Research Article

Functions of Bounded k th p -Variation and Continuity Modulus

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Received 8 April 2015; Revised 1 July 2015; Accepted 28 July 2015

Academic Editor: Mitsuru Sugimoto

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A scale of spaces exists connecting the class of functions of bounded k th p -variation in the sense of Riesz-Merentes with the Sobolev space of functions with p -integrable k th derivative. This scale is generated by the generalized functionals of Merentes type. We prove some limiting relations for these functionals as well as sharp estimates in terms of the fractional modulus of smoothness of order $k - 1/p$.

1. Introduction

Let f be a 1-periodic function on the real line. A set $\Pi = \{x_0, \dots, x_n\}$ of points such that $x_0 < x_1 < \dots < x_n$, where $x_n = x_0 + 1$, will be called a *partition*. For a partition Π we denote by $\|\Pi\| = \max_j (x_{j+1} - x_j)$ its diameter. Let $p > 1$, $p' := p/(p-1)$, and $0 \leq \alpha \leq 1/p'$. For any Π , we set

$$v_{p,\alpha}(f; \Pi) = \left(\sum_{k=0}^{n-1} \frac{|f(x_{k+1}) - f(x_k)|^p}{(x_{k+1} - x_k)^{\alpha p}} \right)^{1/p}. \quad (1)$$

We denote by $V_{p,\alpha}$ the class of all 1-periodic functions f such that $v_{p,\alpha}(f) = \sup_{\Pi} v_{p,\alpha}(f; \Pi) < \infty$, where the supremum is taken over all partitions of $[0, 1]$; see, for example, [1].

If $\alpha = 0$ we get the class $V_{p,0} := V_p$ of functions of bounded p -variation introduced by Wiener [2]. A celebrated result of Riesz (see, e.g., [3, 4]) states that the class W_p^1 of absolutely continuous 1-periodic functions such that the first derivative belongs to the Lebesgue space $L^p[0, 1]$ coincides with the class $V_{p,1/p'}$ and $\|f'\|_p = v_{p,1/p'}(f)$.

Another characterization of the class W_p^1 was given by Terehin in [5] in terms of the fractional continuity modulus of order $1 - 1/p$,

$$\omega_{1-1/p}(f; \delta) = \sup_{\|\Pi\| \leq \delta} v_p(f; \Pi) \quad (0 < \delta \leq 1), \quad (2)$$

introduced by Wiener in [2]. If $f \in W_p^1$, then $\omega_{1-1/p}(f; \delta) \leq \delta^{1/p'} \|f'\|_p$, and conversely if $\omega_{1-1/p}(f; \delta) = O(\delta^{1/p'})$, then $f \in W_p^1$.

The concept of functions of bounded second variation was introduced by de la Vallée Poussin in [6]. By a *refined* partition we mean a partition Π of the form

$$\begin{aligned} x_0 < y_1 \leq z_1 < x_1 < y_2 \leq z_2 < x_2 < \dots < y_n \leq z_n \\ < x_n = x_0 + 1. \end{aligned} \quad (3)$$

Motivated by the result of Riesz, Merentes in [7] introduced, for $p > 1$, the class $V_{p,1/p'}^{(2)}$ of functions of bounded $(p, 1/p')$ second variation as the class of those functions for which

$$\begin{aligned} v_{p,1/p'}^{(2)}(f)^p = \sup_{\Pi} \sum_{k=0}^{n-1} & \left| \frac{f(x_{k+1}) - f(z_{k+1})}{x_{k+1} - z_{k+1}} \right. \\ & \left. - \frac{f(y_{k+1}) - f(x_k)}{y_{k+1} - x_k} \right|^p \frac{1}{(x_{k+1} - x_k)^{p/p'}} \end{aligned} \quad (4)$$

is finite. He proved a similar result to that of Riesz (see, e.g., [7, Theorem, page 122]) and gave a variational characterization of the Sobolev space W_p^2 in terms of functions of bounded $(p, 1/p')$ second variation, namely, a function belongs to W_p^2 if and only if $f \in V_{p,1/p'}^{(2)}$ and $v_{p,1/p'}^{(2)}(f) = \|f''\|_p$. Observe that

from the characterization of Riesz and Merentes it follows that $v_{p,1/p'}(f') = v_{p,1/p'}^{(2)}(f)$. The people interested in this topic can see [8].

The above results motivated Barza and Silvestre to study the scale of spaces $V_{p,\alpha}^{(2)}$ between the class $V_p^{(2)}$ of functions of bounded second p -variation and the Sobolev space W_p^2 (see [9]) as an extension of [1].

Following [9, 10], let $p > 1$ and $0 \leq \alpha \leq 1/p'$. Let Π be a partition of the form

$$\begin{aligned} x_0 &= t_{1,1} < t_{1,2} < \dots < t_{1,k} \leq t_{1,k+1} < \dots < t_{1,2k} \leq \dots \\ &\leq t_{2,k+1} < \dots \leq \dots < t_{3,1} < \dots < t_{j,1} < \dots < t_{j,k} \\ &\leq t_{j,k+1} < \dots < t_{j,2k} \leq \dots \leq t_{m,1} < \dots < t_{m,k} \\ &\leq t_{m,k+1} < \dots < t_{m,2k} = x_0 + 1. \end{aligned} \quad (5)$$

Let for any $k \in \mathbb{N}$

$$\begin{aligned} v_{p,\alpha}^{(k)}(f; \Pi)^p &= \sum_{j=1}^m \left| f[t_{j,k+1}, \dots, t_{j,2k}] - f[t_{j,1}, \dots, t_{j,k}] \right|^p \\ &\cdot \frac{1}{(t_{j,2k} - t_{j,1})^{\alpha p}}, \end{aligned} \quad (6)$$

where the k th divided differences are defined by

$$\begin{aligned} f[t_0, t_1, \dots, t_k] &:= \sum_{j=0}^k \frac{f(t_j)}{(t_j - t_0) \dots (t_j - t_{j-1})(t_j - t_{j+1}) \dots (t_j - t_k)}. \end{aligned} \quad (7)$$

We define the space $V_{p,\alpha}^{(k)}(k \in \mathbb{N})$ as the space of 1-periodic functions f such that

$$v_{p,\alpha}^{(k)}(f) = \sup_{\Pi} v_{p,\alpha}^{(k)}(f; \Pi) < \infty, \quad (8)$$

where the supremum is taken over all partitions of type (5). The number $v_{p,\alpha}^{(k)}(f)$ is called the (p, α) k th variation of f . For $\alpha = 0$, we denote it by $v_p^{(k)}(f)$ and call it the k th p -variation of f . We have that $V_{p,1/p'}^{(k)}$ is the space $RV_{(p,k)}$ in [10]. The spaces $V_{p,\alpha}^{(k)}$ form a scale between the space of functions of bounded k th p -variation and the Sobolev space W_p^k . In [10] the authors extend the results of Riesz and Merentes by showing that $f \in W_p^k$ if and only if $f \in V_{p,1/p'}^{(k)} = RV_{(p,k)}$.

Terehin studied in [5] properties of the function

$$\omega_{k-1/p}(f; \delta) = \sup_{0 < h \leq \delta} \omega_{1-1/p}(\Delta_h^{k-1} f; h) \quad (0 < \delta \leq 1), \quad (9)$$

where $\Delta_h f(x) = f(x+h) - f(x)$, $\Delta_h^k f(x) = \Delta_h \Delta_h^{k-1} f(x)$. The function $\omega_{k-1/p}(f; \delta)$ is called the *fractional modulus of smoothness of order $k - 1/p$* . A characterization of the space

$f \in W_p^k$ in terms of $\omega_{k-1/p}(f; \delta)$ was also given in [5]: $\omega_{k-1/p}(f; \delta) = O(\delta^{k-1/p})$ if and only if $f \in W_p^k$.

We extend the results proved in [1, 9] to the case of functions of bounded k th p -variation. In Section 2 we prove some results which we will use in the proofs of our main theorems. In Section 3 we study sharp estimates of $v_{p,\alpha}^{(k)}(f)$ in terms of the fractional modulus of smoothness of order $k - 1/p$, and we study limits in the scales of spaces $V_{p,\alpha}^{(k)}$ for $k \in \mathbb{N}$.

2. Preliminary Results

Recall that a function f satisfying $\lim_{\delta \rightarrow 0+} \omega_{1-1/p}(f; \delta) = 0$ is called p -continuous and the class of p -continuous functions is denoted by C_p .

Lemma 1 (see [11, Lemma 1]). *Let $p > 1$ and $f \in C_p$ such that $f^{(r)} \in C_p$ for $k, r \in \mathbb{N}$. Then*

$$\omega_{k+r-1/p}(f; \delta) \leq \delta^r \omega_{k-1/p}(f^{(r)}; \delta) \quad (0 < \delta \leq 1). \quad (10)$$

Remark 2. Let Π be a partition as in (5). Since $0 < \alpha \leq 1/p'$ ($p' = p/(p-1)$), then $0 < \alpha p < p-1$ and since $|t_{j,2k} - t_{j,1}| \leq 1$, it follows that

$$|t_{j,2k} - t_{j,1}|^{\alpha p} > |t_{j,2k} - t_{j,1}|^{p-1}. \quad (11)$$

Then, $v_{p,\alpha}^{(k)}(f; [0, 1]) \leq v_{p,1/p'}^{(k)}(f; [0, 1])$. The space $V_{p,1/p'}^{(k)}[0, 1]$ is the space $RV_{(p,k)}[0, 1]$ in [10]. Then, $RV_{(p,k)}[0, 1] \subset V_{p,\alpha}^{(k)}[0, 1]$.

Proposition 3. *Let $1 < p < \infty$, $k \geq 2$, and $0 < \alpha \leq 1/p'$. If $f \in V_{p,\alpha}^{(k)}[0, 1]$, then $f^{(k-1)}$ exists everywhere and is continuous on $[0, 1]$.*

Proof. Suppose that $f \in V_{p,\alpha}^{(k)}[0, 1]$. Then, there exists a constant $M > 0$ such that

$$\begin{aligned} M &\geq v_{p,\alpha}^{(k)}(f; [0, 1]) \\ &\geq \frac{|f[t_{j,k+1}, \dots, t_{j,2k}] - f[t_{j,1}, \dots, t_{j,k}]|^p}{|t_{j,2k} - t_{j,1}|^{\alpha p}}, \end{aligned} \quad (12)$$

for any partition of type (5) and $1 \leq j \leq m$.

