + 1) + \beta_2 \Delta x(b + 1) = 0 \quad (1)$$

and the periodic S-L problem

$$Lx = \Delta[p(t - 1)\Delta x(t - 1)] + [q(t) + \lambda r(t)]x(t) = -f(t),$$

$$U_3(x) = x(a) - x(b + 1) = 0,$$

$$U_4(x) = \Delta x(a) - \Delta x(b + 1) = 0, \quad (2)$$

where $p : [a, b + 1] \rightarrow (0, +\infty)$, $r : [a + 1, b + 1] \rightarrow (0, +\infty)$, and $q(t)$ is defined and real valued on $[a + 1, b + 1]$, and $[a + 1, b + 1] = \{a + 1, a + 2, \dots, b + 1\}$. Consider $\alpha_1^2 + \alpha_2^2 \neq 0$, $\beta_1^2 + \beta_2^2 \neq 0$, in the periodic S-L problem we have $p(a) = p(b + 1)$.

If the parameter λ is not an eigenvalue of the linear problem

$$Lx = 0, \quad U_1(x) = 0, \quad U_2(x) = 0 \quad (3)$$

or the periodic S-L problem

$$Lx = 0, \quad U_3(x) = 0, \quad U_4(x) = 0; \quad (4)$$

then the linear operator is reversible; we can get an equivalent summation form for the problem (1) or problem (2) by using Green's function; there are many results in this case; we refer readers to [1–4]. However, when λ is an eigenvalue, we cannot get the summation form in usual way. So, in this paper we introduce the concept of generalized Green's function and construct the generalized Green function for the two problems. On this base we still can convert the originally irreversible difference operator into summation form. Moreover, we also obtain the necessary and sufficient condition for the solvability of problems (1) and (2). We refer readers to [5–10] for more details about discrete equation.

2. Preliminaries

We first wish to collect some basic lemmas that will be important to us in what follows.

Definition 1 (see [5]). Let $y(t)$ and $z(t)$ be solutions of $Lx = 0$; then

$$\omega[y(t), z(t)] = y(t)\Delta z(t) - z(t)\Delta y(t). \quad (5)$$

Lemma 2 (see [5]). If $y(t)$ and $z(t)$ are defined on $[a, b+2]$, then

$$\begin{aligned} z(t) Ly(t) - y(t) Lz(t) \\ = \Delta \{p(t-1) \omega [z(t-1), y(t-1)]\} \end{aligned} \quad (6)$$

for $t \in [a+1, b+1]$.

By summing both sides of (6) from $a+1$ to $b+1$, we get the following lemma.

Lemma 3 (see [5]). Assume that $y(t)$ and $z(t)$ are defined on $[a, b+2]$. Then

$$\begin{aligned} \sum_{t=a+1}^{b+1} z(t) Ly(t) - \sum_{t=a+1}^{b+1} y(t) Lz(t) \\ = \{p(t) \omega [z(t), y(t)]\}_a^{b+1}. \end{aligned} \quad (7)$$

Lemma 4 (see [5]). If $y(t)$ and $z(t)$ are solutions of $Lx = 0$, then $\omega[y(t), z(t)] = C/p(t)$, $t \in [a, b+1]$, where C is a constant.

Remark 5. If $y(t)$ and $z(t)$ are linearly dependent, then $\omega[y(t), z(t)] = 0$; if they are linearly independent, then $\omega[y(t), z(t)] \neq 0$.

3. The Main Results for Problem (1)

Theorem 6. The semihomogeneous boundary value problem (1) has a solution if and only if

$$\sum_{\tau=a+1}^{b+1} f(\tau) x_n(\tau) = 0, \quad n = 0, 1, 2, \dots, \quad (8)$$

where $x_n(t)$ is an eigenfunction corresponding to the eigenvalue λ .

