Research Article

New Applications of Nunokawa's Lemmas

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We derive several univalent and p -valent conditions on certain meromorphic functions by applying well-known Nunokawa's lemmas.

1. A Sufficient Condition on p -Valent Analytic Functions

Let a function f be univalent in the unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Then, f maps U onto a starlike domain with respect to the origin if and only if

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > 0, \quad z \in U. \quad (1)$$

If a function f is p -valent in U and satisfies (1), then f is called p -valent starlike in U .

In [1], Ozaki proved that if

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad (p \in \mathbb{N} = \{1, 2, 3, \dots\}) \quad (2)$$

is analytic in U and for some real α ,

$$\operatorname{Re} \left\{ \exp(i\alpha) f^{(p)}(z) \right\} > 0, \quad z \in U, \quad (3)$$

then f is at most p -valent in U . Ozaki's p -valent condition is a generalization of the well-known Noshiro-Warschawski univalence condition. Very recently, Nunokawa, Cho, Kwon, and Sokól (an extension of Ozaki's p -valent conditions, preprint) considered similar problems in the unit disc U . In this section, we will extend these p -valent conditions.

To prove our results, we need the following well-known Nunokawa lemmas.

Lemma 1 (see [2, 3]). Let $p(z)$ be analytic function in U of the form

$$p(z) = 1 + \sum_{n=m}^{\infty} c_n z^n, \quad c_m \neq 0, \quad m \geq 1, \quad (4)$$

with $p(z) \neq 0$ in U . If there exists a point z_0 , $|z_0| < 1$, such that

$$|\arg \{p(z)\}| < \frac{\pi\varphi}{2} \quad (5)$$

for $|z| < |z_0|$ and

$$|\arg \{p(z_0)\}| = \frac{\pi\varphi}{2} \quad (6)$$

for some $\varphi > 0$, then

$$\frac{z_0 p'(z_0)}{p(z_0)} = i l \varphi, \quad (7)$$

where

$$l \geq \frac{m}{2} \left(a + \frac{1}{a} \right) \geq m$$

$$\text{when } \arg \{p(z_0)\} = \frac{\pi\varphi}{2},$$

$$l \leq -\frac{m}{2} \left(a + \frac{1}{a} \right) \leq -m \quad (8)$$

$$\text{when } \arg \{p(z_0)\} = -\frac{\pi\varphi}{2},$$

$$\{p(z_0)\}^{1/\varphi} = \pm ia, \quad a > 0.$$

Lemma 2 (see [4]). Let $p \geq 2$ and $1 \leq k \leq p$. Also let $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$ be analytic in U . If there exists $(p-k+1)$ -valent starlike function $g(z) = z^{p-k+1} + \sum_{n=p-k+2}^{\infty} b_n z^n$ that satisfies

$$\operatorname{Re} \left\{ \frac{z f^{(k)}(z)}{g(z)} \right\} > 0, \quad z \in U, \quad (9)$$

then f is p -valent in U .

Theorem 3. Let $p \geq 3$ and also let $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$ be analytic in U . Suppose that, for any positive integer k with $3 \leq k \leq p$,

$$\left| \arg \left\{ \frac{f^{(k)}(z)}{z^{p-k}} \right\} \right| < \frac{\pi}{2} \left\{ \alpha + \frac{2}{\pi} \tan^{-1} \left(\frac{\alpha}{p-k+1} \right) \right\}, \quad (10)$$

$$z \in U,$$

where

$$\alpha = 1 + \frac{2}{\pi} \tan^{-1} \left(\frac{1}{p-k+2} \right). \quad (11)$$

Then, f is p -valent in U .