Let $x_0 \in (0, 1)$. It follows by (12) that

$$\begin{aligned} |t_{j,2k} - t_{j,1}|^\alpha M^{1/p} &\geq |f[t_{j,k+1}, \dots, t_{j,2k}] - f[t_{j,1}, \dots, t_{j,k}]|, \end{aligned} \quad (13)$$

for any partition of type (5) such that $t_{j,k+1} = x_0$ and $1 \leq j \leq m$. So that, Cauchy's convergence criterion establishes the existence of the right-hand derivative $f_+^{(k-1)}(x_0)$ when $0 \leq$

$x_0 < 1$. Similarly, the left-hand derivative $f_-^{(k-1)}(x_0)$ exists when $0 < x_0 \leq 1$. Let us consider the following partition

$$\begin{aligned} 0 &\leq t_{1,1} < \dots < t_{1,k} \leq t_{1,k+1} < \dots < t_{1,2k} \leq \dots < t_{j,k} \\ &= x_0 = t_{j,k+1} < \dots < t_{j,2k} \leq \dots < t_{m-1,2k} \leq t_{m,1} \\ &< \dots < t_{m,2k}. \end{aligned} \quad (14)$$

It follows that

$$\begin{aligned} &\left| f[t_{j,k+1}, \dots, t_{j,2k}] - f[t_{j,1}, \dots, t_{j,k}] \right| \\ &\leq M^{1/p} |t_{j,2k} - t_{j,1}|^\alpha \\ &\leq M^{1/p} (|t_{j,2k} - x_0| + |x_0 - t_{j,1}|)^\alpha \\ &\leq 2^\alpha M^{1/p} (|t_{j,2k} - x_0|^\alpha + |x_0 - t_{j,1}|^\alpha), \end{aligned} \quad (15)$$

which it tends to zero as $t_{j,1} \rightarrow x_0^-$ and $t_{j,2k} \rightarrow x_0^+$. Then, we obtain that $f_+^{(k-1)}(x_0) = f_-^{(k-1)}(x_0)$ and so that, $f^{(k-1)}(x_0)$ exists.

Let us consider $y \in [x_0, 1]$ and the following type of partition

$$\begin{aligned} 0 &\leq t_{1,1} < \dots < t_{1,k} \leq t_{1,k+1} < \dots < t_{j,1} = x_0 - \frac{k-1}{k}h \\ &< t_{j,2} = x_0 - \frac{k-2}{k}h < \dots < t_{j,k} = x_0 \leq t_{j,k+1} = y \\ &< t_{j,k+2} = y + \frac{1}{k}h < \dots < t_{j,2k} = y + \frac{k-1}{k}h \leq \dots \\ &< t_{m,2k}, \end{aligned} \quad (16)$$

where $0 \leq h < \min\{x_0, 1 - y\}$. Then, by (12),

$$\begin{aligned} &\left| f\left[y, y + \frac{1}{k}h, \dots, y + \frac{k-1}{k}h\right] \right. \\ &\quad \left. - f\left[x_0 - \frac{k-1}{k}h, \dots, x_0\right] \right| = |f[t_{j,k+1}, \dots, t_{j,2k}] \\ &\quad - f[t_{j,1}, \dots, t_{j,k}]| \leq M^{1/p} |t_{j,2k} - t_{j,1}|^\alpha = M^{1/p} \left| y \right. \\ &\quad \left. - x_0 + 2\frac{k-1}{k}h \right|^\alpha. \end{aligned} \quad (17)$$

Letting $h \rightarrow 0$, we obtain that

$$|f^{(k-1)}(y) - f^{(k-1)}(x_0)| \leq M^{1/p} |y - x_0|^\alpha, \quad (18)$$

from which the continuity of $f^{(k-1)}$ in $(0, 1)$ follows. Also, $f^{(k-1)}$ is right and left continuous in 0 and 1, respectively. $\square$

Remark 4. Let $1 < p < \infty$, $k \geq 2$, and $0 < \alpha \leq 1/p'$. If $f \in V_{p,\alpha}^{(k)}[0, 1]$, then it follows that

$$f[t_0, t_1, \dots, t_k] = \frac{f^{(k-1)}(\psi)}{(k-1)!} \quad (19)$$

for some $\psi \in \text{conv}\{t_0, \dots, t_k\}$. Indeed, by Proposition 3 we have that $f^{(k-1)}$ exists everywhere on $[0, 1]$ and it is continuous. Then, $f \in C^{(k-1)}[0, 1]$. The equality (19) follows by [10, (2.8)].

Theorem 5. Let $p > 1$ and $0 < \alpha \leq 1/p'$.

(i) If $f \in V_{p,\alpha}^{(k)}[0, 1]$, then

$$v_{p,\alpha}^{(k)}(f; [0, 1]) \geq \frac{1}{(k-1)!} v_{p,\alpha}(f^{(k-1)}; [0, 1]). \quad (20)$$

(ii) If $v_{p,\alpha}(f^{(k-1)}; [0, 1]) < \infty$, then

$$v_{p,\alpha}^{(k)}(f; [0, 1]) \leq \frac{1}{(k-1)!} v_{p,\alpha}(f^{(k-1)}; [0, 1]). \quad (21)$$

Proof. (i) If $f \in V_{p,\alpha}^{(k)}[0, 1]$, then $v_{p,\alpha}^{(k)}(f; [0, 1]) < \infty$ and $f^{(k-1)}$ exists everywhere on $[0, 1]$ by Proposition 3 and it is continuous. To prove the inequality, let $\Pi = \{0 = t_0, \dots, t_n = 1\}$ be a partition of $[0, 1]$ and take $0 < h \leq \min_{0 \leq j \leq m-1} \{(t_{j+1} - t_j)/2\}$. Define the partition Π_j of the subinterval $[t_j, t_{j+1}]$ as

$$\begin{aligned} t_j &= t_{j,1} < t_{j,2} < \dots < t_{j,k} = t_j + h \leq t_{j,k+1} = t_{j+1} - h \\ &< \dots < t_{j,2k} = t_{j+1}. \end{aligned} \quad (22)$$

Then, we have that

$$\begin{aligned} v_{p,\alpha}^{(k)}(f; [0, 1])^p &:= \sup_{\Pi} v_{p,\alpha}^{(k)}(f; \Pi)^p \\ &\geq \sum_{j=1}^m \frac{|f[t_{j,k+1}, \dots, t_{j,2k}] - f[t_{j,1}, \dots, t_{j,k}]|^p}{|t_{j,2k} - t_{j,1}|^{\alpha p}} \\ &= \sum_{j=1}^m \frac{|f[t_{j+1} - h, \dots, t_{j+1}] - f[t_j, \dots, t_j + h]|^p}{|t_{j+1} - t_j|^{\alpha p}}. \end{aligned} \quad (23)$$

Then, by Remark 4, there exist $\psi_j \in \text{conv}\{t_{j+1} - h, \dots, t_{j+1}\}$ and $\psi_{j-1} \in \text{conv}\{t_j, \dots, t_j + h\}$ such that

$$\begin{aligned} &v_{p,\alpha}^{(k)}(f; [0, 1])^p \\ &\geq \frac{1}{(k-1)!^p} \sum_{j=1}^m \frac{|f^{(k-1)}(\psi_j) - f^{(k-1)}(\psi_{j-1})|^p}{|t_{j+1} - t_j|^{\alpha p}}. \end{aligned} \quad (24)$$

By letting $h \rightarrow 0$, since $f^{(k-1)}$ is continuous, it follows that

$$\begin{aligned} &v_{p,\alpha}^{(k)}(f; [0, 1])^p \\ &\geq \frac{1}{(k-1)!^p} \sum_{j=1}^m \frac{|f^{(k-1)}(t_{j+1}) - f^{(k-1)}(t_j)|^p}{|t_{j+1} - t_j|^{\alpha p}}. \end{aligned} \quad (25)$$

By taking then supremum over all Π 's, the inequality follows.

(ii) Given $\varepsilon > 0$, there exists a partition of type (5)

$$0 = t_{1,1} < t_{1,2} < \cdots < t_{1,k} \leq t_{1,k+1} < \cdots < t_{1,2k} \leq \cdots < t_{j,2k} \leq t_{m,1} < \cdots < t_{m,k} \leq t_{m,k+1} < \cdots < t_{m,2k} = 1 \quad (26)$$

such that

$$\begin{aligned} & \nu_{p,\alpha}^{(k)}(f; [0, 1])^p - \varepsilon \\ & < \sum_{j=1}^m \frac{|f[t_{j,k+1}, \dots, t_{j,2k}] - f[t_{j,1}, \dots, t_{j,k}]|^p}{|t_{j,2k} - t_{j,1}|^{\alpha p}}. \end{aligned} \quad (27)$$

Since $\nu_{p,\alpha}(f^{(k-1)}; [0, 1]) < \infty$, it follows that $f^{(k-1)} \in V_{p,\alpha}[0, 1]$. Moreover, as we can see in [9, Theorem 2.3], we have that

$$\begin{aligned} \nu_1^{(2)}(f^{(k-2)}; [0, 1]) & \leq \nu_{p,\alpha}^{(2)}(f^{(k-2)}; [0, 1]) \\ & \leq \nu_{p,\alpha}(f^{(k-1)}; [0, 1]). \end{aligned} \quad (28)$$

Moreover, by [9, Proposition 2.4], it follows that $\nu_1^{(2)}(f^{(k-2)}; [0, 1]) = \nu_1(f^{(k-1)}; [0, 1])$. Hence, $f^{(k-1)} \in BV[0, 1]$. Therefore, $f^{(k-1)}$ is Riemann integrable in $[0, 1]$. Indeed, it can be written as a difference of two monotone functions and then, $f^{(k-1)}$ is Riemann integrable.

By using [10, Proposition 2.5], since $f^{(k-1)}$ is continuous, for $\psi_j \in \text{conv}\{t_{j,k+1}, \dots, t_{j,2k}\}$ and $\psi_{j-1} \in \text{conv}\{t_{j,1}, \dots, t_{j,k}\}$, we have that

$$\begin{aligned} & \nu_{p,\alpha}^{(k)}(f; [0, 1])^p - \varepsilon \\ & < \sum_{j=1}^m \frac{|f[t_{j,k+1}, \dots, t_{j,2k}] - f[t_{j,1}, \dots, t_{j,k}]|^p}{|t_{j,2k} - t_{j,1}|^{\alpha p}} \\ & \leq \frac{1}{(k-1)!^p} \sum_{j=1}^m \frac{|f^{(k-1)}(\psi_j) - f^{(k-1)}(\psi_{j-1})|^p}{|\psi_j - \psi_{j-1}|^{\alpha p}} \\ & \leq \frac{1}{(k-1)!^p} \nu_{p,\alpha}(f^{(k-1)}; [0, 1])^p. \end{aligned} \quad (29)$$

Letting ε tend to zero, the inequality follows. $\square$

Proposition 6. Let $p \geq 1$ and assume that $f^{(k-1)}$ exists everywhere. Then

$$\begin{aligned} \nu_p^{(k)}(f) & := \nu_p^{(k)}(f; [0, 1]) = \frac{1}{(k-1)!} \nu_p(f^{(k-1)}; [0, 1]) \\ & := \frac{1}{(k-1)!} \nu_p(f^{(k-1)}). \end{aligned} \quad (30)$$

(For simplicity, from now on we omit the domain $[0, 1]$ in the functionals $\nu_{p,\alpha}^{(k)}$.)

