Research Article

Choi-Davis-Jensen Inequalities in Semifinite von Neumann Algebras

Turdebek N. Bekjan,¹ Kordan N. Ospanov,² and Asilbek Zulkhazhav²

¹College of Mathematics and Systems Science, Xinjiang University, Urumqi 830046, China
²L.N. Gumilyov Eurasian National University, Astana 010008, Kazakhstan

Correspondence should be addressed to Turdebek N. Bekjan; bekjant@yahoo.com

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We prove the Choi-Davis-Jensen type submajorization inequalities on semifinite von Neumann algebras for concave functions and convex functions.

1. Introduction

Let $A$ and $B$ be $C^*$-algebras, and let $\Phi$ be a linear map between $A$ and $B$. It is said to be positive, if for all positive operators $A \in A$ we have $\Phi(A) \geq 0$. If for all strictly positive operators $A \in A$ ($A > 0$), it follows that $\Phi(A)$ is strictly positive, then $\Phi$ is said to be strictly positive. $\Phi$ is called unital if $\Phi(1) = 1$, where $1$ denotes the unities of the algebras.

Davis [1] and Choi [2] showed that if $\Phi$ is a unital positive linear map on $B(H)$ and if $f$ is an operator convex function on an interval $I$, then the so-called Choi-Davis-Jensen inequality

$$f(\Phi(A)) \leq \Phi(f(A))$$

(1)

holds for every self-adjoint operator $A$ on $H$ whose spectrum is contained in $I$, where $B(H)$ is the $C^*$-algebra of all bounded linear operators on Hilbert space $H$. Khosravi et al. [3] proved that (1) still holds for positive linear map $\Phi: A \to B(H)$ with $0 < \Phi(1) \leq 1$. Antezana et al. [4] obtained the following type of Choi-Davis-Jensen inequality. Let $A, B$ be unital $C^*$-algebras, $\Phi: A \to B$ a positive unital linear map, $f$ a convex function defined on an open interval $I$, and $A \in A$ such that $A$ is self-adjoint and $\sigma(A) \subset I$. If $B$ is a von Neumann algebra and $f$ is monotone, then $f(\Phi(A)) \leq \Phi(f(A))$ (spectral preorder). One can find some related results to these topics in [5–8].

In [3], the authors proved the following refinement of the Choi-Davis-Jensen inequality: let $\Phi_1, \ldots, \Phi_n$ be strictly positive linear maps from a unital $C^*$-algebra $A$ into a unital $C^*$-algebra $B$ and let $\Phi = \sum_{j=1}^n \Phi_j$ be unital. If $f$ is an operator convex function on an interval $I$, then for every self-adjoint operator $A \in A$ with spectrum contained in $I$,

$$f(\Phi_j(A)) = \sum_{i=1}^n \Phi_i(f(\Phi_j(A))) \leq \sum_{i=1}^n \Phi_i(f(A)) = \Phi(f(A))$$

(2)

The purpose of this paper is to extend (2) for measurable operators and for convex functions. Let $\mathcal{A}, \mathcal{M}$ be semifinite von Neumann algebras, $\Phi_1, \ldots, \Phi_n$ positive linear continuous maps from $L_0(\mathcal{A})$ into $L_0(\mathcal{M})$ such that $\Phi_1|_{\mathcal{A}}, \ldots, \Phi_n|_{\mathcal{A}}$ are positive linear maps from $\mathcal{A}$ into $\mathcal{M}$, and $\Phi = \sum_{j=1}^n \Phi_j$ unital, and let $x_k \in L_0(\mathcal{A})^+$ ($k = 1, 2, \ldots, n$). We will prove

$$\sum_{k=1}^n \Phi_k(f(x_k)) \leq f\left(\sum_{k=1}^n \Phi_k(x_k)\right)$$

(3)

for any concave function $f: [0, \infty) \to [0, \infty)$ and

$$g\left(\sum_{k=1}^n \Phi_k(x_k)\right) \leq \sum_{k=1}^n \Phi_k(g(x_k))$$

(4)
for any convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$, where "$\preceq$" mains submajorization.