Proof. Let $x(t)$ be the solution of problem (1) and $x_n(t)$ the nontrivial solution for homogeneous problem (3); using

$$\begin{aligned} x(t) Lx_n(t) - x_n(t) Lx(t) \\ = \Delta \{p(t-1) \omega [x(t-1), x_n(t-1)]\} \end{aligned} \quad (9)$$

and $Lx = -f(t)$, $Lx_n = 0$, we have

$$\begin{aligned} \sum_{\tau=a+1}^{b+1} f(\tau) x_n(\tau) \\ = \{p(t) \omega [x(t), x_n(t)]\}_a^{b+1} \\ = p(b+1) [x(b+1) \Delta x_n(b+1) - x_n(b+1) \Delta x(b+1)] \\ - p(a) [x(a) \Delta x_n(a) - x_n(a) \Delta x(a)]. \end{aligned} \quad (10)$$

Since $x(t)$ and $x_n(t)$ satisfy the boundary condition of (1), then

$$\sum_{\tau=a+1}^{b+1} f(\tau) x_n(\tau) = 0. \quad (11)$$

On the other hand, choose a solution $x(t)$ for $Lx = -f(t)$, satisfying $x(b+1) = x_n(b+1)$, $\Delta x(b+1) = \Delta x_n(b+1)$; then

$$U_2(x) = U_2(x_n) = \beta_1 x(b+1) + \beta_2 \Delta x(b+1) = 0. \quad (12)$$

By conditions $Lx = -f(t)$, $Lx_n = 0$, $\sum_{\tau=a+1}^{b+1} f(\tau) x_n(\tau) = 0$, and (9), observe that

$$\sum_{\tau=a+1}^{b+1} f(\tau) x_n(\tau) = \{p(t) \omega [x(t), x_n(t)]\}_a^{b+1} = 0; \quad (13)$$

then

$$\begin{aligned} p(b+1) [x(b+1) \Delta x_n(b+1) - x_n(b+1) \Delta x(b+1)] \\ = p(a) [x(a) \Delta x_n(a) - x_n(a) \Delta x(a)]; \end{aligned} \quad (14)$$

that is, $U_1(x) = \alpha_1 x(a) + \alpha_2 \Delta x(a) = 0$. Thus, $x(t)$ is a solution of problem (9), and this completes the proof. $\square$

In what follows, we begin to build the generalized Green function.

Theorem 7. If the S-L problem (3) has a solution $x_0(t)$, and $\sum_{t=a+1}^{b+1} x_0^2(t) = 1$, then there exists a unique function $G(t, \tau)$ which satisfies the following properties:

- (1) $LG(t, \tau) = x_0(t)x_0(\tau)$, $U_1(G) = 0$, $t \in [a, \tau]$;
- (2) $LG(t, \tau) = x_0(t)x_0(\tau)$, $U_2(G) = 0$, $t \in [\tau, b+1]$;
- (3) $\Delta_t G(t, \tau)|_{t=\tau+0} - \Delta_t G(t, \tau)|_{t=\tau-0} = -(1/p(\tau))$;
- (4) $\sum_{t=a+1}^{b+1} G(t, \tau)x_0(t) = 0$.

The function having the above properties is called the generalized Green function of the problem (1).

Proof. Choose another solution $x_1(t)$ of $Lx = 0$ which is linearly independent with $x_0(t)$, such that

$$p(t) [x_0(t) \Delta x_1(t) - x_1(t) \Delta x_0(t)] = 1. \quad (15)$$

Let $\omega(t)$ be any one particular solution of $Lx = x_0(t)x_0(\tau)$. Thus, by the structure of general solution for nonhomogeneous difference equations, we know that

$$G(t, \tau) = \begin{cases} \omega(t) + c_1 x_0(t) + c_2 x_1(t), & t \in [a, \tau], \\ \omega(t) + c_3 x_0(t) + c_4 x_1(t), & t \in [\tau, b+1], \end{cases} \quad (16)$$

where c_i ($i = 1, 2, 3, 4$) can be contained by the properties of $G(t, \tau)$.