Proof. The equality follows as in Theorem 5. $\square$

Let $f \in L^1$ and f_h be the Steklov average of the function f (see, e.g., [1]) defined by

$$f_h(x) := \frac{1}{h} \int_0^h f(x+t) dt \quad (0 < h \leq 1). \quad (31)$$

We define the k th Steklov average of f as the Steklov average of the $(k-1)$ th Steklov average of f ,

$$f_{h,\dots,(k),h}(x) := \frac{1}{h} \int_0^h f_{h,\dots,(k-1),h}(x+t) dt \quad (0 < h \leq 1). \quad (32)$$

Lemma 7. Let $p > 1$ and $f \in V_p^{(k)}$. Then

$$(i) \quad \omega_{k-1/p}(f_h; t) \leq \omega_{k-1/p}(f; t) \quad (0 \leq t \leq 1), \quad (33)$$

$$(ii) \quad \|f_{h,\dots,(k),h}^{(k)}\|_p \leq h^{-(k-1/p)} \omega_{k-1/p}(f; h) \quad (0 < h \leq 1), \quad (34)$$

$$(iii) \quad \nu_p^{(k)}(f - f_{h,\dots,(k),h}) \leq \frac{6k}{(k-1)!} \frac{\omega_{k-1/p}(f; h)}{h^{k-1}}. \quad (35)$$

Proof (the argument is inductive). (i) The inequality follows by the first inequality of [1, Lemma 2.2] and the definition of $\omega_{k-1/p}(f; t)$. Indeed, since $(\Delta_\delta^{k-1} f_h)(x) = (\Delta_\delta^{k-1} f)_h(x)$ and $f_h \in V_p^{(k)}$ it follows that

$$\begin{aligned} \omega_{k-1/p}(f_h; t) & = \sup_{0 < \delta \leq t} \omega_{1-1/p}(\Delta_\delta^{k-1} f_h; \delta) \\ & = \sup_{0 < \delta \leq t} \omega_{1-1/p}((\Delta_\delta^{k-1} f)_h; \delta) \\ & \leq \sup_{0 < \delta \leq t} \omega_{1-1/p}(\Delta_\delta^{k-1} f; \delta) \\ & := \omega_{k-1/p}(f; t). \end{aligned} \quad (36)$$

(ii) Let $f \in V_p^{(k)}$. We have that $(f')_h = f'_h$,

$$\begin{aligned} \frac{d}{dx}(f_{h,h})(x) & = \frac{d}{dx} \left(\frac{1}{h} \int_0^h f_h(x+t) dt \right) \\ & = \frac{\int_0^h f'_h(x+u) du}{h} = (f'_h)_h(x), \end{aligned} \quad (37)$$

$$\frac{d^2}{dx^2}(f_{h,h})(x) = \frac{d}{dx}(f'_h)_h(x) = \left(\frac{\Delta_h f(x)}{h} \right)'_h.$$

In the computations in [9, Lemma 2.5], we find that

$$\|f''_{h,h}\|_p \leq h^{-2+1/p} \omega_{1-1/p}(\Delta_h f; h). \quad (38)$$

We have that

$$\begin{aligned} \frac{d}{dx}(f_{h,h,h})(x) & = \frac{d}{dx} \left(\frac{1}{h} \int_0^h f_{h,h}(x+t) dt \right) \\ & = \frac{\int_0^h f'_{h,h}(x+u) du}{h} = (f'_{h,h})_h(x) \\ & = \frac{\Delta_h f_{h,h}(x)}{h} = \frac{(\Delta_h f)_{h,h}(x)}{h}, \end{aligned} \quad (39)$$

since

$$\begin{aligned}\Delta_h f_{h,h}(x) &= \frac{1}{h} \int_0^h \Delta_h f_h(x+t) dt \\ &= \frac{1}{h} \int_0^h (\Delta_h f)_h(x+t) dt = (\Delta_h f)_{h,h}(x).\end{aligned}\quad (40)$$

Then, it follows that

$$\begin{aligned}\frac{d^3}{dx^3} (f_{h,h,h})(x) &= \frac{d^2}{dx^2} \left[\frac{(\Delta_h f)_{h,h}(x)}{h} \right] \\ &= \left(\frac{(\Delta_h f)_{h,h}}{h} \right)''(x).\end{aligned}\quad (41)$$

Therefore, by (38) (see [9, Lemma 2.5])

$$\begin{aligned}\|f_{h,h,h}'''\|_p &= \left\| \frac{d^2}{dx^2} \left[\frac{(\Delta_h f)_{h,h}(x)}{h} \right] \right\|_p = \frac{1}{h} \|(\Delta_h f)_{h,h}''\|_p \\ &\leq h^{-3+1/p} \omega_{1-1/p}(\Delta_h^2 f; h) \\ &\leq h^{-(3-1/p)} \omega_{3-1/p}(f; h),\end{aligned}\quad (42)$$

and the inequality holds for 3. Let us assume that

$$\|f_{h,\dots,(k-1),h}^{(k-1)}\|_p \leq h^{-(k-1)+1/p} \omega_{1-1/p}(\Delta_h^{k-2} f; h) \quad (43)$$

and let us see that the inequality holds for k .

We have that

$$\begin{aligned}\frac{d}{dx} (f_{h,\dots,(k),h})(x) &= \frac{d}{dx} \left(\frac{1}{h} \int_0^h f_{h,\dots,(k-1),h}(x+t) dt \right)(x) \\ &= \frac{\Delta_h f_{h,\dots,(k-1),h}(x)}{h} = \frac{(\Delta_h f)_{h,\dots,(k-1),h}(x)}{h}, \\ \frac{d^k}{dx^k} (f_{h,\dots,(k),h})(x) &= \frac{d^{k-1}}{dx^{k-1}} \left[\frac{(\Delta_h f)_{h,\dots,(k-1),h}(x)}{h} \right] \\ &= \left(\frac{(\Delta_h f)_{h,\dots,(k-1),h}(x)}{h} \right)^{(k-1)}.\end{aligned}\quad (44)$$

Therefore, by (43), it follows that

$$\begin{aligned}\|f_{h,\dots,(k),h}^{(k)}\|_p &= \left\| \left(\frac{(\Delta_h f)_{h,\dots,(k-1),h}(x)}{h} \right)^{(k-1)} \right\|_p \\ &= \frac{1}{h} \|(\Delta_h f)_{h,\dots,(k-1),h}^{(k-1)}\|_p \\ &\leq \frac{1}{h} h^{-(k-1)+1/p} \omega_{1-1/p}(\Delta_h^{k-2} (\Delta_h f); h) \\ &\leq h^{-(k-1/p)} \omega_{k-1/p}(f; h).\end{aligned}\quad (45)$$

(iii) The inequality holds for 2 by [9]. We have that

$$v_p^{(3)}(f - f_{h,h,h}) \leq v_p^{(3)}(f - f_{h,h}) + v_p^{(3)}(f_{h,h} - f_{h,h,h}). \quad (46)$$

By Theorem 5 and [9, Proposition 2.4] it follows that

$$\begin{aligned}v_p^{(3)}(f - f_{h,h,h}) &\leq v_p^{(3)}(f - f_{h,h}) + v_p^{(3)}(f_{h,h} - f_{h,h,h}) \\ &= \frac{1}{2!} [v_p^{(2)}(f' - f'_{h,h}) + v_p^{(2)}(f'_{h,h} - f'_{h,h,h})] \\ &\leq \frac{1}{2!} [2v_p^{(2)}(f'_h - f'_{h,h}) + v_p^{(2)}(f'_{h,h} - f'_{h,h,h})],\end{aligned}\quad (47)$$

because $v_p^{(2)}(f' - f'_{h,h}) \leq 2v_p^{(2)}(f'_h - f'_{h,h})$ since by [9, Lemma 2.5(iii)] $v_p^{(2)}(f - f_{h,h}) \leq 2v_p^{(2)}(f_h - f_{h,h})$. Therefore, since in [9, Lemma 2.5(iii)] we see that $v_p^{(2)}(f - f_h) \leq v_p^{(2)}(f_h - f_{h,h})$, then we have that

$$v_p^{(2)}(f'_h - (f'_h)_h) \leq v_p^{(2)}((f'_h)_h - (f'_h)_{h,h}). \quad (48)$$

Therefore,

$$\begin{aligned}v_p^{(3)}(f - f_{h,h,h}) &\leq \frac{1}{2!} [2v_p^{(2)}(f'_h - f'_{h,h}) + v_p^{(2)}(f'_{h,h} - f'_{h,h,h})] \\ &= \frac{1}{2!} [2v_p^{(2)}(f'_h - (f'_h)_h) + v_p^{(2)}(f'_{h,h} - f'_{h,h,h})] \\ &\leq \frac{1}{2!} [2v_p^{(2)}((f'_h)_h - (f'_h)_{h,h}) + v_p^{(2)}(f'_{h,h} - f'_{h,h,h})] \\ &= \frac{3}{2!} v_p^{(2)}(f'_{h,h} - f'_{h,h,h}) = \frac{3}{2!} v_p(f''_{h,h} - f''_{h,h,h}) \\ &= \frac{3}{2!} v_p \left[\left(\frac{\Delta_h f}{h} \right)'_h - \left(\frac{\Delta_h f}{h} \right)'_{h,h} \right] \\ &= \frac{3}{2!} v_p \left[\left(\frac{\Delta_h f}{h} \right)'_h - \left[\left(\frac{\Delta_h f}{h} \right)'_h \right]_{h,h} \right] \\ &= \frac{3}{2!} v_p(g - g_h)\end{aligned}\quad (*)$$

for $g = (\Delta_h f)'_h/h \in V_p$. By [1, Lemma 2.2 (2.15)] it follows that

$$v_p(g - g_h) \leq 6\omega_{1-1/p}(g; h), \quad (49)$$

and then since $(\Delta_h f)'_h = \Delta_h^2 f/h$, it follows that

$$\begin{aligned} v_p^{(3)}(f - f_{h,h,h}) &\leq \frac{3}{2!} v_p(g - g_h) \\ &\leq \frac{18}{2!} \omega_{1-1/p} \left(\frac{(\Delta_h f)'_h}{h}; h \right) \\ &= \frac{18}{2!} \omega_{1-1/p} \left(\frac{(\Delta_h^2 f)}{h^2}; h \right) \\ &\leq \frac{18}{2!} \frac{1}{h^2} \omega_{1-1/p}((\Delta_h^2 f); h) \\ &\leq \frac{18}{2!} h^{-2} \omega_{3-1/p}(f; h). \end{aligned} \quad (50)$$

Then we see that the inequality holds for 3 as well.

