Research Article

Existence of Solutions for Degenerate Elliptic Problems in Weighted Sobolev Space

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This paper is devoted to the study of the existence of solutions to a general elliptic problem $Au + g(x, u, \nabla u) = f - \operatorname{div} F$, with $f \in L^1(\Omega)$ and $F \in \prod_{i=1}^N L^{p'}(\Omega, \omega_i)$, where A is a Leray-Lions operator from a weighted Sobolev space into its dual and $g(x, s, \xi)$ is a nonlinear term satisfying $g(x, s, \xi) \operatorname{sgn}(s) \geq \rho \sum_{i=1}^N \omega_i |\xi_i|^p$, $|s| \geq h > 0$, and a growth condition with respect to ξ . Here, ω_i, ω_i^* are weight functions that will be defined in the Preliminaries.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ ($N \geq 1$) and let p be a real number with $1 < p < \infty$. Denote by X the weighted Sobolev space $W_0^{1,p}(\Omega, \omega)$, associated with a vector of weight functions $\omega = \{\omega_i(x)\}_{0 \leq i \leq N}$, which is endowed with the usual norm $\|\cdot\|_{1,p,\omega}$. In this paper, we consider a general class of degenerate elliptic problems:

$$\begin{aligned} Au + g(x, u, \nabla u) &= \mu \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \quad (1)$$

where $Au = -\operatorname{div}(a(x, u, \nabla u))$ and the right-hand side term $\mu = f - \operatorname{div} F$, where $f \in L^1(\Omega)$, $F \in \prod_{i=1}^N L^{p'}(\Omega, \omega_i^*)$. We also assume the following:

(H1) The expression

$$\|u\|_X := \left(\sum_{i=1}^N \int_{\Omega} \left| \frac{\partial u(x)}{\partial x_i} \right|^p \omega_i(x) dx \right)^{1/p} \quad (2)$$

is a norm defined on X and it is equivalent to $\|\cdot\|_{1,p,\omega}$.

(H2) There exist a weight function $\sigma(x)$ and a parameter q , $1 < q < \infty$, such that

$$\sigma^{1-q'} \in L^1(\Omega), \quad (3)$$

with $q' = q/(q-1)$. The Hardy inequality

$$\begin{aligned} &\left(\int_{\Omega} |u(x)|^q \sigma dx \right)^{1/q} \\ &\leq C \left(\sum_{i=1}^N \int_{\Omega} \left| \frac{\partial u(x)}{\partial x_i} \right|^p \omega_i(x) dx \right)^{1/p} \end{aligned} \quad (4)$$

holds for every $u \in X$ with a constant $C > 0$ independent of u . Moreover, the embedding

$$X \hookrightarrow L^q(\Omega, \sigma) \quad (5)$$

is compact. Interested reader may refer to [1] for some examples of weights which satisfy the above Hardy inequality (see (4)).

(H3) $a(x, s, \xi) = \{a_i(x, s, \xi)\}_{1 \leq i \leq N} : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a Carathéodory vector-valued function, and for all $i = 1, \dots, N$, there hold

$$|a_i(x, s, \xi)| \leq c_1 \omega_i^{1/p}(x) \cdot \left[k(x) + \sigma^{1/p'} |s|^{q/p'} + \sum_{j=1}^N \omega_j^{1/p'}(x) |\xi_j|^{p-1} \right], \quad (6)$$

$$a(x, s, \xi) \cdot \xi \geq c_0 \sum_{i=1}^N \omega_i(x) |\xi_i|^p, \quad (7)$$

$$[a(x, s, \xi) - a(x, s, \eta)] \cdot (\xi - \eta) > 0, \quad \xi \neq \eta \in \mathbb{R}^N, \quad (8)$$

where $k(x)$ is a positive function in $L^{p'}(\Omega)$, $1/p + 1/p' = 1$, and the constants c_0, c_1 are both positive.

(H4) Let $g(x, s, \xi)$ be a Carathéodory function satisfying the following assumptions:

$$g(x, s, \xi) \operatorname{sgn}(s) \geq \rho \sum_{i=1}^N \omega_i |\xi_i|^p, \quad |s| \geq h, \quad (9)$$

for some $h, \rho > 0$, and

$$|g(x, s, \xi)| \leq b(|s|) \left(\sum_{i=1}^N \omega_i |\xi_i|^p + d(x) \right), \quad (10)$$

with $b : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, a continuous increasing function, and $d(x)$, a nonnegative function in $L^1(\Omega)$.