Firstly, let $G(t, \tau)$ satisfy the boundary conditions

$$U_1(G) = U_1(\omega) + c_1 U_1(x_0) + c_2 U_1(x_1) = 0, \quad (17)$$

$$U_2(G) = U_2(\omega) + c_3 U_2(x_0) + c_4 U_2(x_1) = 0.$$

By the fact that $U_1(x_0) = 0$ and $U_2(x_0) = 0$, we get that

$$\begin{aligned} U_1(G) &= U_1(\omega) + c_2 U_1(x_1) = 0, \\ U_2(G) &= U_2(\omega) + c_4 U_2(x_1) = 0. \end{aligned} \quad (18)$$

Since $x_0(t)$ and $x_1(t)$ are linearly independent, thus, $U_1(x_1) \neq 0$ and $U_2(x_1) \neq 0$, so we get that c_2 and c_4 are

$$c_2 = -\frac{U_1(\omega)}{U_1(x_1)}, \quad c_4 = -\frac{U_2(\omega)}{U_2(x_1)}; \quad (19)$$

then

$$c_2 - c_4 = \frac{U_2(\omega)}{U_2(x_1)} - \frac{U_1(\omega)}{U_1(x_1)}. \quad (20)$$

However, from (16) and $\Delta_t G(t, \tau)|_{t=\tau+0} - \Delta_t G(t, \tau)|_{t=\tau-0} = -(1/p(\tau))$ we get

$$\begin{aligned} (c_1 - c_3) x_0(\tau) + (c_2 - c_4) x_1(\tau) &= 0, \\ (c_1 - c_3) \Delta x_0(\tau) + (c_2 - c_4) \Delta x_1(\tau) &= \frac{1}{p(\tau)}. \end{aligned} \quad (21)$$

By (21), we get

$$c_1 - c_3 = -x_1(\tau), \quad c_2 - c_4 = x_0(\tau). \quad (22)$$

In the following we will prove that (20) is consistent with (22).
By

$$\begin{aligned} p(a) [x_0(a) \Delta x_1(a) - x_1(a) \Delta x_0(a)] &= 1, \\ p(b+1) [x_0(b+1) \Delta x_1(b+1) \\ &\quad - x_1(b+1) \Delta x_0(b+1)] = 1, \\ \alpha_1 x_0(a) + \alpha_2 \Delta x_0(a) &= 0, \\ \beta_1 x_0(b+1) + \beta_2 \Delta x_0(b+1) &= 0, \end{aligned} \quad (23)$$

we get

$$\begin{aligned} x_0(a) &= \frac{\alpha_2}{p(a) U_1(x_1)}, \\ \Delta x_0(a) &= \frac{-\alpha_1}{p(a) U_1(x_1)}, \\ x_0(b+1) &= \frac{\beta_2}{p(b+1) U_2(x_1)}, \\ \Delta x_0(b+1) &= \frac{-\beta_1}{p(b+1) U_2(x_1)}. \end{aligned} \quad (24)$$

From

$$x_0 L \omega - \omega L x_0 = \Delta [p(x_0 \Delta \omega - \omega \Delta x_0)](t-1) \quad (25)$$

and $\sum_{t=a+1}^{b+1} x_0^2(t) = 1$ we get

$$\begin{aligned} x_0(\tau) &= [p(x_0 \Delta \omega - \omega \Delta x_0)]_a^{b+1} \\ &= p(b+1) [x_0(b+1) \Delta \omega(b+1) \\ &\quad - \omega(b+1) \Delta x_0(b+1)] \\ &\quad - p(a) [x_0(a) \Delta \omega(a) - \omega(a) \Delta x_0(a)]. \end{aligned} \quad (26)$$

Now, we get

$$x_0(\tau) = \frac{U_2(\omega)}{U_2(x_1)} - \frac{U_1(\omega)}{U_1(x_1)} = c_2 - c_4; \quad (27)$$

that is, (20) and (22) are consistent.