In particular we have by [9, Lemma 2.5(iii)] that

$$\begin{aligned} v_p^{(2)}(f - f_{h,h}) &= v_p^{(2)}(f - f_{h,\dots^{(2)},h}) \leq 2v_p(f'_h - f'_{h,h}) \\ &= 2v_p(f_{h,\dots^{(2-1)},h} - f_{h,\dots^{(2)},h}^{(2-1)}), \end{aligned} \quad (51)$$

and we have proven that

$$v_p^{(3)}(f - f_{h,\dots^{(3)},h}) \leq \frac{3}{2!} v_p(f_{h,\dots^{(3-1)},h} - f_{h,\dots^{(3)},h}^{(3-1)}). \quad (52)$$

Let us assume that the inequality holds for $k-1$ and let us take k . We can see that the inequality holds also for k . Indeed, by the same kind of argument than for 3 we see that (recall (*))

$$\begin{aligned} v_p^{(k)}(f - f_{h,\dots^{(k)},h}) &\leq v_p^{(k)}(f - f_{h,\dots^{(k-1)},h}) + v_p^{(k)}(f_{h,\dots^{(k-1)},h} - f_{h,\dots^{(k)},h}) \\ &\leq v_p^{(k)}(f - f_{h,\dots^{(k-1)},h}) + \frac{v_p(f_{h,\dots^{(k-1)},h} - f_{h,\dots^{(k)},h}^{(k-1)})}{(k-1)!} \\ &\leq \frac{k}{(k-1)!} v_p[f_{h,\dots^{(k-1)},h} - (f_{h,\dots^{(k-1)},h})_h] \\ &\leq \frac{6k}{(k-1)!} \omega_{1-1/p}(f_{h,\dots^{(k-1)},h}; h), \end{aligned} \quad (53)$$

since $f_{h,\dots^{(k-1)},h}^{(k-1)} \in V_p$.

Recall that

$$(\Delta_h f)'_h = \frac{\Delta_h(\Delta_h f)}{h}. \quad (54)$$

Then $f''_{h,h}(x) = (\Delta_h f)'_h/h = \Delta_h^2 f/h^2$. Hence, $f'''_{h,h,h}(x) = (\Delta_h^2 f/h^2)'_h = \Delta_h^3 f/h^3$. Let us assume that this type of identity holds for $k-2$, that is, $f_{h,\dots^{(k-2)},h}^{(k-2)} = \Delta_h^{k-2} f/h^{k-2}$. By (54), we see that

$$\begin{aligned} f_{h,\dots^{(k-1)},h}^{(k-1)} &= (f_{h,\dots^{(k-2)},h})'_h = \left(\frac{\Delta_h^{k-2} f}{h^{k-2}} \right)'_h \\ &= \frac{\Delta_h(\Delta_h^{k-2} f/h^{k-2})}{h} = \frac{\Delta_h^{k-1} f}{h^{k-1}} \end{aligned} \quad (55)$$

and the same identity holds for any $k \in \mathbb{N}$.

Therefore, by (53) and the previous identity, we obtain that

$$\begin{aligned} v_p^{(k)}(f - f_{h,\dots^{(k)},h}) &\leq \frac{6k}{(k-1)!} \omega_{1-1/p} \left(\frac{\Delta_h^{k-1} f}{h^{k-1}}; h \right) \\ &\leq \frac{6k}{(k-1)!} \frac{\omega_{k-1/p}(f; h)}{h^{k-1}}. \end{aligned} \quad (56)$$

□

In the next proposition we give an exact description of Peetre's K -functional for the pair $(V_p^{(k)}, W_p^k)$ for $k \in \mathbb{N}$. By [12, page 172]

$$K(f, t; V_p^{(k)}, W_p^k) := \inf_{g \in W_p^k} \{ \|f - g\|_{V_p^{(k)}} + t \|g^{(k)}\|_p \} \quad (57)$$

($t > 0$),

with $\|f\|_{V_p^{(k)}} = |f(0)| + |f'(0)| + \dots + |f^{(k-1)}(0)| + v_p^{(k)}(f)$ for any $f \in V_p^{(k)}$.

Proposition 8. Let $f \in V_p^{(k)}$. For any $t \in (0, 1]$

$$\begin{aligned} \frac{1}{(k-1)!} \frac{\omega_{k-1/p}(f; t^{p'})}{t^{p'(k-1)}} &\leq K(f, t; V_p^{(k)}, W_p^k) \\ &\leq k \left(1 + \frac{6}{(k-1)!} \right) \frac{\omega_{k-1/p}(f; t^{p'})}{t^{p'(k-1)}}. \end{aligned} \quad (58)$$

Proof. For $t \in (0, 1]$ let $h = t^{p'}$ and $g = f_{h,\dots^{(k)},h}$. Since $f \in W_p^k$, then $g \in W_p^k$. Hence, by (ii) and (iii) in Lemma 7 it follows that

$$\begin{aligned} &|f(0) - g(0)| + \dots + |f^{(k-1)}(0) - g^{(k-1)}(0)| \\ &+ v_p^{(k)}(f - g) + h^{1/p'} \|g^{(k)}\|_p \\ &\leq (k-1) \frac{\omega_{k-1/p}(f; h)}{h^{k-1}} + \frac{6k}{(k-1)!} \frac{\omega_{k-1/p}(f; h)}{h^{k-1}} \\ &+ h^{1/p'} h^{-(k-1/p)} \omega_{k-1/p}(f; h) \leq k \frac{\omega_{k-1/p}(f; h)}{h^{k-1}} \\ &+ \frac{6k}{(k-1)!} \frac{\omega_{k-1/p}(f; h)}{h^{k-1}}. \end{aligned} \quad (59)$$

Therefore,

$$\begin{aligned} &\|f - g\|_{V_p^{(k)}} + h^{1/p'} \|g^{(k)}\|_p \\ &\leq k \left(1 + \frac{6}{(k-1)!} \right) \frac{\omega_{k-1/p}(f; h)}{h^{k-1}}, \end{aligned} \quad (60)$$

and by taking infimum over all $g \in W_p^k$, it follows that

$$K(f, t; V_p^{(k)}, W_p^k) \leq k \left(1 + \frac{6}{(k-1)!} \right) \frac{\omega_{k-1/p}(f; t^{p'})}{t^{p'(k-1)}}. \quad (61)$$

We prove now the left inequality. Let $g \in W_p^k$. By [11, page 19], it follows that

$$\begin{aligned}
 \frac{\omega_{k-1/p}(f; t^{p'})}{t^{p'(k-1)}} &\leq \frac{\omega_{k-1/p}(f - g; t^{p'})}{t^{p'(k-1)}} \\
 &\quad + \frac{\omega_{k-1/p}(g; t^{p'})}{t^{p'(k-1)}} \\
 &\leq \frac{t^{p'(k-1)} \omega_{1-1/p}((f - g)^{(k-1)}; t^{p'})}{t^{p'(k-1)}} \\
 &\quad + \frac{t^{p'(k-1)} \omega_{1-1/p}(g^{(k-1)}; t^{p'})}{t^{p'(k-1)}} \quad (62) \\
 &= \omega_{1-1/p}(f^{(k-1)} - g^{(k-1)}; t^{p'}) \\
 &\quad + t \|g^{(k)}\|_p \\
 &\leq \nu_p(f^{(k-1)} - g^{(k-1)}) + t \|g^{(k)}\|_p \\
 &= (k-1)! \nu_p^{(k)}(f - g) + t \|g^{(k)}\|_p,
 \end{aligned}$$

since $g^{(k-1)} \in W_p$. Taking infimum over all g , we obtain that

$$\frac{\omega_{k-1/p}(f; t^{p'})}{t^{p'(k-1)}} \leq (k-1)! K(f, t; V_p^{(k)}, W_p^k). \quad (63)$$

□

3. Main Results

Proposition 9. Let $p > 1$ and $f \in W_p^k$. Then

$$\lim_{\delta \rightarrow 0_+} \frac{\omega_{k-1/p}(f; \delta)}{\delta^{k-1/p}} = \|f^{(k)}\|_p. \quad (64)$$

Proof. By Lemma 1 and [1, Proposition 3.1] it follows that

$$\begin{aligned}
 \lim_{\delta \rightarrow 0_+} \frac{\omega_{k-1/p}(f; \delta)}{\delta^{k-1/p}} &\leq \lim_{\delta \rightarrow 0_+} \frac{\omega_{1-1/p}(f^{(k-1)}; \delta)}{\delta^{1-1/p}} \\
 &= \|f^{(k)}\|_p.
 \end{aligned} \quad (65)$$

Now we prove the reverse inequality. Let $\Delta_\delta f(x) = f(x+\delta) - f(x)$, $\Delta_\delta^2 f(x) = \Delta_\delta(\Delta_\delta f)(x)$, and $\Delta_\delta^k f(x) = \Delta_\delta(\Delta_\delta^{k-1} f)(x)$. Then

$$\begin{aligned}
 \|f^{(k)}\|_p &\leq \left\| f^{(k)} - \frac{\Delta_\delta f^{(k-1)}}{\delta} \right\|_p \\
 &\quad + \left\| \frac{\Delta_\delta f^{(k-1)}}{\delta} - \frac{\Delta_\delta^2 f^{(k-2)}}{\delta^2} \right\|_p + \dots \quad (66) \\
 &\quad + \left\| \frac{\Delta_\delta^{k-1} f'}{\delta^{k-1}} - \frac{\Delta_\delta^k f}{\delta^k} \right\|_p + \left\| \frac{\Delta_\delta^k f}{\delta^k} \right\|_p.
 \end{aligned}$$

We set $\omega_k(f; \delta)_p = \sup_{0 < h \leq \delta} \|\Delta_h^k f\|_p$, which is the k th L^p -modulus of continuity; see [1, 9]. By Hölder's inequality and Fubini's theorem

$$\begin{aligned}
 &\left\| f^{(k)} - \frac{\Delta_\delta f^{(k-1)}}{\delta} \right\|_p \\
 &= \left(\int_0^1 \left| f^{(k)}(x) - \frac{1}{\delta} \int_0^\delta f^{(k)}(x+t) dt \right|^p dx \right)^{1/p} \\
 &\leq \left(\frac{1}{\delta} \int_0^\delta \left(\int_0^1 |f^{(k)}(x) - f^{(k)}(x+t)|^p dx \right) dt \right)^{1/p} \quad (67) \\
 &\leq \left(\frac{1}{\delta} \int_0^\delta \|\Delta_t f^{(k)}\|_p^p dt \right)^{1/p} \\
 &\leq \left(\frac{1}{\delta} \int_0^\delta \sup_{0 \leq t \leq \delta} \|\Delta_t f^{(k)}\|_p^p dt \right)^{1/p} \leq \omega(f^{(k)}; \delta)_p.
 \end{aligned}$$