This paper is organized as follows. Section 2 contains some preliminary definitions. In Section 3, we prove the main result and related results.

2. Preliminaries

We use standard notions from theory of noncommutative $L_p$-spaces. Our main references are [9, 10] (see also [9] for more historical references). Throughout the paper, let $\mathcal{M}$ be a semifinite von Neumann algebra acting on a Hilbert space $\mathcal{H}$ with a normal semifinite faithful trace $\tau$. Let $L_0(\mathcal{M})$ denote the topological $\ast$-algebra of measurable operators with respect to $(\mathcal{M}, \tau)$. The topology of $L_0(\mathcal{M})$ is determined by the convergence in measure. The trace $\tau$ can be extended to the positive cone $L_0^+(\mathcal{M})$ of $L_0(\mathcal{M})$:

$$\tau(x) = \int_0^\infty \lambda \, d\tau(e_\lambda(x)), \quad (5)$$

where $x = \int_0^\infty \lambda \, d\mathcal{E}_\lambda(x)$ is the spectral decomposition of $x$.

For $x \in L_0(\mathcal{M})$ we define

$$\lambda_s(x) = \tau(e_s^+(\lambda(x))) \quad (s > 0),$$

$$\mu_t(x) = \inf \{ s > 0 : \lambda_s(x) \leq t \} \quad (t > 0), \quad (6)$$

where $e_s^+(\lambda(x)) = e_{(s,\infty)}(\lambda(x))$ is the spectral projection of $\lambda(x)$ associated with the interval $(s, \infty)$. The function $s \mapsto \lambda_s(x)$ is called the distribution function of $x$ and $\mu_t(x)$ is the generalized $s$-number of $x$. We will denote simply by $\lambda(x)$ and $\mu(x)$ the functions $s \mapsto \lambda_s(x)$ and $t \mapsto \mu_t(x)$, respectively.

It is easy to check that both are decreasing and continuous from the right on $(0, \infty)$. For further information we refer the reader to [11].

If $x, y \in L_0(\mathcal{M})$, then we say that $y$ is submajorised by $x$ (in the sense of Hardy, Littlewood, and Polya) and write $y \preceq x$ if and only if

$$\int_0^t \mu_s(y) \, ds \leq \int_0^t \mu_s(x) \, ds, \quad \forall t > 0. \quad (7)$$

We remark that if $\mathcal{M} = \mathcal{B}(\mathcal{H})$ and $\tau$ is the standard trace, then

$$\mu_t(x) = \tau(e_{[t,\infty)}(x)), \quad t \in [j-1, j), \quad j = 1, 2, \ldots \quad (8)$$

and if $x, y \in \mathcal{M}$, then $y \preceq x$ is equivalent to

$$\sum_{j=1}^k s_j(y) \leq \sum_{j=1}^k s_j(x), \quad 1 \leq k \leq n. \quad (9)$$

For further information we refer the reader to [11–13].

Let $x, y$ be self-adjoint elements of $\mathcal{M}$; we say that $x$ spectrally dominates $y$, denoted by $y \preceq x$, if $e_{(\alpha,\infty)}(y)$ is equivalent, in the sense of Murray-von Neumann, to a subprojection of $e_{(\alpha,\infty)}(x)$ for every real number $\alpha$ (see [6]). It is clear that if $x$ spectrally dominates $y$, then $y$ is submajorised by $x$.

3. Main Results

Lemma 1. Let $\mathcal{N}$ and $\mathcal{M}$ be semifinite von Neumann algebras. Let $\Phi$ be a positive linear continuous map from $L_0(\mathcal{N})$ into $L_0(\mathcal{M})$ such that the restriction of $\Phi$ on $\mathcal{N}$ is a unital positive linear map from $\mathcal{N}$ into $\mathcal{M}$.