In order to determine c_1 and c_3 , by $c_1 - c_3 = -x_1(\tau)$, we obtain $c_1 = c_3 - x_1(\tau)$; putting c_1 into $\sum_{t=a+1}^{b+1} G(t, \tau) x_0(t) = 0$, we get

$$\begin{aligned} \sum_{t=a+1}^{b+1} \omega(t) x_0(t) + c_3 \sum_{t=a+1}^{\tau-1} x_0^2(t) - x_1(\tau) \sum_{t=a+1}^{\tau-1} x_0^2(t) \\ - \frac{U_1(\omega)}{U_1(x_1)} \sum_{t=a+1}^{\tau-1} x_0(t) x_1(t) + c_3 \sum_{t=\tau}^{b+1} x_0^2(t) \\ - \frac{U_2(\omega)}{U_2(x_1)} \sum_{t=\tau}^{b+1} x_0(t) x_1(t) = 0; \end{aligned} \quad (28)$$

then

$$\begin{aligned} c_3 = - \sum_{t=a+1}^{b+1} \omega(t) x_0(t) + x_1(\tau) \sum_{t=a+1}^{\tau-1} x_0^2(t) \\ + \frac{U_1(\omega)}{U_1(x_1)} \sum_{t=a+1}^{\tau-1} x_0(t) x_1(t) \\ + \frac{U_2(\omega)}{U_2(x_1)} \sum_{t=\tau}^{b+1} x_0(t) x_1(t). \end{aligned} \quad (29)$$

All these show that we can get c_1 , c_2 , c_3 , and c_4 uniquely one by one; then $G(t, \tau)$ is unique too; that is, the function $G(t, \tau)$ constructed by the above steps has all the properties of generalized Green's function. This completes the proof. $\square$

Example 8. Consider the generalized Green function of the S-L problem

$$\begin{aligned} \Delta^2 x(t-1) + 2x(t) &= 0, \\ x(0) &= 0, \quad x(2) = 0. \end{aligned} \quad (30)$$

Proof. From the conditions $\sum_{t=1}^2 x_0^2(t) = 1$ and $p(t)[x_0(t) \Delta x_1(t) - x_1(t) \Delta x_0(t)] = 1$, we can get

$$x_0(t) = \sin \frac{\pi}{2} t, \quad x_1(t) = -\cos \frac{\pi}{2} t. \quad (31)$$

It is easy to verify that

$$\omega(t) = \frac{1}{4} \sin \frac{\pi}{2} \tau \sin \frac{\pi}{2} t - \frac{t}{2} \sin \frac{\pi}{2} \tau \cos \frac{\pi}{2} t \quad (32)$$

is a particular solution of the following equation:

$$\Delta^2 \omega(t-1) + 2\omega(t) = \sin \frac{\pi}{2} \tau \sin \frac{\pi}{2} t. \quad (33)$$

Note that

$$\begin{aligned} U_1(x_1) &= x_1(0) = -1, \quad U_1(\omega) = \omega(0) = 0, \\ U_2(x_1) &= x_1(2) = 1, \quad U_2(\omega) = \omega(2) = \sin \frac{\pi}{2} \tau. \end{aligned} \quad (34)$$

So

$$\begin{aligned} c_2 &= 0, & c_4 &= -\sin \frac{\pi}{2} \tau, \\ c_3 &= -\frac{1}{2} \left(\tau \cos \frac{\pi}{2} \tau + \frac{1}{2} \sin \frac{\pi}{2} \tau \right) - \frac{1}{4} \cos \frac{\pi}{2} \tau, \\ c_1 &= c_3 - x_1(\tau) = -\frac{1}{2} \left(\tau \cos \frac{\pi}{2} \tau + \frac{1}{2} \sin \frac{\pi}{2} \tau \right) + \frac{3}{4} \cos \frac{\pi}{2} \tau; \end{aligned} \quad (35)$$

then we can get the generalized Green function

$$G(t, \tau) = \begin{cases} -\frac{t}{2} \sin \frac{\pi}{2} \tau \cos \frac{\pi}{2} t \\ \quad + \frac{3-2\tau}{4} \sin \frac{\pi}{2} t \cos \frac{\pi}{2} \tau, & t \in [0, \tau], \\ \frac{2-t}{2} \sin \frac{\pi}{2} \tau \cos \frac{\pi}{2} t \\ \quad - \frac{2\tau+1}{4} \sin \frac{\pi}{2} t \cos \frac{\pi}{2} \tau, & t \in [\tau, 2]. \end{cases} \quad (36)$$

Now, we prove that by the extended Green function it still can convert the originally irreversible difference operator into reversible form, and its summation form is unique. $\square$