Moreover, it follows by Hölder's inequality and Fubini's theorem that

$$\begin{aligned}
 &\left\| \frac{\Delta_\delta f^{(k-1)}}{\delta} - \frac{\Delta_\delta^2 f^{(k-2)}}{\delta^2} \right\|_p = \left\| \Delta_\delta \left(\frac{f^{(k-1)}}{\delta} \right) - \Delta_\delta^2 \left(\frac{f^{(k-2)}}{\delta^2} \right) \right\|_p \\
 &= \left(\int_0^1 \left| \frac{\Delta_\delta f^{(k-1)}}{\delta}(x) - \frac{1}{\delta^2} \int_0^\delta \Delta_\delta f^{(k-1)}(x+t) dt \right|^p dx \right)^{1/p} \\
 &= \left(\int_0^1 \frac{1}{\delta^p} \left| \int_0^\delta \left[\frac{\Delta_\delta f^{(k-1)}}{\delta}(x) - \frac{\Delta_\delta f^{(k-1)}(x+t)}{\delta} \right] dt \right|^p dx \right)^{1/p} \\
 &\leq \left(\int_0^1 \frac{1}{\delta^p} \left(\int_0^\delta \left| \frac{\Delta_\delta f^{(k-1)}}{\delta}(x) - \frac{\Delta_\delta f^{(k-1)}(x+t)}{\delta} \right| dt \right)^p dx \right)^{1/p} \\
 &\quad \cdot \left(\int_0^\delta dt \right)^{p-1} = \left(\frac{1}{\delta} \right. \\
 &\quad \cdot \int_0^\delta \int_0^1 \left| \frac{\Delta_\delta f^{(k-1)}}{\delta}(x) - \frac{\Delta_\delta f^{(k-1)}(x+t)}{\delta} \right|^p dx dt \Big)^{1/p} \leq \left(\frac{1}{\delta} \right. \\
 &\quad \cdot \int_0^\delta \left\| \Delta_t \left[\frac{\Delta_\delta f^{(k-1)}}{\delta} \right] \right\|_p^p dt \Big)^{1/p} \leq \left(\frac{1}{\delta} \right. \\
 &\quad \cdot \int_0^\delta \sup_{0 \leq t \leq \delta} \left\| \Delta_t \left[\frac{\Delta_\delta f^{(k-1)}}{\delta} \right] \right\|_p^p dt \Big)^{1/p} \leq \left(\frac{1}{\delta} \right. \\
 &\quad \cdot \int_0^\delta \omega \left(\frac{\Delta_\delta f^{(k-1)}}{\delta}; \delta \right)_p^p dt \Big)^{1/p} \leq \omega \left(\frac{\Delta_\delta f^{(k-1)}}{\delta}; \delta \right)_p.
 \end{aligned} \quad (68)$$

We repeat this type of argument to the other terms in the sum in the right hand side of inequality (66), except for the last one. For instance, in the last case we see that

$$\left\| \frac{\Delta_\delta^{k-1} f'}{\delta^{k-1}} - \frac{\Delta_\delta^k f}{\delta^k} \right\|_p \leq \omega \left(\frac{\Delta_\delta^{k-1} f'}{\delta^{k-1}}; \delta \right)_p. \quad (69)$$

Since $f^{(k)}$, $\Delta_\delta f^{(k-1)}/\delta, \dots, \Delta_\delta^{k-1} f'/\delta^{k-1}$ belong to L^p , then by all the previous inequalities it follows that all the intermediate

terms except the last one in the sum of the right hand side in (66) tend to zero as $\delta \rightarrow 0_+$. It was proved in [13] that $\omega_k(f; \delta)_p \leq \delta^{1/p} \omega_{k-1/p}(f; \delta)$. Then

$$\begin{aligned} \left\| \frac{\Delta_\delta^k f}{\delta^k} \right\|_p &\leq \frac{\omega_k(f; \delta)_p}{\delta^k} \leq \frac{\delta^{1/p} \omega_{k-1/p}(f; \delta)}{\delta^k} \\ &= \frac{\omega_{k-1/p}(f; \delta)}{\delta^{k-1/p}}, \end{aligned} \quad (70)$$

$$\|f^{(k)}\|_p \leq \lim_{\delta \rightarrow 0_+} \frac{\omega_{k-1/p}(f; \delta)}{\delta^{k-1/p}}.$$

□

Theorem 10. Let $1 < p < \infty$ and f be a 1-periodic function.

(i) If $f \in W_p^k$, then

$$\begin{aligned} \lim_{s \rightarrow 1/p' -} \left(\frac{1}{p'} - s \right)^{1/p} \\ \cdot \left(\int_0^1 [t^{-s-(k-1)} \omega_{k-1/p}(f; t)]^p \frac{dt}{t} \right)^{1/p} &= \left(\frac{1}{p} \right)^{1/p} \\ \cdot \|f^{(k)}\|_p. \end{aligned} \quad (71)$$

(ii) If $\lim_{t \rightarrow 0_+} (\omega_{k-1/p}(f; t)/t^{k-1}) = 0$ and

$$\begin{aligned} \overline{\lim}_{s \rightarrow 1/p' -} \left(\frac{1}{p'} - s \right)^{1/p} \\ \cdot \left(\int_0^1 [t^{-s-(k-1)} \omega_{k-1/p}(f; t)]^p \frac{dt}{t} \right)^{1/p} < \infty, \end{aligned} \quad (72)$$

then $f \in W_p^k$.

Proof. (i) Let $f \in W_p^k$ and $0 < s < 1/p'$. By Proposition 9 we have that, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\begin{aligned} \|f^{(k)}\|_p^p - \varepsilon < \frac{\omega_{k-1/p}(f; t)^p}{t^{kp-1}} < \|f^{(k)}\|_p^p + \varepsilon \\ (0 < t < \delta). \end{aligned} \quad (73)$$

Multiplying (73) by t^{p-sp-2} and then integrating from 0 to δ and subtracting $\|f^{(k)}\|_p^p$ from each side, it follows that

$$\begin{aligned} (\delta^{p-sp-1} - 1) \|f^{(k)}\|_p^p - \varepsilon \delta^{p-sp-1} &\leq (p-sp-1) \\ \cdot \int_0^\delta (t^{-s-(k-1)} \omega_{k-1/p}(f; t))^p \frac{dt}{t} - \|f^{(k)}\|_p^p \\ &\leq (\delta^{p-sp-1} - 1) \|f^{(k)}\|_p^p + \varepsilon \delta^{p-sp-1}. \end{aligned} \quad (74)$$

On the other hand, since $f \in W_p^k$, then $f \in V_p^{(k)}$ and by Lemma 1 and Theorem 5 we have that

$$\begin{aligned} \omega_{k-1/p}(f; t) &\leq t^{k-1} \omega_{1-1/p}(f^{(k-1)}; t) \leq t^{k-1} v_p(f^{(k-1)}) \\ &:= t^{k-1} v_{p,0}(f^{(k-1)}) \\ &\leq t^{k-1} (k-1)! v_{p,0}^{(k)}(f) \\ &= (k-1)! t^{k-1} v_p^{(k)}(f) < \infty. \end{aligned} \quad (75)$$

Hence, it follows that

$$\begin{aligned} p \left(\frac{1}{p'} - s \right) \int_\delta^1 (t^{-s-(k-1)} \omega_{k-1/p}(f; t))^p \frac{dt}{t} \\ \leq p \left(\frac{1}{p'} - s \right) \delta^{-sp-1} v_p^{(k)}(f)^p. \end{aligned} \quad (76)$$

By adding up (74) and (76) we get

$$\begin{aligned} \left| p \left(\frac{1}{p'} - s \right) \int_0^1 [t^{-s-(k-1)} \omega_{k-1/p}(f; t)]^p \frac{dt}{t} - \|f^{(k)}\|_p^p \right| \\ \leq (1 - \delta^{p-sp-1}) \|f^{(k)}\|_p^p + \varepsilon \delta^{p-1-sp} \\ + p \left(\frac{1}{p'} - s \right) \delta^{-sp-1} v_p^{(k)}(f)^p. \end{aligned} \quad (77)$$

As $s \rightarrow 1/p' -$, the right hand side tends to ε . Since $\varepsilon > 0$ was arbitrary, by letting it tend to zero we get the first part of the theorem.

(ii) Applying the first part of the theorem to f_h and Lemma 7(i), we have that there exists a constant $C > 0$ such that

$$\begin{aligned} \frac{1}{p} \|f_h^{(k)}\|_p^p &= \lim_{s \rightarrow (1-1/p)-} \left(\frac{1}{p'} - s \right) \\ \cdot \int_0^1 [t^{-s-(k-1)} \omega_{k-1/p}(f_h; t)]^p \frac{dt}{t} \\ &\leq \overline{\lim}_{s \rightarrow (1-1/p)-} \left(\frac{1}{p'} - s \right) \\ \cdot \int_0^1 [t^{-s-(k-1)} \omega_{k-1/p}(f; t)]^p \frac{dt}{t} &= C \end{aligned} \quad (78)$$

and then

$$\begin{aligned} \|f_h^{(k)}\|_p^p &= \int_0^1 \left(\frac{|f^{(k-1)}(x+h) - f^{(k-1)}(x)|}{h} \right)^p dx \\ &\leq C \quad (0 < h \leq 1). \end{aligned} \quad (79)$$

Since $\sup_h h^{-1} \|\Delta_h f^{(k-1)}\|_p < \infty$, then by a deep result of Hardy and Littlewood (see [14, Theorem 24]) it follows that $f \in W_p^k$. □

In the following theorem we prove the continuity at the end points of the scale determined by $v_{p,\alpha}^{(k)}(f)$ in the same manner as in [1, 9].

Theorem 11. Let $1 < p < \infty$. If f is 1-periodic function, then

- (i) $\lim_{\alpha \rightarrow 1/p'} v_{p,\alpha}^{(k)}(f) = v_{p,1/p'}^{(k)}(f)$, and
- (ii) if $f \in V_{p,a}^{(k)}$, for some $a > 0$, then

$$\lim_{\alpha \rightarrow 0_+} v_{p,\alpha}^{(k)}(f) = v_p^{(k)}(f). \quad (80)$$

Proof. The proof of (i) and (ii) is similar to the proof of [1, Theorem 3.4] (see also [9, Theorem 3.3]). It is enough to define for a partition Π of type (5), $t_{1,1} < \dots < t_{1,2k} \leq t_{2,1} < \dots \leq t_{m,k+1} < \dots < t_{m,2k}$, $I_l = \{j : 2^{-l-1} < t_{j,2k} - t_{j,1} \leq 2^{-l}\}$, and

$$S_l^{(k)}(f) = \left(\sum_{l \in I_l} |f[t_{l,k+1}, \dots, t_{l,2k}] - f[t_{l,1}, \dots, t_{l,k}]|^p \right)^{1/p}. \quad (81)$$

By summation by parts, the result follows as in [9, Theorem 3.3]. $\square$

The condition $f \in V_{p,a}^{(k)}$ for some $a > 0$ is necessary for (80) to hold. If $f \in V_p^{(k)}$ has a discontinuous $(k-1)$ derivative, then $v_{p,\alpha}(f^{(k-1)}) = \infty$ for any $\alpha > 0$. Therefore, $v_{p,\alpha}^{(k)}(f) = \infty$ for all $\alpha > 0$, while $v_p^{(k)}(f) < \infty$.