(i) If $f : [0, +\infty) \to [0, +\infty)$ is a concave function, then

$$\Phi(f(x)) \leq f(\Phi(x)), \quad \forall x \in L_0(\mathcal{N})^+. \quad (10)$$

(ii) If $g : [0, +\infty) \to [0, +\infty)$ is a convex function with $g(0) = 0$, then

$$g(\Phi(x)) \leq \Phi(g(x)), \quad \forall x \in L_0(\mathcal{N})^+. \quad (11)$$

Proof. (i) We may assume $f(0) = 0$. It implies $f$ is nondecreasing. First assume that $x \in \mathcal{N}^+$. We use same method as in the proof of Theorem 2 [4] (see Remark 3.2 in [4]). Let $a > 0$. If $\xi \in e_{[t,f(\xi)]\leq a}(\Phi(\mathcal{N}))(\mathcal{H}) \cap e_{(\alpha,\infty)}(\Phi(f(\xi)))(\mathcal{H})$ with $\|\xi\| = 1$, then $f(\Phi(\mathcal{N})(\xi), \xi_\alpha) \leq a$ and $(\Phi(f(\xi))(\xi_\alpha))_\alpha > a$. On the other hand, using Jensen's inequality for the state $(\Phi(\mathcal{N})(\xi_\alpha), \xi_\alpha)$ and nondecreasing concave function $f$, we get $(\Phi(f(\xi))(\xi_\alpha), \xi_\alpha) \leq f((\Phi(\mathcal{N})(\xi_\alpha)), \xi_\alpha) \leq a$. Therefore $e_{[t,f(\xi)]\leq a}(\Phi(x))(\mathcal{H}) \cap e_{(\alpha,\infty)}(\Phi(f(\xi)))(\mathcal{H}) = \{ 0 \}$. Thus

$$e_{(\alpha,\infty)}(\Phi(f(\xi))) = e_{[t,f(\xi)]\leq a}(\Phi(x)) \wedge e_{(\alpha,\infty)}(\Phi(f(\xi))) = 0$$

that is, $\Phi(f(\xi)) \leq f(\Phi(\xi))$. Hence (10) holds.

Now let $x \in L_0(\mathcal{N})^+$. For each $m = 1, 2, \ldots$, observe that $x \wedge m 1 \in \mathcal{M}$, and so, using the first case, it follows that

$$\Phi(f(x \wedge m 1)) \leq f(\Phi(x \wedge m 1)). \quad (13)$$

Using the functional calculus and Corollary 1.2 in [13] observe that

$$x \wedge m 1 \uparrow_m x, \quad f(x \wedge m 1) \uparrow_m f(x) \quad (14)$$

and so, by continuity of $\Phi$, it follows that

$$\Phi(x \wedge m 1) \uparrow_m \Phi(x), \quad \Phi(f(x \wedge m 1)) \uparrow_m \Phi(f(x)). \quad (15)$$
Using (vi) of Lemma 2.5 in [11] we obtain that
\[
\int_0^t \mu_s(f(\Phi(x))) \, ds = \int_0^t f(\mu_s(\Phi(x))) \, ds = \int_0^t \mu_s(\Phi(f(x))) \, ds, \quad \forall t > 0;
\]
that is, (10) holds.

(ii) The proof is similar to the proof of (i).

**Theorem 2.** Let \( N, M \) be semifinite von Neumann algebras, \( \Phi_1, \ldots, \Phi_n \) positive linear continuous maps from \( L_0(N) \) into \( L_0(M) \) such that the restriction of \( \Phi_k \) on \( N \) is a positive linear map from \( N \) into \( M \) \((k = 1, 2, \ldots, n)\), and \( \Phi = \sum_{j=1}^n \Phi_j \) unital.

(i) If \( f : [0, +\infty) \rightarrow [0, +\infty) \) is a concave function, then, for \( x_k \in L_0(N) \) \((k = 1, 2, \ldots, n)\),
\[
\sum_{k=1}^n \Phi_k(f(x_k)) \leq f\left( \sum_{k=1}^n \Phi_k(x_k) \right). \quad (17)
\]

(ii) If \( g : [0, +\infty) \rightarrow [0, +\infty) \) is a convex function with \( g(0) = 0 \), then, for \( x_k \in L_0(N) \) \((k = 1, 2, \ldots, n)\),
\[
g\left( \sum_{k=1}^n \Phi_k(x_k) \right) \leq \sum_{k=1}^n \Phi_k(g(x_k)). \quad (18)
\]