In the past decade, much attention has been devoted to nonlinear elliptic equations because of their wide application to physical models such as non-Newtonian fluids, boundary layer phenomena for viscous fluids, and chemical heterogeneous model. When $-\operatorname{div} F = 0$, Akdim et al. [2] proved in the variational setting, under assumptions (H1)–(H4), that, for every $f \in W^{-1,p'}(\Omega, \omega^*)$, with g satisfying the sign condition

$$g(x, s, \xi) \operatorname{sgn}(s) \geq 0. \quad (11)$$

Problem (1) has a solution $u \in W_0^{1,p}(\Omega, \omega)$, where the authors used the approach based on the strong convergence of the positive part u_n^+ (negative part u_n^-) of u_n (the approximating sequence of u). Ammar [3] extended this existence result to problems with general data $f \in L^1(\Omega)$, under hypotheses (H1)–(H4). They also used a similar approach to prove the existence of renormalized solutions. When $-\operatorname{div} F \neq 0$, Aharouch et al. [4] proved the existence result for problem (1), by assuming the sign condition (11). For more details on weighted Sobolev spaces, the readers may refer to [5].

Boccardo et al. [6] considered the nonlinear boundary value problem

$$Au + g(x, u, \nabla u) = \mu, \quad (12)$$

where $\mu \in L^1(\Omega) + W^{-1,p'}(\Omega)$ and $g(x, u, \nabla u) \in L^1(\Omega)$ with sign condition (9) for large values of s . By combining

the truncation technique with some delicate test functions, the authors showed that the problem has a solution $u \in W_0^{1,p}(\Omega)$. Mainly motivated by [4, 6], we investigate the elliptic problem (1) in weighted Sobolev space. By choosing test functions different from those employed in [4, 6], we show that problem (1) admits at least one weak solution with (9) instead of the sign condition (11). It is worth pointing out that (9) gives a sign condition on $g(x, s, \xi)$ only for large values of s , which brings about many difficulties. The essential one of those is that we have to construct some new test functions to obtain the a priori estimates of the approximation solutions u_n since the usual one $T_k(u_n)$ is not a proper test function for our problem. The outline of this paper is as follows. In Section 2, we give some preliminaries and some technical lemmas. The main results will be stated and proved in Section 3.

2. Preliminaries

In this section, we give some preliminaries (see [5]). Throughout this section, we assume that the vector field $a(x, s, \xi) = \{a_i(x, s, \xi)\}_{1 \leq i \leq N} : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ satisfies assumptions (6)–(8) and g satisfies (9)–(10). Let $\omega = \{\omega_i(x)\}_{0 \leq i \leq N}$ be a vector of measurable weight functions strictly positive a.e. in Ω , such that

$$\begin{aligned} \omega_i &\in L^1(\Omega), \\ \omega_i^{-1/(p-1)} &\in L^1(\Omega). \end{aligned} \quad (13)$$

We define the weighted space with weight γ on Ω as

$$L^p(\Omega, \gamma) = \{u = u(x) : u\gamma^{1/p} \in L^p(\Omega)\}. \quad (14)$$

With this space, we equip the norm

$$\|u\|_{p,\gamma} = \left(\int_{\Omega} |u(x)|^p \gamma(x) dx \right)^{1/p}. \quad (15)$$

We denote by $W_0^{1,p}(\Omega, \omega)$ the space of all real-valued functions $u \in L^p(\Omega, \omega_0)$ such that the derivatives (see [5]) in the sense of distributions satisfy

$$\frac{\partial u}{\partial x_i} \in L^p(\Omega, \omega_i) \quad \forall i = 1, \dots, N, \quad (16)$$

endowed with the norm

$$\begin{aligned} \|u\|_{1,p,\omega} &= \left(\int_{\Omega} |u(x)|^p \omega_0(x) dx \right. \\ &\quad \left. + \sum_{i=1}^N \int_{\Omega} \left| \frac{\partial u(x)}{\partial x_i} \right|^p \omega_i(x) dx \right)^{1/p}. \end{aligned} \quad (17)$$

Let $X := W_0^{1,p}(\Omega, \omega)$ be the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{1,p,\omega}$. Then, $(X, \|\cdot\|_{1,p,\omega})$ is a reflexive Banach space whose dual is equivalent to $W^{-1,p'}(\Omega, \omega^*)$, where $\omega^* = \{\omega_i^* = \omega_i^{1-p'} = \omega_i^{-p'/p}\}$, $i = 1, \dots, N$, and $p' = p/(p-1)$. As usual, for

s, k in $\mathbb{R}$, with $k \geq 0$, we denote $T_k(s) = \max(-k, \min(k, s))$ and $G_k(s) = s - T_k(s)$.

The following lemmas will be needed throughout this paper (refer to [2, 7]).

Lemma 1. Let a and b be two nonnegative real numbers, and let

$$\varphi(s) = se^{\theta s^2}, \quad (18)$$

with $\theta = b^2/4a^2$. Then,

$$a\varphi'(s) - b|\varphi(s)| \geq \frac{a}{2}, \quad s \in \mathbb{R}. \quad (19)$$

Lemma 2. Let $g \in L^r(\Omega, \gamma)$ and $g_n \in L^r(\Omega, \gamma)$, with $\|g_n\|_{L^r(\Omega, \gamma)} \leq c$, $1 < r < \infty$. If $g_n \rightarrow g$ a.e. in Ω , then $g_n \rightarrow g$ weakly in $L^r(\Omega, \gamma)$, where γ is a weight function on Ω .