Terehin in [13] showed that, for any function $f \in C_p$, the modulus $\omega_{1-1/p}(f; t)$ is equivalent to the nondecreasing function on $[0, 1]$ given by

$$\omega_1(f; t) = t^{1/p'} \inf_{0 < u \leq t} \frac{\omega_{1-1/p}(f; u)}{u^{1/p'}}. \quad (82)$$

Clearly, $t^{-1/p'} \omega_1(f; t)$ is a decreasing function and (see, e.g., [1])

$$\omega_1(t) \leq \omega_{1-1/p}(f; t) \leq 2^{1/p'} \omega_1(t) \quad (0 \leq t \leq 1). \quad (83)$$

Similarly, for any function f such that $\lim_{t \rightarrow 0_+} (\omega_{k-1/p}(f; t)/t^{k-1}) = 0$, it is possible to prove that the following function ω defined by

$$\omega(t) := \omega(f; t) = t^{1/p'} \inf_{0 < u \leq t} \frac{\omega_{k-1/p}(f; u)}{u^{(k-1)+1/p'}} \quad (84)$$

is increasing and $t^{-1/p'} \omega(t)$ is decreasing. Moreover, since $\omega_{k-1/p}(f; t)/t^{(k-1)+1/p'} \leq 2^{k-1/p} (\omega_{k-1/p}(f; s)/s^{(k-1)+1/p'})$ for any $s < t$ (see [15]), we have that

$$\omega(t) \leq \frac{\omega_{k-1/p}(f; t)}{t^{k-1}} \leq 2^{k-1/p} \omega(t) \quad (0 \leq t \leq 1). \quad (85)$$

In the next theorem we prove a sharp estimate of $v_{p,\alpha}^{(k)}(f)$ in terms of $\omega_{k-1/p}(f; t)$, which in particular implies the result of Merentes [7].

Theorem 12. Let $1 < p < \infty$, $0 < \alpha < 1/p'$, and f be a 1-periodic function in $V_p^{(k)}$. Assume that

$$I_{p,\alpha}^{(k)}(f) = \left(\int_0^1 [t^{-\alpha-(k-1)} \omega_{k-1/p}(f; t)]^p \frac{dt}{t} \right) < \infty. \quad (86)$$

Then, $f \in V_{p,\alpha}^{(k)}$ and

$$v_{p,\alpha}^{(k)}(f) \leq A \left(v_p^{(k)} + p' \alpha^{1/p} \left(\frac{1}{p'} - \alpha \right)^{1/p} I_{p,\alpha}^{(k)}(f) \right), \quad (87)$$

where A is constant dependent of k and p .

Proof. Condition (86) implies that $\lim_{t \rightarrow 0_+} (\omega_{k-1/p}(f; t)/t^{k-1}) = 0$. Without loss of generality (see [1, Theorem 4.1]), we may suppose that

$$\lim_{t \rightarrow 0_+} \frac{\omega(f; t)}{t^{1/p'}} = \infty. \quad (88)$$

Let $\omega_n = \omega(f; 2^{-n})$ and $\bar{\omega}_n = 2^{n/p'} \omega(f; 2^{-n})$, $n \in \mathbb{N}$. As in [1, page 5], we define the sequence of natural numbers n_k . Set $n_0 = 0$ and

$$n_{l+1} = \min \left(n; \max \left(\frac{\omega_n}{\omega_{n_l}}, \frac{\bar{\omega}_{n_l}}{\bar{\omega}_n} \right) \leq \frac{1}{2} \right), \quad (89)$$

$$l = 0, 1, \dots$$

By the monotonicity properties of $\omega(f; t)$ we have that either

$$\omega_{n_l} < 4\omega_{n_{l+1}} \quad (90)$$

or

$$\bar{\omega}_{n_{l+1}} < 4\bar{\omega}_{n_l}. \quad (91)$$

Assume first that (90) holds. Let us define

$$\begin{aligned} g_l(x) &= f_{2^{-n_l}, \dots, (k), 2^{-n_l}}(x) \\ &= 2^{n_l} \int_0^{2^{-n_l}} f_{2^{-n_l}, \dots, (k-1), 2^{-n_l}}(x+t) dt. \end{aligned} \quad (92)$$

For a fixed partition Π , we set

$$J_l = \{j : 2^{-n_{l+1}} < t_{j,2k} - t_{j,1} \leq 2^{-n_l}\}. \quad (93)$$

We define

$$\begin{aligned} R_{l,\alpha}(f) &:= \left(\sum_{j \in J_l} |f[t_{j,k+1}, \dots, t_{j,2k}] \right. \\ &\quad \left. - f[t_{j,1}, \dots, t_{j,k}] \right|^p \frac{1}{(t_{j,2k} - t_{j,1})^{\alpha p}} \Big)^{1/p}, \\ R_l(f) &:= \left(\sum_{j \in J_l} |f[t_{j,k+1}, \dots, t_{j,2k}] \right. \\ &\quad \left. - f[t_{j,1}, \dots, t_{j,k}] \right|^p \Big)^{1/p}. \end{aligned} \quad (94)$$

For any $j \in J_l$, we have that $g_l \in C^{(k)}$ and, by [10, Proposition 2.3],

$$g_l[t_0, \dots, t_k] = \frac{g_l^{(k)}(\psi)}{k!} \quad (\psi \in \text{conv}\{t_0, \dots, t_k\}). \quad (95)$$

Then, by Hölder's inequality it follows that

$$\begin{aligned} & |g_l[t_{j,k+1}, \dots, t_{j,2k}] - g_l[t_{j,1}, \dots, t_{j,k}]|^p \\ & \cdot \frac{1}{(t_{j,2k} - t_{j,1})^{\alpha p}} = \frac{|g_l^{(k)}(\psi'_j) - g_l^{(k)}(\psi_j)|^p}{k! |t_{j,2k} - t_{j,1}|^{\alpha p}} \\ & \leq \frac{2^{-n_l(p-1-\alpha p)}}{k!} \int_{\psi_j}^{\psi'_j} |g_l^{(k)}|^p, \end{aligned} \quad (96)$$

where $\psi_j \in \text{conv}\{t_{j,1}, \dots, t_{j,k}\}$ and $\psi'_j \in \text{conv}\{t_{j,k+1}, \dots, t_{j,2k}\}$.

Thus, by Lemma 7(ii) and by (85), it follows that

$$\begin{aligned} R_{l,\alpha}(g_l) & \leq 2^{-n_l(1/p'-\alpha)} \|g_l^{(k)}\|_p \\ & \leq 2^{-n_l(1/p'-\alpha)} 2^{n_l/p'} 2^{(k-1)n_l} \omega_{k-1/p}(f; 2^{-n_l}) \\ & \leq 2^{-n_l(1/p'-\alpha)} 2^{n_l/p'} 2^{k-1/p} \omega_{n_l}. \end{aligned} \quad (97)$$

Then

$$R_{l,\alpha}(g_l) \leq 2^{n_l\alpha+k} \omega_{n_l} \quad (98)$$

and moreover

$$R_{l,\alpha}(f - g_l) \leq 2^{n_{l+1}\alpha} R_l(f - g_l). \quad (99)$$

By (98) and (99), it follows that

$$\begin{aligned} v_{p,\alpha}^{(k)}(f, \Pi) & \leq 2^k \left(\sum_{l=0}^{\infty} 2^{n_l\alpha p} \omega_{n_l}^p \right)^{1/p} \\ & + \left(\sum_{l=0}^{\infty} 2^{n_{l+1}\alpha p} R_l(f - g_l)^p \right)^{1/p}. \end{aligned} \quad (100)$$

We estimate the second term of the above sum. Obviously, $R_l(f - g_k) \leq v_p^{(k)}(f - g_l)$. By Lemma 7(iii), (85), and (90), it follows that

$$\begin{aligned} 2^{n_{l+1}\alpha p} R_l(f - g_l)^p & \leq 2^{n_{l+1}\alpha p} v_p^{(k)}(f - g_l)^p \\ & \leq 2^{n_{l+1}\alpha p} \left(\frac{6k}{(k-1)!} \right)^p \left(\frac{\omega_{k-1/p}(f; 2^{-n_l})}{2^{-n_l(k-1)}} \right)^p \\ & \leq 2^{n_{l+1}\alpha p} \left(\frac{6k}{(k-1)!} \right)^p 2^{kp-1+2p} \omega_{n_{l+1}}^p. \end{aligned} \quad (101)$$

Hence,

$$\begin{aligned} v_{p,\alpha}^{(k)}(f, \Pi) & \leq 2^k \left(\sum_{l=0}^{\infty} 2^{n_l\alpha p} \omega_{n_l}^p \right)^{1/p} \\ & + \left(\sum_{l=0}^{\infty} 2^{n_{l+1}\alpha p} R_l(f - g_l)^p \right)^{1/p} \\ & \leq \left(2^k + 2^{k-1/p+2} \frac{6k}{(k-1)!} \right) \left(\sum_{l=0}^{\infty} 2^{n_l\alpha p} \omega_{n_l}^p \right)^{1/p}. \end{aligned} \quad (102)$$

Assume now that (91) holds. Let us define

$$\begin{aligned} \widehat{g}_l(x) & = f_{2^{-n_{l+1}}, \dots, (k), 2^{-n_{l+1}}}(x) \\ & = 2^{n_{l+1}} \int_0^{2^{-n_{l+1}}} f_{2^{-n_{l+1}}, \dots, (k-1), 2^{-n_{l+1}}}(x+t) dt. \end{aligned} \quad (103)$$

Thus by Lemma 7(ii) and (85) and by the definition of $\omega(f, t)$ and (91)

$$\begin{aligned} R_{l,\alpha}(\widehat{g}_l) & \leq 2^{-n_l(1/p'-\alpha)} 2^{n_{l+1}(k-1/p)} \omega_{k-1/p}(f; 2^{-n_{l+1}}) \\ & \leq 2^{-n_l(1/p'-\alpha)} 2^{n_{l+1}(k-1/p)} 2^{k-1/p} \omega_{n_{l+1}} \\ & \leq 4 \cdot 2^{-n_l(1/p'-\alpha)} 2^{k-1/p} \overline{\omega}_{n_l} \\ & \leq 4 \cdot 2^{-n_l(1/p'-\alpha)} 2^{k-1/p} 2^{n_l/p'} \omega_{n_l} \\ & \leq 4 \cdot 2^{n_l\alpha+k} \omega_{n_l}. \end{aligned} \quad (104)$$