Proof. First we prove that

$$\left| \arg \left\{ \frac{f^{(k-1)}(z)}{z^{p-k+1}} \right\} \right| < \frac{\pi\alpha}{2}, \quad z \in U. \quad (12)$$

For $3 \leq k \leq p$, we define

$$p_1(z) = \frac{f^{(k-1)}(z)}{p(p-1) \cdots (p-k+2) z^{p-k+1}}, \quad (13)$$

$$p_1(0) = 1.$$

If there exists a point $z_0 \in U$ such that

$$\left| \arg \{p_1(z)\} \right| < \frac{\pi\alpha}{2} \quad \text{for } |z| < |z_0|, \quad (14)$$

$$\left| \arg \{p_1(z_0)\} \right| = \frac{\pi\alpha}{2},$$

then, from Lemma 1, we have

$$\frac{z_0 p_1'(z_0)}{p_1(z_0)} = i\alpha. \quad (15)$$

For the case $\arg\{p_1(z_0)\} = \pi\alpha/2$, we have

$$\begin{aligned} & \arg \left\{ \frac{f^{(k)}(z_0)}{z_0^{p-k}} \right\} \\ &= \arg \left\{ (p-k+1) p_1(z_0) + z_0 p_1'(z_0) \right\} \\ &= \arg \left\{ p_1(z_0) \left(1 + \frac{z_0 p_1'(z_0)}{(p-k+1) p_1(z_0)} \right) \right\} \\ &= \arg \{p_1(z_0)\} + \arg \left\{ 1 + \frac{i\alpha}{p-k+1} \right\} \\ &\geq \frac{\pi}{2} \left\{ \alpha + \frac{2}{\pi} \tan^{-1} \left(\frac{\alpha}{p-k+1} \right) \right\}. \end{aligned} \quad (16)$$

This contradicts (10). For the case $\arg\{p_1(z_0)\} = -(\pi\alpha/2)$, applying the same method as the above, we have

$$\begin{aligned} & \arg \left\{ \frac{f^{(k)}(z_0)}{z_0^{p-k}} \right\} \\ &= \arg \left\{ p_1(z_0) \left(1 + \frac{z_0 p_1'(z_0)}{(p-k+1) p_1(z_0)} \right) \right\} \\ &= \arg \{p_1(z_0)\} + \arg \left\{ 1 - \frac{i\alpha}{p-k+1} \right\} \\ &\leq -\frac{\pi}{2} \left\{ \alpha + \frac{2}{\pi} \tan^{-1} \left(\frac{\alpha}{p-k+1} \right) \right\}. \end{aligned} \quad (17)$$

This also contradicts (10). Therefore,

$$\left| \arg \left\{ \frac{f^{(k-1)}(z)}{z^{p-k+1}} \right\} \right| < \frac{\pi\alpha}{2}, \quad z \in U. \quad (18)$$

Because $\alpha = 1 + (2/\pi) \tan^{-1}(1/(p-k+2)) > 1$, we cannot apply Lemma 2 at this moment. We need the following step.

Put

$$p_2(z) = \frac{f^{(k-2)}(z)}{p(p-1) \cdots (p-k+3) z^{p-k+2}}, \quad (19)$$

$$p_2(0) = 1.$$

Then, it follows that

$$\begin{aligned} & \frac{f^{(k-1)}(z)}{p(p-1) \cdots (p-k+2) z^{p-k+1}} \\ &= p_2(z) \left(1 + \frac{z p_2'(z)}{(p-k+2) p_2(z)} \right). \end{aligned} \quad (20)$$

We want to show $\operatorname{Re}\{p_2(z)\} > 0$ ($z \in U$). If there exists a point $z_1 \in U$ such that

$$\begin{aligned} & \left| \arg \{p_2(z)\} \right| < \frac{\pi}{2} \quad \text{for } |z| < |z_1|, \\ & \left| \arg \{p_2(z)\} \right| = \frac{\pi}{2}, \end{aligned} \quad (21)$$