Lemma 3 (assume (H1)). Let $G : \mathbb{R} \rightarrow \mathbb{R}$ be uniformly Lipschitzian, with $G(0) = 0$. Let $u \in W_0^{1,p}(\Omega, \omega)$. Moreover, if the set D of discontinuity points of G' is finite, then

$$\begin{aligned} & \frac{\partial G \circ u}{x_i} \\ &= \begin{cases} 0, & \text{a.e. in } \{x \in \Omega : u(x) \in D\}, \\ G'(u) \frac{\partial u}{\partial x_i}, & \text{a.e. in } \{x \in \Omega : u(x) \notin D\}. \end{cases} \end{aligned} \quad (20)$$

Lemma 4 (assume (H1)). Let $u \in W_0^{1,p}(\Omega, \omega)$ and $T_k(u)$, $k \in \mathbb{R}^+$, be the usual truncation. Then, $T_k(u) \in W_0^{1,p}(\Omega, \omega)$. Moreover, one has

$$T_k(u) \rightarrow u \quad \text{strongly in } W_0^{1,p}(\Omega, \omega). \quad (21)$$

Lemma 5 (assume (H1) and (H2)). Let $\{u_n\}$ be a sequence of functions in $W_0^{1,p}(\Omega, \omega)$ such that $u_n \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega, \omega)$ and

$$\begin{aligned} & \lim_{n \rightarrow \infty} \int_{\Omega} [a(x, u_n, \nabla u_n) - a(x, u, \nabla u)] \cdot \nabla (u_n - u) dx \\ &= 0. \end{aligned} \quad (22)$$

Then, $u_n \rightarrow u$ strongly in $W_0^{1,p}(\Omega, \omega)$.

3. Main Results

Firstly, we give the definition of weak solution for problem (1).

Definition 6. One says $u \in W_0^{1,p}(\Omega, \omega)$ is a weak solution to problem (1), provided that

$$\begin{aligned} & \int_{\Omega} a(x, u, \nabla u) \cdot \nabla v dx + \int_{\Omega} g(x, u, \nabla u) v dx \\ &= \int_{\Omega} f v dx + \int_{\Omega} F \cdot \nabla v dx, \end{aligned} \quad (23)$$

for every $v \in W_0^{1,p}(\Omega, \omega) \cap L^\infty(\Omega)$.

Now, we will state and prove our main result on the existence of weak solutions to problem (1).

Theorem 7. Let f be in $L^1(\Omega)$ and $F \in \prod_{i=1}^N L^{p'}(\Omega, \omega_i^*)$. Then, there exists at least one solution u to problem (1).

Proof. The proof will be divided into 5 steps.

Step 1 (the approximation equation). We introduce the following approximation equation of problem (1). Let f_n be a sequence of $L^\infty(\Omega)$ functions that converges to f strongly in $L^1(\Omega)$ and let $n \in N$,

$$g_n(x, s, \xi) = \frac{g(x, s, \xi)}{1 + (1/n)|g(x, s, \xi)|}; \quad (24)$$

then $g_n(x, s, \xi)$ is bounded and satisfies (10) and

$$g_n(x, s, \xi) \cdot \text{sgn}(s) \geq 0, \quad (25)$$

for almost every x in Ω , for every ξ in $\mathbb{R}^N$, and for every s in $\mathbb{R}$ with $|s| \geq h$. By the results of [2], there exists a solution $u_n \in W_0^{1,p}(\Omega, \omega)$ of

$$\begin{aligned} A(u_n) + g_n(x, u_n, \nabla u_n) &= f_n - \text{div} F \quad \text{in } \Omega, \\ u_n &= 0 \quad \text{on } \partial\Omega, \end{aligned} \quad (26)$$

which satisfies

$$\begin{aligned} & \int_{\Omega} a(x, u_n, \nabla u_n) \cdot \nabla v dx + \int_{\Omega} g_n(x, u_n, \nabla u_n) v dx \\ &= \int_{\Omega} f_n v dx + \int_{\Omega} F \cdot \nabla v dx, \end{aligned} \quad (27)$$

for every $v \in W_0^{1,p}(\Omega, \omega) \cap L^\infty(\Omega)$.

Step 2 (the weak convergence $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega, \omega)$). Take $v = \varphi(T_h(u_n))$ as a test function in (27), where $h > 0$ is defined in (9) and $\varphi(s)$ is as in (19). Writing $\varphi'_h = \varphi'(T_h(u_n))$ and $\varphi_h = \varphi(T_h(u_n))$ for simplicity, we have

$$\begin{aligned} & \int_{\Omega} (a(x, u_n, \nabla u_n) \cdot \nabla T_h(u_n)) \varphi'_h dx \\ &+ \int_{\Omega} g_n(x, u_n, \nabla u_n) \varphi_h dx \\ &= \int_{\Omega} f_n \varphi_h dx + \int_{\Omega} F \cdot \nabla T_h(u_n) \varphi'_h dx. \end{aligned} \quad (28)$$