Then

$$R_{l,\alpha}(\widehat{g}_l) \leq 4 \cdot 2^{n_l\alpha+k} \omega_{n_l} \quad (105)$$

and moreover

$$R_{l,\alpha}(f - \widehat{g}_l) \leq 2^{n_{l+1}\alpha} R_l(f - \widehat{g}_l). \quad (106)$$

By (105) and (106), it follows that

$$\begin{aligned} v_{p,\alpha}^{(k)}(f, \Pi) & \leq 4 \cdot 2^k \left(\sum_{l=0}^{\infty} 2^{n_l\alpha p} \omega_{n_l}^p \right)^{1/p} \\ & + \left(\sum_{l=0}^{\infty} 2^{n_{l+1}\alpha p} R_l(f - \widehat{g}_l)^p \right)^{1/p}. \end{aligned} \quad (107)$$

We estimate now the second term of the above sum. Obviously, we have $R_l(f - \widehat{g}_l) \leq v_p^{(k)}(f - \widehat{g}_l)$. By Lemma 7(iii) and (85) we get that

$$\begin{aligned} 2^{n_{l+1}\alpha p} R_l(f - \widehat{g}_l)^p & \leq 2^{n_{l+1}\alpha p} v_p^{(k)}(f - \widehat{g}_l)^p \\ & \leq 2^{n_{l+1}\alpha p} \left(\frac{6k}{(k-1)!} \right)^p \left(\frac{\omega_{k-1/p}(f; 2^{-n_{l+1}})}{2^{-n_{l+1}(k-1)}} \right)^p \\ & \leq \left(\frac{2^k 6k}{(k-1)!} \right)^p 2^{n_{l+1}\alpha p} \omega_{n_{l+1}}^p. \end{aligned} \quad (108)$$

Hence,

$$\begin{aligned} v_{p,\alpha}^{(k)}(f, \Pi) &\leq 4 \cdot 2^k \left(\sum_{l=0}^{\infty} 2^{n_l \alpha p} \omega_{n_l}^p \right)^{1/p} \\ &\quad + \left(\sum_{l=0}^{\infty} 2^{n_{l+1} \alpha p} R_l (f - \widehat{g}_l)^p \right)^{1/p} \\ &\leq 2^k \left(4 + \frac{6k}{(k-1)!} \right) \left(\sum_{l=0}^{\infty} 2^{n_l \alpha p} \omega_{n_l}^p \right)^{1/p}. \end{aligned} \quad (109)$$

By (102) and (109),

$$v_{p,\alpha}^{(k)}(f, \Pi) \leq 2^{k+2} \left(1 + \frac{6k}{(k-1)!} \right) \left(\sum_{l=0}^{\infty} 2^{n_l \alpha p} \omega_{n_l}^p \right)^{1/p}. \quad (110)$$

Applying [1, Lemma 2.1] with $q = p$, $\beta = p\alpha$ and $\gamma = 1/p'$, it follows that

$$\begin{aligned} v_{p,\alpha}^{(k)}(f, \Pi) &\leq 2^{k+2} \left(1 + \frac{6k}{(k-1)!} \right) \left(\sum_{l=0}^{\infty} 2^{n_l \alpha p} \omega_{n_l}^p \right)^{1/p} \\ &\leq 2^{k+2} \left(1 + \frac{6k}{(k-1)!} \right) \left[2\omega_0^p \right. \\ &\quad \left. + \frac{2^{p+2} p'}{p} p\alpha \left(\frac{p}{p'} - \alpha p \right) \int_0^1 t^{-\alpha p} \omega(t)^p \frac{dt}{t} \right]^{1/p} \\ &\leq 2^{k+6} \left(1 + \frac{6k}{(k-1)!} \right) \left[\omega_0 \right. \\ &\quad \left. + p' \alpha^{1/p} \left(\frac{1}{p'} - \alpha \right)^{1/p} \left(\int_0^1 t^{-\alpha p} \omega(t)^p \frac{dt}{t} \right)^{1/p} \right], \end{aligned} \quad (111)$$

since $p^{1/p} \leq 2$ and $(p')^{1/p} \leq p'$.

On the other hand, by (85),

$$\begin{aligned} \omega_0 &:= \omega(1) \leq \frac{\omega_{k-1/p}(f; 1)}{1} \leq \omega_{1-1/p}(f^{(k-1)}; 1) \\ &\leq v_p(f^{(k-1)}) = v_p^{(k)}(f), \end{aligned} \quad (112)$$

and also

$$\begin{aligned} &\left(\int_0^1 t^{-\alpha p} \omega(t)^p \frac{dt}{t} \right)^{1/p} \\ &\leq \left(\int_0^1 (t^{-\alpha-(k-1)} \omega_{k-1/p}(f; t))^p \frac{dt}{t} \right)^{1/p} \\ &= I_{p,\alpha}^{(k)}(f). \end{aligned} \quad (113)$$

Therefore,

$$\begin{aligned} v_{p,\alpha}^{(k)}(f, \Pi) &\leq 2^{k+6} \left(1 + \frac{6k}{(k-1)!} \right) \\ &\quad \cdot \left[v_p^{(k)}(f) + p' \alpha^{1/p} \left(\frac{1}{p'} - \alpha \right)^{1/p} I_{p,\alpha}^{(k)}(f) \right]. \end{aligned} \quad (114)$$

□

Remarks. If $f \in W_p^k$, then, by Theorem 10(i), we have that

$$\begin{aligned} &\lim_{s \rightarrow 1/p' -} \left(\frac{1}{p'} - s \right)^{1/p} \\ &\quad \cdot \left(\int_0^1 [t^{-s-(k-1)} \omega_{k-1/p}(f; t)]^p \frac{dt}{t} \right)^{1/p} = \left(\frac{1}{p} \right)^{1/p} \\ &\quad \cdot \|f^{(k)}\|_p. \end{aligned} \quad (115)$$

Moreover, by the celebrated result of Riesz and Theorem 5,

$$\begin{aligned} v_p^{(k)}(f) &= \lim_{\alpha \rightarrow 0} v_{p,\alpha}^{(k)}(f) \leq v_{p,1/p'}^{(k)}(f) \\ &= \frac{1}{(k-1)!} v_{p,1/p'}(f^{(k-1)}) \\ &= \frac{1}{(k-1)!} \|f^{(k)}\|_p. \end{aligned} \quad (116)$$

The upper limit as $\alpha \rightarrow 1/p' -$ of the right hand side of (87) is less than

$$\begin{aligned} &2^{k+6} \left(1 + \frac{6k}{(k-1)!} \right) \\ &\quad \cdot \left[\frac{1}{(k-1)!} + p' \left(\frac{1}{p'} \right)^{1/p} \left(\frac{1}{p} \right)^{1/p} \right] \|f^{(k)}\|_p. \end{aligned} \quad (117)$$

Also, by Theorem 11(i), the left hand side of (87) tends to $v_{p,1/p'}^{(k)}(f)$. Thus, taking limit as $\alpha \rightarrow 1/p' -$ in it, it follows that

$$v_{p,1/p'}^{(k)}(f) \leq C(p, k) \|f^{(k)}\|_p. \quad (118)$$

This implies the result of Merentes [7] that states that $W_p^k \subset V_{p,1/p'}^{(k)}$, and it shows that the order of the constant is optimal.

Let us assume that $I_{p,\alpha_0}^{(k)}(f) < \infty$ for some $0 < \alpha_0 < 1/p'$. Since $I_{p,\alpha}^{(k)}(f) \leq I_{p,\alpha_0}^{(k)}(f)$ for $0 < \alpha < \alpha_0$, it follows that

$$\lim_{\alpha \rightarrow 0_+} \alpha^{1/p} I_{p,\alpha}^{(k)}(f) \leq \lim_{\alpha \rightarrow 0_+} \alpha^{1/p} I_{p,\alpha_0}^{(k)}(f) = 0. \quad (119)$$

Moreover, since $I_{p,\alpha_0}^{(k)}(f) < \infty$, by Theorem 12, $f \in V_{p,\alpha_0}^{(k)}$. Hence, $v_{p,\alpha}^{(k)}(f)$ tends to $v_p^{(k)}(f)$ as $\alpha \rightarrow 0_+$ by Theorem 11(ii). Thus, the left hand side of (87) agree in behavior with the right hand side when $\alpha \rightarrow 0_+$ which means that the order of the constant is also optimal.

4. Further Results and Applications

Lemma 13. Let $1 < p < \infty$, $0 < \alpha < 1/p'$, and f be a 1-periodic function. Assume that

$$\begin{aligned} I_{p,\alpha}^{(k)}(f) &= \left(\int_0^1 [t^{-\alpha-(k-1)} \omega_{k-1/p}(f; t)]^p \frac{dt}{t} \right)^{1/p} \\ &< \infty. \end{aligned} \quad (120)$$

Then

$$\begin{aligned} I_{p,\alpha}^{(k)}(f) &\leq \left(\int_0^1 \left[t^{-\alpha} \omega_{1-1/p}(f^{(k-1)}; t) \right]^p \frac{dt}{t} \right)^{1/p} \\ &= I_{p,\alpha}(f^{(k-1)}) \leq C(p, \alpha, k) I_{p,\alpha}^{(k)}(f), \end{aligned} \quad (121)$$

where

$$C(p, \alpha, k) = C 2^{1-1/p} \left(\frac{(k-1)^p}{\alpha^p} + 2^{kp-1} C(p) \right)^{1/p} \quad (122)$$

if $1 \leq p \leq 2$ and $C(p, \alpha, k) = C 2^{1-1/p} ((k-1)^p / \alpha^p + 1 / (p - \alpha p - 1))^{1/p}$ if $p \geq 2$.

Proof. From the pointwise estimate of Lemma 1, we have that

$$\begin{aligned} I_{p,\alpha}^{(k)}(f) &\leq \left(\int_0^1 \left[t^{-\alpha-(k-1)} t^{k-1} \omega_{1-1/p}(f^{(k-1)}; t) \right]^p \frac{dt}{t} \right)^{1/p} \\ &= I_{p,\alpha}(f^{(k-1)}). \end{aligned} \quad (123)$$

We prove now the second inequality. Since $I_{p,\alpha}^{(k)}(f) < \infty$, it follows that $I_{p,\alpha}(f^{(k-1)}) < \infty$. Then, $f^{(k-1)} \in C_p$. We use the following Marchaud-Timan type inequality in [15, Theorem 4],

$$\begin{aligned} \omega_{1-1/p}(f^{(k-1)}; t) &\leq C \left((k-1) \int_0^t \frac{\omega_{k-1/p}(f; u)}{u^k} du \right. \\ &\quad \left. + t^{1-1/p} \left(\int_t^1 \frac{\omega_{k-1/p}(f; u)^s}{u^{s(k-1/p)+1}} du \right)^{1/s} \right), \end{aligned} \quad (124)$$

where C is an absolute constant independent on p and $s = \max(p, 2)$.