Theorem 9. Let $G(t, \tau)$ be the generalized Green function of the S-L problem (2), and $x_0(t)$ satisfying $\sum_{t=a+1}^{b+1} x_0^2(t) = 1$ is a solution of the S-L problem, and $\sum_{\tau=a+1}^{b+1} f(\tau)x_0(\tau) = 0$. Then the solution of the semihomogeneous boundary value problem

$$Lx = -f(t), \quad U_1(x) = 0, \quad U_2(x) = 0 \quad (37)$$

can be expressed by

$$x(t) = \sum_{\tau=a+1}^{b+1} G(t, \tau) f(\tau). \quad (38)$$

Proof. By the fact that

$$\begin{aligned} \Delta x(t-1) &= \sum_{\tau=a+1}^{b+1} \Delta G(t-1, \tau) f(\tau), \\ \Delta^2 x(t-1) &= \Delta \left[\sum_{\tau=a+1}^{t-1} \Delta G(t-1, \tau) f(\tau) \right. \\ &\quad \left. + \sum_{\tau=t}^{b+1} \Delta G(t-1, \tau) f(\tau) \right] \\ &= \Delta \left\{ \sum_{\tau=a+1}^{t-1} [G(t, \tau) f(\tau) - G(t-1, \tau) f(\tau)] \right. \\ &\quad \left. + \sum_{\tau=t}^{b+1} [G(t, \tau) f(\tau) - G(t-1, \tau) f(\tau)] \right\} \end{aligned}$$

$$\begin{aligned} &= \Delta [G(t, t-1) f(t-1) \\ &\quad - G(t-1, a+1) f(a+1) \\ &\quad + G(t, b+1) f(b+1) - G(t-1, t) f(t)] \\ &= [\Delta G(t, b+1) f(b+1) \\ &\quad - \Delta G(t-1, a+1) f(a+1)] \\ &\quad + [\Delta G(t, t-1) f(t-1) - \Delta G(t-1, t) f(t)] \\ &= \sum_{\tau=a+1}^{b+1} \Delta^2 G(t-1, \tau) f(\tau) \\ &\quad + [\Delta_t G(\tau, t)|_{\tau=t+0} - \Delta_t G(\tau, t)|_{\tau=t-0}] f(t) \\ &= \sum_{\tau=a+1}^{b+1} \Delta^2 G(t-1, \tau) f(\tau) - \frac{f(t)}{p(t)}, \end{aligned} \quad (39)$$

then

$$\begin{aligned} Lx &= \Delta p(t-1) \Delta x(t-1) + p(t) \Delta^2 x(t-1) \\ &\quad + q(t) x(t) + \lambda r(t) x(t) \\ &= \sum_{\tau=a+1}^{b+1} (LG) f(\tau) - f(t) \\ &= \sum_{\tau=a+1}^{b+1} x_0(t) x_0(\tau) f(\tau) - f(t) = -f(t). \end{aligned} \quad (40)$$

To prove the uniqueness, it needs to be proved that we cannot find $\tilde{x}(t)$ and $f(t)$ which satisfy $\sum_{\tau=a+1}^{b+1} f(\tau)x_0(\tau) = 0$ except $x_0(t)$ and $f(t)$; if there exists $\tilde{x}(t)$, then by Theorem 6, we know that $\tilde{x}(t)$ is also an eigenfunction of the S-L problem; this shows that $\tilde{x}(t)$ and $x_0(t)$ are linearly dependent, which completes the proof. $\square$

4. The Main Results for Problem (2)

In what follows, we build the generalized Green function of the periodic S-L problem.

Theorem 10. If the periodic S-L problem (2) has a solution $x_0(t)$, and choosing a linearly independent solution $x_1(t)$ with $x_0(t)$ for $Lx = 0$, then there exists a function $G(t, \tau)$ which satisfies the following properties:

- (1) $LG(t, \tau) = x_0(t)x_0(\tau)$, $U_3(G) = 0$, $t \in [a, \tau]$;
- (2) $LG(t, \tau) = x_0(t)x_0(\tau)$, $U_4(G) = 0$, $t \in [\tau, b+1]$;
- (3) $\Delta_t G(t, \tau)|_{t=\tau+0} - \Delta_t G(t, \tau)|_{t=\tau-0} = -(1/p(\tau))$;
- (4) $\sum_{t=a+1}^{b+1} G(t, \tau)x_0(t) = 0$ and $\sum_{t=a+1}^{b+1} G(t, \tau)x_1(t) = 0$.