Thanks to Young's inequality and (7), we have

$$\begin{aligned} & c_0 \int_{\Omega} \sum_{i=1}^N \left| \frac{\partial T_h(u_n)}{\partial x_i} \right|^p \omega_i(x) \varphi'_h dx \\ &+ \int_{\Omega} g_n(x, u_n, \nabla u_n) \varphi_h dx \\ &\leq \varphi(h) \int_{\Omega} |f_n| dx + \varphi'(h) \int_{\Omega} \sum_{i=1}^N |F_i \cdot \omega_i^{-1/p}|^{p'} dx \\ &+ \frac{c_0}{2} \int_{\Omega} \sum_{i=1}^N \left| \frac{\partial T_h(u_n)}{\partial x_i} \right|^p \omega_i(x) \varphi'_h dx. \end{aligned} \quad (29)$$

Since $\{f_n\}$ is bounded in $L^1(\Omega)$ and $F \in \prod_{i=1}^N L^{p'}(\Omega, \omega_i^*)$, it follows from the above inequality that

$$\begin{aligned} & \frac{c_0}{2} \int_{\Omega} \sum_{i=1}^N \left| \frac{\partial T_h(u_n)}{\partial x_i} \right|^p \omega_i(x) \varphi'_h dx \\ & + \int_{\Omega} g_n(x, u_n, \nabla u_n) \varphi_h dx \leq C, \end{aligned} \quad (30)$$

where C is independent of n . Splitting the second term on the left-hand side where $|u_n| < h$ and $|u_n| \geq h$, we can write

$$\begin{aligned} & \int_{\Omega} g_n(x, u_n, \nabla u_n) \varphi_h dx \\ & = \int_{\{|u_n| < h\}} g_n(x, u_n, \nabla u_n) \varphi_h dx \\ & + \int_{\{|u_n| \geq h\}} g_n(x, u_n, \nabla u_n) \varphi_h dx. \end{aligned} \quad (31)$$

Using (9) and (10), we get

$$\begin{aligned} & \int_{\{|u_n| \geq h\}} g_n(x, u_n, \nabla u_n) \varphi_h dx \geq \rho \varphi(h) \\ & \cdot \int_{\{|u_n| \geq h\}} \sum_{i=1}^N \left| \frac{\partial u_n}{\partial x_i} \right|^p \omega_i(x) dx, \\ & \left| \int_{\{|u_n| < h\}} g_n(x, u_n, \nabla u_n) \varphi_h dx \right| \leq b(h) \\ & \cdot \left(\int_{\Omega} \sum_{i=1}^N \left| \frac{\partial T_h(u_n)}{\partial x_i} \right|^p \omega_i(x) \varphi_h dx \right. \\ & \left. + \varphi(h) \|d\|_{L^1(\Omega)} \right). \end{aligned} \quad (32)$$

Hence,

$$\begin{aligned} & \int_{\Omega} \sum_{i=1}^N \left| \frac{\partial T_h(u_n)}{\partial x_i} \right|^p \omega_i(x) \left[\frac{c_0}{2} \varphi'_h - b(h) |\varphi_h| \right] dx \\ & + \rho \varphi(h) \int_{\{|u_n| \geq h\}} \sum_{i=1}^N \left| \frac{\partial u_n}{\partial x_i} \right|^p \omega_i(x) dx \leq C. \end{aligned} \quad (33)$$

Recalling (19) in Lemma 1, let $a = c_0/2$, $b = b(h)$; we then obtain

$$\begin{aligned} & \int_{\Omega} \sum_{i=1}^N \left| \frac{\partial T_h(u_n)}{\partial x_i} \right|^p \omega_i(x) dx \\ & + \int_{\{|u_n| \geq h\}} \sum_{i=1}^N \left| \frac{\partial u_n}{\partial x_i} \right|^p \omega_i(x) dx \leq C, \end{aligned} \quad (34)$$

which implies

$$\int_{\Omega} \sum_{i=1}^N \left| \frac{\partial u_n}{\partial x_i} \right|^p \omega_i(x) dx \leq C, \quad (35)$$

or equivalently

$$\|u_n\|_X \leq C, \quad (36)$$

where C is some positive constant. Therefore, we can extract a subsequence, still denoted by itself, such that

$$u_n \rightharpoonup u \quad \text{weakly in } W_0^{1,p}(\Omega, \omega). \quad (37)$$

By (5) and (37), we have for a subsequence $u_n \rightarrow u$ strongly in $L^q(\Omega, \omega)$ and a.e. in Ω . Then, $T_k(u_n)$ is bounded in $W_0^{1,p}(\Omega, \omega)$. Hence, by the results of [8], we have

$$T_k(u_n) \rightharpoonup T_k(u) \quad \text{weakly in } W_0^{1,p}(\Omega, \omega). \quad (38)$$