then, from Lemma 1, we have

$$\frac{z_1 p_2'(z_1)}{p_2(z_1)} = i\ell. \quad (22)$$

For the case $\arg\{p_2(z_1)\} = \pi/2$, by (20), we have

$$\begin{aligned} & \arg \left\{ \frac{f^{(k-1)}(z_1)}{z_1^{p-k+1}} \right\} \\ &= \arg \left\{ p_2(z_1) \left(1 + \frac{z_1 p_2'(z_1)}{(p-k+2) p_2(z_1)} \right) \right\} \\ &= \arg \{p_2(z_1)\} + \arg \left\{ 1 + \frac{i\ell}{p-k+2} \right\} \\ &\geq \frac{\pi}{2} \left\{ 1 + \frac{2}{\pi} \tan^{-1} \left(\frac{1}{p-k+2} \right) \right\} = \frac{\pi\alpha}{2}. \end{aligned} \quad (23)$$

It contradicts (18). For the case $\arg \{p_2(z_1)\} = -(\pi/2)$, applying the same method as the above, we have

$$\arg \left\{ \frac{f^{(k-1)}(z_1)}{z_1^{p-k+1}} \right\} \leq -\frac{\pi}{2} \left\{ 1 + \frac{2}{\pi} \tan^{-1} \left(\frac{1}{p-k+2} \right) \right\} = -\frac{\pi\alpha}{2}. \quad (24)$$

It also contradicts (18). This shows that

$$\left| \arg \left\{ \frac{f^{(k-2)}(z)}{z^{p-k+2}} \right\} \right| < \frac{\pi}{2}, \quad z \in U, \quad (25)$$

or

$$\operatorname{Re} \left\{ \frac{f^{(k-2)}(z)}{z^{p-k+2}} \right\} = \operatorname{Re} \left\{ \frac{zf^{(k-2)}(z)}{z^{p-k+3}} \right\} > 0, \quad z \in U. \quad (26)$$

Therefore, from Lemma 2, we obtain that f is p -valent in U . Now the proof is complete. $\square$

Taking $k = p$ in Theorem 3, we have the following corollary.

Corollary 4. Let $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$ be analytic in U . Suppose that, for positive integer $p \geq 3$,

$$\left| \arg \{f^{(p)}(z)\} \right| < \frac{\pi}{2} \left\{ \alpha + \frac{2}{\pi} \tan^{-1} \alpha \right\}, \quad z \in U, \quad (27)$$

where

$$\alpha = 1 + \frac{2}{\pi} \tan^{-1} \left(\frac{1}{2} \right) = 1.2951672353 \dots, \quad (28)$$

$$\alpha + \frac{2}{\pi} \tan^{-1} \alpha = 1.876592028 \dots$$

Then, f is p -valent in U .

Example 5. The function $f(z) = z^p$ ($p \geq 3$) satisfies condition (10) in Theorem 3 and is p -valent in U .

2. A Sufficient Condition on Univalence of Meromorphic Functions

Let A be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (29)$$

which are analytic in U . For two functions f and g analytic in U , the function f is said to be subordinate to g , written as $f(z) < g(z)$ ($z \in U$), if there exists a function w analytic in U with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$. In particular, if the function g is univalent in U , then $f(z) < g(z)$ is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$.

Also let Σ denote the class of functions of the form

$$f(z) = \frac{1}{z} + a_0 + a_1 z + \dots \quad (30)$$

which are analytic in the punctured unit disc $U^* = U \setminus \{0\}$.

In [5], Nunokawa et al. proved the following theorem.

Theorem A. Let $f \in A$ with $f(z)/z \neq 0$ for $0 < |z| < 1$ and let

$$\left| \left(\frac{z}{f(z)} \right)'' \right| \leq 1, \quad z \in U. \quad (31)$$

Then, f is univalent in U .

In [6], Yang and Liu generalized the result of Theorem A and obtained sharp bound 2 of (31). Many interesting properties of analytic functions were considered by several authors in some recent works (e.g., [7–9]). In this section, we will derive a sufficient condition on univalence of meromorphic functions in Σ .

For our purpose, we need the following lemmas.