The function having the above properties is called the generalized Green function to the problem (2).

Proof. Choose $x_1(t)$ so that

$$p(t) [x_0(t) \Delta x_1(t) - x_1(t) \Delta x_0(t)] = 1. \quad (41)$$

Let $\omega(t)$ be any one particular solution to $Lx = x_0(t)x_0(\tau)$; then

$$G(t, \tau) = \begin{cases} \omega(t) + c_1 x_0(t) + c_2 x_1(t), & t \in [a, \tau], \\ \omega(t) + c_3 x_0(t) + c_4 x_1(t), & t \in [\tau, b+1], \end{cases} \quad (42)$$

where c_i ($i = 1, 2, 3, 4$) can be contained by the properties of $G(t, \tau)$.

Firstly, let $G(t, \tau)$ satisfy the boundary conditions

$$\begin{aligned} U_3(G) &= U_3(\omega) + c_1 U_3(x_0) + c_2 U_3(x_1) = 0, \\ U_4(G) &= U_4(\omega) + c_3 U_4(x_0) + c_4 U_4(x_1) = 0. \end{aligned} \quad (43)$$

By the fact that $U_3(x_0) = 0$, $U_4(x_0) = 0$ and in the periodic problem we have $U_3(x_1) = U_4(x_1) = 0$; then $U_3(\omega) = U_4(\omega) = 0$.

Thus, c_i can be contained; that is, by the conditions (3) and (4), we get

$$c_1 - c_3 = -x_1(\tau), \quad c_2 - c_4 = x_0(\tau). \quad (44)$$

In order to determine c_i , by $c_1 - c_3 = -x_1(\tau)$, we have $c_1 = c_3 - x_1(\tau)$; by $c_2 - c_4 = x_0(\tau)$, we have $c_2 = c_4 + x_0(\tau)$; putting c_1, c_2 into $\sum_{t=a+1}^{b+1} G(t, \tau)x_0(t) = 0$ and $\sum_{t=a+1}^{b+1} G(t, \tau)x_1(t) = 0$ respectively, it shows that

$$\begin{aligned} & c_3 \sum_{t=a+1}^{b+1} x_0^2(t) + c_4 \sum_{t=a+1}^{b+1} x_0(t) x_1(t) \\ &= - \sum_{t=a+1}^{b+1} \omega(t) x_0(t) + x_1(\tau) \sum_{t=a+1}^{\tau-1} x_0^2(t) \\ & \quad - x_0(\tau) \sum_{t=a+1}^{\tau-1} x_0(t) x_1(t), \\ & c_3 \sum_{t=a+1}^{b+1} x_0(t) x_1(t) + c_4 \sum_{t=a+1}^{b+1} x_1^2(t) \\ &= - \sum_{t=a+1}^{b+1} \omega(t) x_1(t) + x_1(\tau) \sum_{t=a+1}^{\tau-1} x_0(t) x_1(t) \\ & \quad - x_0(\tau) \sum_{t=a+1}^{\tau-1} x_1^2(t). \end{aligned} \quad (45)$$

Combine the above equations; c_3 and c_4 can be determined; then c_1 and c_2 can be determined, so $G(t, \tau)$ also can be determined; that is, the function $G(t, \tau)$ set by the above steps has all the properties of generalized Green's function. This completes the proof. $\square$

Remark 11. Because the choice of $x_0(t)$ in the above structure of generalized Green's function is not unique, so the generalized Green function of periodic S-L problem (2) is changed with $x_0(t)$, to have different form.