Step 3 (the strong convergence $u_n \rightarrow u$ in $W_0^{1,p}(\Omega, \omega)$). For every $k \geq h$, we will prove that $T_k(u_n)$ converges strongly to $T_k(u)$ in $W_0^{1,p}(\Omega, \omega)$. We first prove that

$$\lim_{k \rightarrow +\infty} \sup_{n \in N^*} \int_{\{|u_n| \geq k\}} \sum_{i=1}^N \left| \frac{\partial u_n}{\partial x_i} \right|^p \omega_i dx = 0. \quad (39)$$

Here, we denote by N^* the set of natural numbers. Choosing $v = T_k(u_n) - T_{k-1}(u_n)$ as a test function in (27) with $k \geq h+1$, using (7) and Young's inequality, we obtain

$$\begin{aligned} & \frac{c_0}{2} \int_{\Omega} \sum_{i=1}^N \omega_i(x) \left| \frac{\partial (T_k(u_n) - T_{k-1}(u_n))}{\partial x_i} \right|^p dx \\ & + \int_{\Omega} g_n(x, u_n, \nabla u_n) (T_k(u_n) - T_{k-1}(u_n)) dx \\ & \leq \int_{\{|u_n| \geq k-1\}} f_n(T_k(u_n) - T_{k-1}(u_n)) dx \\ & + \int_{\{k-1 \leq |u_n| \leq k\}} \sum_{i=1}^N |F_i \cdot \omega_i^{-1/p}|^{p'} dx. \end{aligned} \quad (40)$$

Noticing (9) and that $T_k(u_n) - T_{k-1}(u_n)$ has the same sign as $g_n(x, u_n, \nabla u_n)$ if $|u_n| > h$ and is zero if $|u_n| \leq h$, we get

$$\begin{aligned} & g_n(x, u_n, \nabla u_n) (T_k(u_n) - T_{k-1}(u_n)) \\ & \geq |g_n(x, u_n, \nabla u_n)| \chi_{\{|u_n| \geq k\}}. \end{aligned} \quad (41)$$

Dropping the nonnegative term, we have

$$\begin{aligned} & \int_{\{|u_n| \geq k\}} |g_n(x, u_n, \nabla u_n)| dx \\ & \leq \int_{\{|u_n| \geq k-1\}} |f_n| dx \\ & + \int_{\{k-1 \leq |u_n| \leq k\}} \sum_{i=1}^N |F_i \cdot \omega_i^{-1/p}|^{p'} dx. \end{aligned} \quad (42)$$

Since

$$\begin{aligned} \text{meas} \{|u_n| \geq k\} & \leq \int_{\Omega} \frac{|u_n| \omega_0^{1/p}}{k} (\omega_0^{-1}(x))^{1/p} dx \\ & \leq \frac{1}{k} \|u_n\|_{L^p(\Omega, \omega_0)} \|(\omega_0^{-1})^{1/p}\|_{L^{p'}(\Omega)} \\ & \leq \frac{1}{k} \|u_n\|_{L^p(\Omega, \omega_0)} \|(\omega_0^{-1})^{1/p}\|_{L^{1/(p-1)}(\Omega)}, \end{aligned} \quad (43)$$

we obtain

$$\lim_{k \rightarrow +\infty} \sup_{n \in N^*} \text{meas}(\{|u_n| \geq k-1\}) = 0. \quad (44)$$

Taking into account the fact that $\{f_n\}$ is compact in $L^1(\Omega)$ and $F \in \prod_{i=1}^N L^{p'}(\Omega, \omega_i^*)$, we deduce that

$$\begin{aligned} & \lim_{k \rightarrow +\infty} \sup_{n \in N^*} \int_{\{|u_n| \geq k-1\}} |f_n| dx \\ & + \int_{\{k-1 \leq |u_n| \leq k\}} \sum_{i=1}^N |F_i \cdot \omega_i^{-1/p}|^{p'} dx = 0. \end{aligned} \quad (45)$$

Hence,

$$\lim_{k \rightarrow +\infty} \sup_{n \in N^*} \int_{\{|u_n| \geq k\}} |g_n(x, u_n, \nabla u_n)| dx = 0. \quad (46)$$

Noticing that $k \geq h$ and (9), this completes the proof of assertion (39).

Let $k \geq h$ be fixed, $0 < \varepsilon < k$, and choose $v = \varphi(T_\varepsilon(u_n - T_k(u)))$ as a test function in (27), where $\varphi(s)$ is defined in Lemma 1 (refer to [8–10]). We thus obtain

$$\begin{aligned} & \underbrace{\int_{\Omega} a(x, u_n, \nabla u_n) \cdot \nabla (T_\varepsilon(u_n - T_k(u))) \varphi'(T_\varepsilon(u_n - T_k(u))) dx}_{(A)} \\ & + \underbrace{\int_{\Omega} g_n(x, u_n, \nabla u_n) \varphi(T_\varepsilon(u_n - T_k(u))) dx}_{(B)} \\ & = \underbrace{\int_{\Omega} f_n \varphi(T_\varepsilon(u_n - T_k(u))) dx}_{(C)} \\ & + \underbrace{\int_{\Omega} F \cdot \nabla (T_\varepsilon(u_n - T_k(u))) \varphi'(T_\varepsilon(u_n - T_k(u))) dx}_{(D)}. \end{aligned} \quad (47)$$