**Proof.** Let \( N_n \) be the von Neumann algebra:
\[
N_n = \left\{ \begin{pmatrix} x_1 & 0 & \cdots & 0 \\ 0 & x_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_n \end{pmatrix} : x_k \in N, \quad k = 1, 2, \ldots, n \right\}
\]
\[
= 1, 2, \ldots, n
\]
on Hilbert space \( H \oplus \cdots \oplus H \). Define \( \Phi : N_n \rightarrow M \) by
\[
\Phi \left( \begin{pmatrix} x_1 & 0 & \cdots & 0 \\ 0 & x_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_n \end{pmatrix} \right) = \sum_{k=1}^n \Phi_k(x_k); \quad (20)
\]
then \( \Phi \) is a unital positive linear map from \( N_n \) into \( M \). By Lemma 1, we obtain desired result.

Using Theorem 5.3 in [14] and Theorem 2 we obtain the following.

**Proposition 3.** Let \( N, M \) be semifinite von Neumann algebras, \( \Phi_1, \ldots, \Phi_n \) positive linear continuous maps from \( L_0(N) \) into \( L_0(M) \) such that the restriction of \( \Phi_k \) on \( N \) is a positive linear map from \( N \) into \( M \) \((k = 1, 2, \ldots, n)\), and \( \Phi = \sum_{j=1}^n \Phi_j \) unital.

(i) If \( f : [0, +\infty) \rightarrow [0, +\infty) \) is a concave function, then, for \( x_k \in N^+(k = 1, 2, \ldots, n)\),
\[
\sum_{k=1}^n \Phi_k(f(x_k)) \leq f\left( \sum_{k=1}^n \Phi_k(x_k) \right) \leq \sum_{k=1}^n f(\Phi_k(x_k)). \quad (21)
\]

(ii) If \( g : [0, +\infty) \rightarrow [0, +\infty) \) is a convex function with \( g(0) = 0 \), then, for \( x_k \in N^+(k = 1, 2, \ldots, n)\),
\[
\sum_{k=1}^n g\left( \Phi_k(x_k) \right) \leq g\left( \sum_{k=1}^n \Phi_k(x_k) \right) \leq \sum_{k=1}^n g(\Phi_k(x_k)). \quad (22)
\]

**Proposition 4.** Let \( N, M \) be semifinite von Neumann algebras and \( \Phi_1, \ldots, \Phi_n \) positive linear continuous maps from \( L_0(N) \) into \( L_0(M) \) such that the restriction of \( \Phi_k \) on \( N \) is a positive linear map from \( N \) into \( M \) \((k = 1, 2, \ldots, n)\). Suppose \( \Phi = \sum_{j=1}^n \Phi_j \) is unital trace-preserving positive linear map from \( N \) into \( M \). If \( g : [0, +\infty) \rightarrow [0, +\infty) \) is a convex function with \( g(0) = 0 \), then
\[
\sum_{k=1}^n g\left( \Phi_k(x_k) \right) \leq g\left( \sum_{k=1}^n \Phi_k(x_k) \right) \leq \sum_{k=1}^n g(\Phi_k(x_k)). \quad (23)
\]

**Proof.** By Corollary 2.9 in [15] we have that \( \sum_{j=1}^n \Phi_j \) is a trace-preserving positive contraction. Using Theorem 5.3 in [14], Lemma 3.1 in [16] (it is also holds for the semifinite case), and Theorem 2 we obtain the desired result.