Example 12. Consider the generalized Green function of the periodic S-L problem

$$\begin{aligned} \Delta^2 x(t-1) + 2x(t) &= 0, \\ x(0) &= x(4), \quad \Delta x(0) = \Delta x(4). \end{aligned} \quad (46)$$

Proof. Let $x_0(t) = \sin(\pi/2)t$, and from

$$p(t) [x_0(t) \Delta x_1(t) - x_1(t) \Delta x_0(t)] = 1, \quad (47)$$

we can let $x_1(t) = -\cos(\pi/2)t$, (where $p(t) = 1$). It is easy to verify that

$$\omega(t) = \frac{1}{4} \sin \frac{\pi}{2} \tau \sin \frac{\pi}{2} t - \frac{t}{2} \sin \frac{\pi}{2} \tau \cos \frac{\pi}{2} t \quad (48)$$

is a particular solution of the following equation:

$$\Delta^2 \omega(t-1) + 2\omega(t) = \sin \frac{\pi}{2} \tau \sin \frac{\pi}{2} t. \quad (49)$$

By Theorem 10, we have

$$\begin{aligned} c_3 &= -\frac{1}{4} \sin \frac{\pi}{2} \tau - \frac{1}{2} \cos \frac{\pi}{2} \tau \left(\frac{\tau}{2} + \frac{1}{4} \cos \pi \tau - \frac{1}{4} \right) \\ & \quad - \frac{1}{8} \sin \frac{\pi}{2} \tau \sin \pi \tau, \\ c_4 &= -\frac{1}{8} \cos \frac{\pi}{2} \tau \sin \pi \tau - \frac{1}{2} \sin \frac{\pi}{2} \tau \left(-\frac{1}{4} \cos \pi \tau + \frac{\tau}{2} + \frac{9}{4} \right); \end{aligned} \quad (50)$$

then

$$\begin{aligned} c_1 &= -\frac{1}{4} \sin \frac{\pi}{2} \tau - \frac{1}{2} \cos \frac{\pi}{2} \tau \left(\frac{\tau}{2} - \frac{9}{4} + \frac{1}{4} \cos \pi \tau \right) \\ & \quad - \frac{1}{8} \sin \frac{\pi}{2} \tau \sin \pi \tau, \\ c_2 &= -\frac{1}{8} \cos \frac{\pi}{2} \tau \sin \pi \tau - \frac{1}{2} \sin \frac{\pi}{2} \tau \left(-\frac{1}{4} \cos \pi \tau + \frac{\tau}{2} + \frac{1}{4} \right). \end{aligned} \quad (51)$$

So we can get the generalized Green function

$$G(t, \tau) = \begin{cases} -\frac{1}{2} \sin \frac{\pi}{2} t \cos \frac{\pi}{2} \tau \left(\frac{\tau}{2} + \frac{1}{4} \cos \pi \tau - \frac{9}{4} \right) - \frac{1}{8} \sin \frac{\pi}{2} \tau \sin \pi \tau \sin \frac{\pi}{2} t \\ \quad + \frac{1}{8} \cos \frac{\pi}{2} t \cos \frac{\pi}{2} \tau \sin \pi \tau \\ \quad + \frac{1}{2} \sin \frac{\pi}{2} \tau \cos \frac{\pi}{2} t \left(-\frac{1}{4} \cos \pi \tau + \frac{\tau}{2} + \frac{1}{4} - t \right), & t \in [0, \tau], \\ -\frac{1}{2} \sin \frac{\pi}{2} t \cos \frac{\pi}{2} \tau \left(\frac{\tau}{2} - \frac{1}{4} + \frac{1}{4} \cos \pi \tau \right) - \frac{1}{8} \sin \frac{\pi}{2} \tau \sin \pi \tau \sin \frac{\pi}{2} t \\ \quad + \frac{1}{8} \cos \frac{\pi}{2} t \cos \frac{\pi}{2} \tau \sin \pi \tau \\ \quad + \frac{1}{2} \sin \frac{\pi}{2} \tau \cos \frac{\pi}{2} t \left(-\frac{1}{4} \cos \pi \tau + \frac{\tau}{2} + \frac{9}{4} - t \right), & t \in [\tau, 4]. \end{cases} \quad (52)$$