In the following, $\delta(\varepsilon, n)$ represents a quantity which converges to zero as firstly $n \rightarrow \infty$ and secondly $\varepsilon \rightarrow 0$. For convenience, we write

$$\begin{aligned} \varphi'_{\varepsilon, n} &= \varphi'(T_\varepsilon(u_n - T_k(u))), \\ \varphi_{\varepsilon, n} &= \varphi(T_\varepsilon(u_n - T_k(u))). \end{aligned} \quad (48)$$

Observe that, in the weak* topology of $L^\infty(\Omega)$ and almost everywhere in Ω , we have

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \varphi_{\varepsilon, n} &= 0, \\ \lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \varphi'_{\varepsilon, n} &= 1. \end{aligned} \quad (49)$$

Now, as $\{f_n\}$ is compact in $L^1(\Omega)$ and (49), we have

$$(C) = \delta(\varepsilon, n). \quad (50)$$

Thanks to $T_{k+\varepsilon}(u_n) \rightharpoonup T_k(u)$ weakly in $W_0^{1,p}(\Omega, \omega)$, $F \in \prod_{i=1}^N L^{p'}(\Omega, \omega_i^*)$, and (49), we obtain

$$\begin{aligned} (D) &= \int_{\Omega} F \cdot \nabla (T_{k+\varepsilon}(u_n) - T_k(u)) \chi_{\varepsilon} \{|u_n - T_k(u)|\} \varphi'_{\varepsilon, n} dx \\ &= \delta(\varepsilon, n), \end{aligned} \quad (51)$$

where $\chi_{\varepsilon} \{|u_n - T_k(u)|\} = \chi\{|u_n - T_k(u)| \leq \varepsilon\}$. We can decompose (A) as

$$\begin{aligned} & \underbrace{\int_{\Omega} a(x, T_{k+\varepsilon}(u_n), \nabla T_{k+\varepsilon}(u_n)) \cdot \nabla (T_\varepsilon(u_n - T_k(u))) \varphi'_{\varepsilon, n} dx}_{(E)} \\ & + \underbrace{\int_{\Omega} a(x, u_n, \nabla G_{k+\varepsilon}(u_n)) \chi\{|u_n| \geq k + \varepsilon\} \nabla (T_\varepsilon(u_n - T_k(u))) \varphi'_{\varepsilon, n} dx}_{(F)}. \end{aligned} \quad (52)$$

Owing to $\{|u_n - T_k(u)| \leq \varepsilon\} \subset \{|u_n| \leq k + \varepsilon\}$, we get

$$\begin{aligned} (F) &= \int_{\Omega} a(x, u_n, \nabla G_{k+\varepsilon}(u_n)) \chi\{|u_n| \geq k + \varepsilon\} \\ & \cdot \nabla (T_{k+\varepsilon}(u_n) - T_k(u)) \chi_{\varepsilon} \{|u_n - T_k(u)|\} \varphi'_{\varepsilon, n} dx. \end{aligned} \quad (53)$$

Since $\nabla T_{k+\varepsilon}(u_n)$ is zero whenever $\nabla G_{k+\varepsilon}(u_n)$ is not zero, hence,

$$\begin{aligned} (F) &= - \int_{\Omega} a(x, u_n, \nabla G_{k+\varepsilon}(u_n)) \\ & \cdot \nabla T_k(u) \chi\{|u_n| > k + \varepsilon\} \chi_{\varepsilon} \{|u_n - T_k(u)|\} \varphi'_{\varepsilon, n} dx. \end{aligned} \quad (54)$$

Since $\nabla T_k(u) \equiv 0$ on the set $\{|u| > k + \varepsilon\}$, we see that, as $n \rightarrow \infty$,

$$\nabla T_k(u) \chi_{\{|u_n| > k + \varepsilon\}} \rightarrow 0, \quad \text{a.e. in } \Omega. \quad (55)$$

As $\nabla T_k(u) \in \prod_{i=1}^N L^p(\Omega, \omega_i)$, Lebesgue's dominated convergence theorem guarantees that

$$\begin{aligned} \nabla T_k(u) \chi_{\{|u_n| > k + \varepsilon\}} &\rightarrow 0 \\ \text{strongly in } \prod_{i=1}^N L^p(\Omega, \omega_i), & \quad n \rightarrow \infty. \end{aligned} \quad (56)$$

By (6), (49), and the fact that $a(x, u_n, \nabla G_{k+\varepsilon}(u_n))$ is bounded in $\prod_{i=1}^N L^{p'}(\Omega, \omega_i^*)$, we obtain

$$(F) = \delta(\varepsilon, n). \quad (57)$$

Now we split (E) into

$$\begin{aligned} & \underbrace{\int_{\Omega} [a(x, T_{k+\varepsilon}(u_n), \nabla T_{k+\varepsilon}(u_n)) - a(x, T_{k+\varepsilon}(u_n), \nabla T_k(u))] \cdot \nabla (T_{k+\varepsilon}(u_n) - T_k(u)) \varphi'_{\varepsilon, n} dx}_{(G)} \\ & + \underbrace{\int_{\Omega} a(x, T_{k+\varepsilon}(u_n), \nabla T_k(u)) \cdot \nabla (T_{k+\varepsilon}(u_n) - T_k(u)) \varphi'_{\varepsilon, n} dx}_{(H)}. \end{aligned} \quad (58)$$