Lemma 6 (see [10]). Let f and g be analytic in U with $f(0) = g(0)$. If $h(z) = zg'(z)$ is starlike in U and $zf'(z) < h(z)$, then

$$f(z) < g(z) = g(0) + \int_0^z \frac{h(t)}{t} dt. \quad (32)$$

Lemma 7 (see [11]). Let g be analytic and convex univalent in U and let $f(z) = g(0) + \sum_{k=n}^{\infty} a_k z^k$ ($n \in \mathbb{N} = \{1, 2, 3, \dots\}$) be analytic in U . If $f(z) < g(z)$, then

$$z^{-c} \int_0^z t^{c-1} f(t) dt < \frac{1}{n} z^{-c/n} \int_0^z t^{c/n-1} g(t) dt, \quad (33)$$

where $\operatorname{Re}(c) \geq 0$ and $c \neq 0$.

Theorem 8. Let $f(z) = (1/z) + a_0 + a_1 z + \dots \in \Sigma$ with $zf(z) \neq 0$ for $0 < |z| < 1$. If

$$\left| \left(\frac{1}{zf(z)} \right)'' \right| \leq \alpha, \quad z \in U, \quad (34)$$

where $0 < \alpha < M = (1 - 2|a_0|)/2$ and $|a_0| < 1/2$, then f is meromorphic univalent in U^* .

Proof. Let

$$p(z) = \left(\frac{1}{zf(z)} \right)''. \quad (35)$$

Then, the function p is analytic in U . By integration from 0 to z , we get

$$\frac{1}{zf(z)} = 1 - a_0 z + \int_0^z dv \int_0^v p(u) du. \quad (36)$$

Thus, we have

$$f(z) = \frac{1}{z - a_0 z^2 + z\varphi(z)}, \quad (37)$$

where $\varphi(z) = \int_0^z dv \int_0^v p(u) du$.

One can see easily from (34) and (35) that $|\varphi(z)| \leq \alpha$ and $|z\varphi'(z)| \leq \alpha$, which shows that $|(z\varphi(z))'| \leq 2\alpha$ for $z \in U$. Therefore,

$$\begin{aligned} |z_2\varphi(z_2) - z_1\varphi(z_1)| &= \left| \int_{z_1}^{z_2} (z\varphi(z))' dz \right| \\ &\leq 2\alpha |z_2 - z_1| \end{aligned} \quad (38)$$

for $z_1, z_2 \in U^*$ and $z_1 \neq z_2$.

If $z_1 \neq z_2$, it follows from (37) and (38) that

$$\begin{aligned} |f(z_1) - f(z_2)| &= \frac{|(z_2 - z_1) - a_0(z_2^2 - z_1^2) + (z_2\varphi(z_2) - z_1\varphi(z_1))|}{|z_1 - a_0z_1^2 + z_1\varphi(z_1)| |z_2 - a_0z_2^2 + z_2\varphi(z_2)|} \\ &\geq \frac{|z_2 - z_1| - 2|a_0||z_2 - z_1| - 2\alpha|z_2 - z_1|}{|z_1 - a_0z_1^2 + z_1\varphi(z_1)| |z_2 - a_0z_2^2 + z_2\varphi(z_2)|} > 0. \end{aligned} \quad (39)$$

Hence, f is meromorphic univalent in U^* . The proof is complete. $\square$

Taking $a_0 = 0$ in Theorem 8, we obtain the following.

Corollary 9. Let $f(z) = 1/z + a_1z + \dots \in \Sigma$ with $zf(z) \neq 0$ for $0 < |z| < 1$. If

$$\left| \left(\frac{1}{zf(z)} \right)'' \right| < \frac{1}{2}, \quad z \in U, \quad (40)$$

then f is meromorphic univalent in U^* .