**Corollary 5.** Let \( a_k \in M \) \((k = 1, 2, \ldots, n)\) and \( \sum_{k=1}^n a_k^*a_k = 1 \).

(i) If \( f : [0, +\infty) \rightarrow [0, +\infty) \) is a concave function, then, for \( x_k \in L_0(M)^+(k = 1, 2, \ldots, n)\),
\[
\sum_{k=1}^n a_k^*f(x_k) a_k \leq f\left( \sum_{k=1}^n a_k^*a_k x_k \right) \leq \sum_{k=1}^n f(a_k^*x_k a_k). \quad (24)
\]

(ii) If \( g : [0, +\infty) \rightarrow [0, +\infty) \) is a convex function with \( g(0) = 0 \), then
\[
\sum_{k=1}^n g(a_k^*x_k a_k) \leq g\left( \sum_{k=1}^n a_k^*a_k x_k \right) \leq \sum_{k=1}^n g(a_k^*x_k a_k) \quad (25)
\]

Let \( M_n \) be von Neumann algebra of \( n \times n \) complex matrices, and let \( P_1, P_2, \ldots, P_r \) be a family of mutually orthogonal
projections in $\mathbb{C}^n$ such that $\varphi_{f_j} = P_j = I$, where $I$ is unit matrix in $M_n$. Then the operation of taking $A$ to $\varphi(A) = \sum_{j=1}^r P_j AP_j$ is called a pinching of $A$. The pinching $\varphi : M_n \to M_n$ is a trace-preserving positive map (see [17, 18]).

**Proposition 6.** Let $\varphi : M_n \to M_n$ be a pinching.

(i) If $f : [0, +\infty) \to [0, +\infty)$ is a concave function, then
$$\varphi(f(A)) \leq f(\varphi(A)), \quad \forall A \in M_n^+.$$  \hspace{1cm} (26)

(ii) If $g : [0, +\infty) \to [0, +\infty)$ is a convex function with $g(0) = 0$, then
$$g(\varphi(A)) \leq \varphi(g(A)) \leq g(A), \quad \forall A \in M_n^+.$$  \hspace{1cm} (27)

Let $x = (x_k)_{k \geq 1}$ be a sequence in $L_0(\mathcal{M})$. Define
$$\text{diag}(x_k) = \begin{pmatrix} x_1 & 0 & \cdots & 0 \\ 0 & x_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_k \end{pmatrix}.$$  \hspace{1cm} (28)

**Proposition 7.** Let $x = (x_k)_{k \geq 1}$ be a sequence in $L_0(\mathcal{M})$.

(i) If $f : [0, +\infty) \to [0, +\infty)$, where $f(\sqrt{t})$ is a concave function, then
$$\left\| f\left( \sum_{k \geq 1} |x_k|^2 \right)^{1/2} \right\|_p \leq \sum_{k \geq 1} \left\| f(|x_k|) \right\|^{1/p},$$  \hspace{1cm} (29)

$$\forall 0 < p \leq 1.$$  

(ii) If $g : [0, +\infty) \to [0, +\infty)$, where $g(\sqrt{t})$ is a convex function with $g(0) = 0$, then
$$\left\| \sum_{k \geq 1} g(|x_k|) \right\|^{1/p} \leq g\left( \left( \sum_{k \geq 1} |x_k|^2 \right)^{1/2} \right\|_p,$$  \hspace{1cm} (30)

$$\forall 1 \leq p < \infty.$$  

**Proof.** (i) Since
$$\begin{pmatrix} |x_1|^2 & 0 & \cdots \\ 0 & |x_2|^2 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} = \varphi\begin{pmatrix} |x_1|^2 & x_1 x_2^* & \cdots \\ x_2^* x_1 & |x_2|^2 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$  \hspace{1cm} (31)

and $f(\sqrt{t})^p$ is concave, by Lemma 1, we get
$$\varphi\begin{pmatrix} f^p\left( \begin{pmatrix} x_1^* & x_2^* & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \right) \\ f^p\left( \begin{pmatrix} x_1^* & x_2^* & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \right) \\ \vdots \end{pmatrix} \leq \begin{pmatrix} f(|x_1|^p) & 0 & \cdots \\ 0 & f(|x_2|^p) & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}.$$  \hspace{1cm} (32)

Hence
$$\left\| f\left( \sum_{k \geq 1} |x_k|^2 \right)^{1/2} \right\|_p \leq \sum_{k \geq 1} \left\| f(|x_k|) \right\|^{1/p},$$  \hspace{1cm} (33)

Using the same arguments, we can prove (ii).

**Conflict of Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References