We will prove that

$$a_i(x, T_{k+\varepsilon}(u_n), \nabla T_k(u)) \longrightarrow a_i(x, T_{k+\varepsilon}(u), \nabla T_k(u)) \quad (59)$$

strongly in $L^{p'}(\Omega, \omega_i^*)$.

In fact,

$$\begin{aligned} & |a_i(x, T_{k+\varepsilon}(u_n), \nabla T_k(u))|^{p'} \omega_i^{-p'/p} \leq c_1 \left[k(x) \right. \\ & \quad \left. + \sigma^{1/p'} |T_{k+\varepsilon}(u_n)|^{q/p'} \right. \\ & \quad \left. + \sum_{i=1}^N \omega_i^{1/p'}(x) \left| \frac{\partial T_k(u)}{\partial x_i} \right|^{p-1} \right]^{p'} \leq c_2 \left[k(x)^{p'} \right. \\ & \quad \left. + \sigma |T_{k+\varepsilon}(u_n)|^q + \sum_{i=1}^N \omega_i(x) \left| \frac{\partial T_k(u)}{\partial x_i} \right|^p \right], \end{aligned} \quad (60)$$

where c_1, c_2 are positive constants. Since $T_{k+\varepsilon}(u_n) \rightharpoonup T_{k+\varepsilon}(u)$ weakly in $W_0^{1,p}(\Omega, \omega)$ and $W_0^{1,p}(\Omega, \omega) \hookrightarrow L^q(\Omega, \sigma)$ is compact, then $T_{k+\varepsilon}(u_n) \rightarrow T_{k+\varepsilon}(u)$ strongly in $L^q(\Omega, \sigma)$ and a.e. in Ω . Hence,

$$\begin{aligned} & |a_i(x, T_{k+\varepsilon}(u_n), \nabla T_k(u))|^{p'} \omega_i^* \\ & \longrightarrow |a_i(x, T_{k+\varepsilon}(u), \nabla T_k(u))|^{p'} \omega_i^* \quad \text{a.e. in } \Omega, \\ & c_2 \left[k(x)^{p'} + \sigma |T_{k+\varepsilon}(u_n)|^q + \sum_{i=1}^N \omega_i(x) \left| \frac{\partial T_k(u)}{\partial x_i} \right|^p \right] \\ & \longrightarrow c_2 \left[k(x)^{p'} + \sigma |T_{k+\varepsilon}(u)|^q \right. \\ & \quad \left. + \sum_{i=1}^N \omega_i(x) \left| \frac{\partial T_k(u)}{\partial x_i} \right|^p \right] \quad \text{strongly in } L^q(\Omega, \sigma). \end{aligned} \quad (61)$$

Then, by the generalized Lebesgue dominated convergence theorem, we deduce (59). By $\partial T_{k+\varepsilon}(u_n)/\partial x_i \rightharpoonup \partial T_k(u)/\partial x_i$ weakly in $L^p(\Omega, \omega_i)$ and (49), we have

$$(H) = \delta(\varepsilon, n). \quad (62)$$

Using (57) and (62), we have

$$\begin{aligned} (A) &= \int_{\Omega} [a(x, T_{k+\varepsilon}(u_n), \nabla T_{k+\varepsilon}(u_n)) \\ & \quad - a(x, T_{k+\varepsilon}(u_n), \nabla T_k(u))] \cdot \nabla (T_{k+\varepsilon}(u_n) \\ & \quad - T_k(u)) \varphi'_{\varepsilon, n} dx + \delta(\varepsilon, n). \end{aligned} \quad (63)$$

As for the term (B), we decompose it as

$$\begin{aligned} & \underbrace{\int_{\{|u_n| \geq k+\varepsilon\}} g_n(x, u_n, \nabla u_n) \varphi_{\varepsilon, n} dx}_{(I)} \\ & + \underbrace{\int_{\{|u_n| < k+\varepsilon\}} g_n(x, u_n, \nabla u_n) \varphi_{\varepsilon, n} dx}_{(J)}. \end{aligned} \quad (64)$$

It is clear that on the set $\{u_n \geq k + \varepsilon\}$ we get

$$\varphi_{\varepsilon, n} = \varphi(T_{\varepsilon}(u_n - T_k(u))) = \varphi(\varepsilon) \geq 0, \quad (65)$$

while on the set $\{u_n < -k - \varepsilon\}$ we get

$$\varphi_{\varepsilon, n} = \varphi(T_{\varepsilon}(u_n - T_k(u))) = \varphi(-\varepsilon) \leq 0. \quad (66)$$