However, if we put $x_0(t) = -\cos(\pi/2)t$, then the generalized Green function is

$$G(t, \tau) = \begin{cases} \frac{1}{2} \sin \frac{\pi}{2} t \cos \frac{\pi}{2} \tau \left(\frac{1}{4} \cos \pi \tau - \frac{17-t-2\tau}{4} \right) - \frac{1}{8} \cos \frac{\pi}{2} \tau \sin \pi \tau \cos \frac{\pi}{2} t \\ \quad + \frac{1}{8} \sin \frac{\pi}{2} t \sin \frac{\pi}{2} \tau \sin \pi \tau \\ \quad + \frac{1}{4} \cos \frac{\pi}{2} \tau \cos \frac{\pi}{2} t \left(-\frac{1}{4} \cos \pi \tau + \frac{\tau}{2} - \frac{15}{4} \right), & t \in [0, \tau], \\ \frac{1}{2} \sin \frac{\pi}{2} t \cos \frac{\pi}{2} \tau \left(\frac{2\tau-9+4t}{4} + \frac{1}{4} \cos \pi \tau \right) - \frac{1}{8} \cos \frac{\pi}{2} \tau \sin \pi \tau \cos \frac{\pi}{2} t \\ \quad + \frac{1}{8} \sin \frac{\pi}{2} t \sin \frac{\pi}{2} \tau \sin \pi \tau \\ \quad - \frac{1}{2} \sin \frac{\pi}{2} \tau \cos \frac{\pi}{2} t \left(-\frac{1}{4} \cos \pi \tau + \frac{\tau}{2} - \frac{3}{4} \right), & t \in [\tau, 4]. \end{cases} \quad (53)$$

□

Be similar to Theorem 9, we can get the following result; however, because the generalized Green function of the periodic S-L problem (2) is not unique, summation form of difference operator is also not unique.

Theorem 13. *Let $G(t, \tau)$ be the generalized Green function of the periodic S-L problem (2); then the solution of the semihomogeneous boundary value problem*

$$Lx = -f(t), \quad U_3(x) = 0, \quad U_4(x) = 0 \quad (54)$$

can be expressed by

$$x(t) = \sum_{\tau=a+1}^{b+1} G(t, \tau) f(\tau). \quad (55)$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The authors are grateful to the anonymous referees for their valuable suggestions. This research was supported by the National Natural Science Foundation of China (11101335 and 11201378).

References

- [1] D. Bai and Y. Xu, "Nontrivial solutions of boundary value problems of second-order difference equations," *Journal of Mathematical Analysis and Applications*, vol. 326, no. 1, pp. 297–302, 2007.

- [2] G. Zhang and Z. Yang, "Positive solutions of a general discrete boundary value problem," *Journal of Mathematical Analysis and Applications*, vol. 339, no. 1, pp. 469–481, 2008.
- [3] R. Ma, Y. Lu, and T. Chen, "Existence of one-signed solutions of discrete second-order periodic boundary value problems," *Abstract and Applied Analysis*, vol. 2012, Article ID 437912, 13 pages, 2012.
- [4] C. Gao, "Necessary and sufficient condition for the existence of solutions to a discrete second-order boundary value problem," *Abstract and Applied Analysis*, vol. 2012, Article ID 951251, 12 pages, 2012.
- [5] W. G. Kelley and A. C. Peterson, *Difference Equations: An Introduction with Applications*, Academic Press, New York, NY, USA, 2001.
- [6] C. S. Goodrich, "Continuity of solutions to discrete fractional initial value problems," *Computers & Mathematics with Applications*, vol. 59, no. 11, pp. 3489–3499, 2010.
- [7] C. S. Goodrich, "Solutions to a discrete right-focal fractional boundary value problem," *International Journal of Difference Equations*, vol. 5, no. 2, pp. 195–216, 2010.
- [8] C. S. Goodrich, "On a discrete fractional three-point boundary value problem," *Journal of Difference Equations and Applications*, vol. 18, no. 3, pp. 397–415, 2010.
- [9] R. Agarwal and P. Y. Wong, *Advanced Topics in Difference Equations*, Springer, New York, NY, USA, 1997.
- [10] C. Gao and J. Xu, "Bifurcation techniques and positive solutions of discrete Lidstone boundary value problems," *Applied Mathematics and Computation*, vol. 218, no. 2, pp. 434–444, 2011.