Assume first that $s = p$, that is, $p \geq 2$. Let $t > 0$. By raising (124) to p , then multiplying both sides of it by $t^{-\alpha p-1}$, and then integrating it from 0 to 1 we obtain

$$\begin{aligned} \int_0^1 \left[t^{-\alpha} \omega_{1-1/p}(f^{(k-1)}; t) \right]^p \frac{dt}{t} &\leq C^p 2^{p-1} \left[(k-1)^p \right. \\ &\quad \cdot \int_0^1 t^{-\alpha p-1} \left(\int_0^t \frac{\omega_{k-1/p}(f; u)}{u^k} du \right)^p dt \\ &\quad \left. + \int_0^1 t^{p-\alpha p-2} \left(\int_t^1 \frac{\omega_{k-1/p}(f; u)^p}{u^{kp}} du \right) dt \right]. \end{aligned} \quad (125)$$

We estimate now the first term of the above sum by the use of the sharp weighted Hardy inequality (see, e.g., [16, Lemma 3.9]) and we get

$$\begin{aligned} \int_0^1 t^{-\alpha p-1} \left(\int_0^t \frac{\omega_{k-1/p}(f; u)}{u^k} du \right)^p dt \\ \leq \frac{1}{\alpha^p} \int_0^1 \left[t^{-\alpha-(k-1)} \omega_{k-1/p}(f; t) \right]^p \frac{dt}{t}. \end{aligned} \quad (126)$$

For the second term, by Fubini's theorem we have

$$\begin{aligned} \int_0^1 t^{p-\alpha p-2} \left(\int_t^1 \frac{\omega_{k-1/p}(f; u)^p}{u^{kp}} du \right) dt \\ = \frac{1}{p - \alpha p - 1} \int_0^1 \left[t^{-\alpha} \frac{\omega_{k-1/p}(f; t)}{t^{k-1}} \right]^p \frac{dt}{t}. \end{aligned} \quad (127)$$

Therefore, (126) and (127) imply the following estimate

$$\begin{aligned} \int_0^1 \left[t^{-\alpha} \omega_{1-1/p}(f^{(k-1)}; t) \right]^p \frac{dt}{t} \\ \leq C^p 2^{p-1} \left(\frac{(k-1)^p}{\alpha^p} + \frac{1}{p - \alpha p - 1} \right) \\ \cdot \int_0^1 \left[t^{-\alpha-(k-1)} \omega_{k-1/p}(f; t) \right]^p \frac{dt}{t}. \end{aligned} \quad (128)$$

Let now $s = 2$, that is, $1 < p \leq 2$. By raising (124) to p , then multiplying both sides of it by $t^{-\alpha p-1}$, and then integrating it from 0 to 1 we get

$$\begin{aligned} \int_0^1 \left[t^{-\alpha} \omega_{1-1/p}(f^{(k-1)}; t) \right]^p \frac{dt}{t} &\leq C^p 2^{p-1} \left[(k-1)^p \right. \\ &\quad \cdot \int_0^1 t^{-\alpha p-1} \left(\int_0^t \frac{\omega_{k-1/p}(f; u)}{u^k} du \right)^p dt \\ &\quad \left. + \int_0^1 t^{p-\alpha p-2} \left(\int_t^1 \frac{\omega_{k-1/p}(f; u)^2}{u^{2(k-1/p)+1}} du \right)^{p/2} dt \right]. \end{aligned} \quad (129)$$

The first term of the right-hand side of the inequality can be estimated as before (see (126)). The second term instead can be estimated by the fact that for any $u > t$ the following inequality holds

$$\begin{aligned} \left(\frac{\omega_{k-1/p}(f; u)}{u^{k-1/p}} \right)^2 &= \left(\frac{\omega_{k-1/p}(f; u)}{u^{(k-1)+1/p'}} \right)^2 \\ &\leq \left(2^{k-1/p} \frac{\omega_{k-1/p}(f; t)}{t^{(k-1)+1/p'}} \right)^2. \end{aligned} \quad (130)$$

Then,

$$\begin{aligned} \int_0^1 t^{p-\alpha p-2} \left(\int_t^1 \left(\frac{\omega_{k-1/p}(f; u)}{u^{k-1/p}} \right)^2 \frac{du}{u} \right)^{p/2} dt \\ \leq 2^{kp-1} C(p) \int_0^1 \left[t^{-\alpha-(k-1)} \omega_{k-1/p}(f; u) \right]^p \frac{dt}{t}. \end{aligned} \quad (131)$$

Therefore,

$$\begin{aligned} & \int_0^1 \left[t^{-\alpha} \omega_{1-1/p} (f^{(k-1)}; t) \right]^p \frac{dt}{t} \\ & \leq C^p 2^{p-1} \left(\frac{(k-1)^p}{\alpha^p} + 2^{kp-1} C(p) \right) \\ & \cdot \int_0^1 \left[t^{-\alpha-(k-1)} \omega_{k-1/p} (f; t) \right]^p \frac{dt}{t}. \end{aligned} \quad (132)$$

□

Theorem 12 can be derived by [1, Theorem 4.1] and the previous lemma, but the constant will not have right asymptotic behaviour. Indeed, since $f \in V_p^{(k)}$, then, as in Proposition 14,

$$v_{p,\alpha}^{(k)}(f) \leq A \left(v_p^{(k)}(f) + c_{p,\alpha} \frac{C(p, \alpha, k)}{(k-1)!} I_{p,\alpha}^{(k)}(f) \right). \quad (133)$$

Proposition 14. Let $1 < p < \infty$, $0 < \alpha < 1/p'$, and f be a 1-periodic function such that $f^{(k-1)}$ belongs to V_p . Assume that

$$I_{p,\alpha}^{(k)}(f) = \left(\int_0^1 \left[t^{-\alpha-(k-1)} \omega_{k-1/p} (f; t) \right]^p \frac{dt}{t} \right) < \infty. \quad (134)$$

Then, $f \in V_{p,\alpha}^{(k)}$ and

$$v_{p,\alpha}^{(k)}(f) \leq A \left(v_p^{(k)}(f) + \frac{c_{p,\alpha} C(p, \alpha, k)}{(k-1)!} I_{p,\alpha}^{(k)}(f) \right), \quad (135)$$

where A is an absolute constant and $C(p, \alpha, k)$ is in Lemma 13.

Proof. The result follows by [1, Theorem 4.1], Lemma 13, and Theorem 5. Indeed, since $f^{(k-1)} \in V_p$ and $I_{p,\alpha}^{(k)}(f) < \infty$, then by [1, Theorem 4.1] it follows that $f^{(k-1)} \in V_{p,\alpha}$. Then, $f \in V_{p,\alpha}^{(k)}$ by Theorem 5. By [1, Theorem 4.1] and Lemma 13, it follows that

$$\begin{aligned} v_{p,\alpha}^{(k)}(f^{(k-1)}) & \leq A \left(v_p(f^{(k-1)}) + c_{p,\alpha} I_{p,\alpha}(f^{(k-1)}) \right) \\ & \leq A \left((k-1)! v_p^{(k)}(f) + c_{p,\alpha} C(p, \alpha, k) I_{p,\alpha}^{(k)}(f) \right), \end{aligned} \quad (136)$$

where A is the absolute constant in [1, Theorem 4.1]. Finally, applying again Theorem 5, it follows that

$$v_{p,\alpha}^{(k)}(f) \leq A \left(v_p^{(k)}(f) + c_{p,\alpha} \frac{C(p, \alpha, k)}{(k-1)!} I_{p,\alpha}^{(k)}(f) \right). \quad (137)$$

The inequality is sharp in the sense that (134) cannot be removed. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The authors are very grateful to the professors N. Merentes and J. L. Sánchez for useful discussions. They would like to thank Professor V. Kolyada for sharing [15] and the referees by their appropriate comments. This research has been partially supported by the Central Bank of Venezuela. They want to thank the library staff of B.C.V for compiling the references.

References

- [1] M. Lind, "On fractional smoothness of functions related to p -variation," *Mathematical Inequalities & Applications*, vol. 16, no. 1, pp. 21–39, 2013.
- [2] N. Wiener, "The quadratic variation of a function and its Fourier coefficients," *Journal of Mathematical Physics*, vol. 3, pp. 72–94, 1924.
- [3] I. P. Natanson, *Theory of Functions of a Real Variable*, Frederick Ungar Publishing Company, New York, NY, USA, 1955.
- [4] F. Riesz, "Untersuchungen über systeme integrierbarer funktionen," *Mathematische Annalen*, vol. 69, no. 4, pp. 449–497, 1910.
- [5] A. P. Terehin, "Approximation of functions of bounded p -variations," *Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika*, vol. 2, pp. 171–187, 1965 (Russian).
- [6] C. J. de la Vallée Poussin, "Sur la convergence des formules d'interpolation entre ordonnées equidistantes," *Bulletins de l'Académie Royale des Sciences, des Lettres et des Beaux Arts de Belgique*, pp. 314–410, 1908.
- [7] N. Merentes, "On functions of bounded $(p, 2)$ -variation," *Collectanea Mathematica*, vol. 43, no. 2, pp. 117–123, 1992.
- [8] J. Appel, J. Banas, and N. Merentes, *Bounded Variation and Around*, De Gruyter, Boston, Mass, USA, 2014.
- [9] S. Barza and P. Silvestre, "Functions of bounded second p -variation," *Revista Matemática Complutense*, vol. 27, no. 1, pp. 69–91, 2014.
- [10] N. Merentes, S. Rivas, and J. L. Sánchez, "On functions of bounded (p, k) -variation," *Journal of Function Spaces and Applications*, vol. 2012, Article ID 202987, 9 pages, 2012.
- [11] S. S. Volosivets, "Asymptotic properties of one compact set of smooth functions in the space of functions of bounded p -variation," *Mathematical Notes*, vol. 57, no. 1–2, pp. 148–157, 1995.
- [12] R. A. DeVore and G. G. Lorentz, *Constructive Approximation*, Springer, Berlin, Germany, 1993.
- [13] A. P. Terehin, "Functions of bounded p -variations with a given modulus of continuity," *Matematicheskie Zametki*, vol. 53, no. 6, pp. 523–530, 1972 (Russian), translated in *Mathematical Notes*, vol. 12, pp. 751–755, 1972.
- [14] G. H. Hardy and J. E. Littlewood, "Some properties of fractional integrals. I," *Mathematische Zeitschrift*, vol. 27, no. 1, pp. 565–606, 1928.
- [15] S. S. Volosivets, "Polynomials of best approximation and relations between moduli of continuity in spaces of functions of bounded p -variation," *Russian Mathematics*, vol. 40, no. 9, pp. 21–26, 1996.
- [16] C. Bennett and R. Sharpley, *Interpolation of Operators*, Academic Press, Boston, Mass, USA, 1988.