Theorem 10. Let $f(z) = 1/z + a_0 + a_1z + \dots \in \Sigma$ with $zf(z) \neq 0$ for $0 < |z| < 1$. Also let f satisfy condition (34). Then, for $z \in U^*$,

$$\begin{aligned} \left| \frac{1}{zf(z)} - 1 \right| &\leq |z| \left(|a_0| + \frac{\alpha}{2} |z| \right), \\ 1 - |z| \left(|a_0| + \frac{\alpha}{2} |z| \right) &\leq \operatorname{Re} \left(\frac{1}{zf(z)} \right) \\ &\leq 1 + |z| \left(|a_0| + \frac{\alpha}{2} |z| \right), \\ |f(z)| &\geq \frac{1}{|z| (1 + |a_0||z| + (\alpha/2)|z|^2)}. \end{aligned} \quad (41)$$

Equalities in (41) are attained if we take

$$f(z) = \frac{1}{z(1 \pm a_0z + (\alpha/2)z^2)}. \quad (42)$$

Proof. In view of (34), we have

$$z \left(\frac{1}{zf(z)} \right)'' < \alpha z. \quad (43)$$

Applying Lemma 6 to (43), we find that

$$\left(\frac{1}{zf(z)} \right)' + a_0 < \alpha z. \quad (44)$$

By using Lemma 7, (44) gives that

$$\frac{1}{z} \int_0^z \left[\left(\frac{1}{tf(t)} \right)' + a_0 \right] dt < \frac{\alpha}{2} z; \quad (45)$$

that is,

$$\frac{1}{zf(z)} = 1 - a_0z + \frac{\alpha}{2}zw(z), \quad (46)$$

where w is analytic in U and $|w(z)| \leq |z|$ ($z \in U$) by Schwarz lemma.

Now, from (46), we can easily derive inequalities (41). The proof is complete. $\square$

Example 11. The function $f(z) = 1/z$ satisfies the conditions in Theorems 8 and 10.

An Open Problem. The maximum value of M in Theorem 8 for which (34) implies meromorphic univalence is unknown.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

- [1] S. Ozaki, "On the theory of multivalent functions II," *Science Reports of the Tokyo Kyoiku Daigaku, Section A*, vol. 4, pp. 45–87, 1941.
- [2] M. Nunokawa, "On properties of non-Carathéodory functions," *Proceedings of the Japan Academy Series A*, vol. 68, no. 6, pp. 152–153, 1992.
- [3] M. Nunokawa, "On the order of strongly starlikeness of strongly convex functions," *Proceedings of the Japan Academy, Series A: Mathematical Sciences*, vol. 69, no. 7, pp. 234–237, 1993.
- [4] M. Nunokawa, "On the theory of multivalent functions," *Tsukuba Journal of Mathematics*, vol. 11, no. 2, pp. 273–286, 1987.
- [5] M. Nunokawa, M. Obradović, and S. Owa, "One criterion for univalence," *Proceedings of the American Mathematical Society*, vol. 106, no. 4, pp. 1035–1037, 1989.
- [6] D.-G. Yang and J.-L. Liu, "On a class of univalent functions," *International Journal of Mathematics and Mathematical Sciences*, vol. 22, no. 3, pp. 605–610, 1999.
- [7] H. M. Srivastava, S. Bulut, M. Çağlar, and N. Yağmur, "Coefficient estimates for a general subclass of analytic and bi-univalent functions," *Filomat*, vol. 27, no. 5, pp. 831–842, 2013.
- [8] H. M. Srivastava, J. K. Prajapat, G. I. Oros, and R. Sendrutiu, "Geometric properties of a certain general family of integral operators," *Filomat*, vol. 28, no. 4, pp. 745–754, 2014.
- [9] Q.-H. Xu, C.-B. Lv, N.-C. Luo, and H. M. Srivastava, "Sharp coefficient estimates for a certain general class of spirallike functions by means of differential subordination," *Filomat*, vol. 27, no. 7, pp. 1351–1356, 2013.

- [10] T. J. Suffridge, "Some remarks on convex maps of the unit disk," *Duke Mathematical Journal*, vol. 37, pp. 775–777, 1970.
- [11] D. J. Hallenbeck and S. Ruscheweyh, "Subordination by convex functions," *Proceedings of the American Mathematical Society*, vol. 52, pp. 191–195, 1975.