By (9) and the fact that $k \geq h$, we obtain

$$(I) \geq 0. \quad (67)$$

Using (7) and (10) and noticing that $\varepsilon < k$, we have

$$\begin{aligned} |(J)| &\leq \int_{\{|u_n| \leq k+\varepsilon\}} b(2k) \left[d(x) + \sum_{i=1}^N \left| \frac{\partial u_n}{\partial x_i} \right|^p \omega_i(x) \right] \\ &\quad \cdot |\varphi_{\varepsilon, n}| dx \leq b(2k) \int_{\{|u_n| \leq k+\varepsilon\}} d(x) |\varphi_{\varepsilon, n}| dx \\ &\quad + \frac{b(2k)}{c_0} \int_{\{|u_n| \leq k+\varepsilon\}} a(x, u_n, \nabla u_n) \nabla u_n |\varphi_{\varepsilon, n}| dx \\ &= \delta(\varepsilon, n) + \frac{b(2k)}{c_0} \int_{\Omega} a(x, T_{k+\varepsilon}(u_n), \nabla T_{k+\varepsilon}(u_n)) \\ &\quad \cdot \nabla T_{k+\varepsilon}(u_n) |\varphi_{\varepsilon, n}| dx \leq \frac{b(2k)}{c_0} \\ &\quad \cdot \int_{\Omega} [a(x, T_{k+\varepsilon}(u_n), \nabla T_{k+\varepsilon}(u_n)) \end{aligned}$$

$$\begin{aligned}
& -a(x, T_{k+\varepsilon}(u_n), \nabla T_k(u)) \cdot \nabla(T_{k+\varepsilon}(u_n) \\
& - T_k(u)) |\varphi_{\varepsilon,n}| dx + \int_{\Omega} a(x, T_{k+\varepsilon}(u_n), \nabla T_k(u)) \\
& \cdot \nabla(T_{k+\varepsilon}(u_n) - T_k(u)) |\varphi_{\varepsilon,n}| dx \\
& + \int_{\Omega} a(x, T_{k+\varepsilon}(u_n), \nabla T_{k+\varepsilon}(u_n)) \cdot \nabla T_k(u) \\
& \cdot |\varphi_{\varepsilon,n}| dx + \delta(\varepsilon, n).
\end{aligned} \tag{68}$$

By the weak convergence of $\partial T_{k+\varepsilon}(u_n)/\partial x_i \rightarrow \partial T_k(u)/\partial x_i$ in $L^p(\Omega, \omega_i)$, (49), and (59), we have

$$\begin{aligned}
& \int_{\Omega} a(x, T_{k+\varepsilon}(u_n), \nabla T_k(u)) \\
& \cdot \nabla(T_{k+\varepsilon}(u_n) - T_k(u)) |\varphi_{\varepsilon,n}| dx = \delta(\varepsilon, n).
\end{aligned} \tag{69}$$

Since $a(x, T_{k+\varepsilon}(u_n), \nabla T_{k+\varepsilon}(u_n))$ is bounded in $\prod_{i=1}^N L^{p'}(\Omega, \omega_i^*)$ and (49),

$$\begin{aligned}
& \int_{\Omega} a(x, T_{k+\varepsilon}(u_n), \nabla T_{k+\varepsilon}(u_n)) \cdot \nabla T_k(u) |\varphi_{\varepsilon,n}| dx \\
& = \delta(\varepsilon, n).
\end{aligned} \tag{70}$$

We have

$$\begin{aligned}
|J| & \leq \frac{b(2k)}{c_0} \int_{\Omega} [a(x, T_{k+\varepsilon}(u_n), \nabla T_{k+\varepsilon}(u_n)) \\
& - a(x, T_{k+\varepsilon}(u_n), \nabla T_k(u))] \cdot \nabla(T_{k+\varepsilon}(u_n) \\
& - T_k(u)) |\varphi_{\varepsilon,n}| dx + \delta(\varepsilon, n).
\end{aligned} \tag{71}$$

Invoking (63), we have

$$\begin{aligned}
& \int_{\Omega} [a(x, T_{k+\varepsilon}(u_n), \nabla T_{k+\varepsilon}(u_n)) \\
& - a(x, T_{k+\varepsilon}(u_n), \nabla T_k(u))] \cdot \nabla(T_{k+\varepsilon}(u_n) \\
& - T_k(u)) \left[\varphi'_{\varepsilon,n} - \frac{b(2k)}{c_0} |\varphi_{\varepsilon,n}| \right] dx = \delta(\varepsilon, n).
\end{aligned} \tag{72}$$

Consequently, by (19) and letting $a = 1$ and $b = b(2k)/c_0$, it yields

$$\begin{aligned}
& \int_{\Omega} [a(x, T_{k+\varepsilon}(u_n), \nabla T_{k+\varepsilon}(u_n)) \\
& - a(x, T_{k+\varepsilon}(u_n), \nabla T_k(u))] \cdot \nabla(T_{k+\varepsilon}(u_n) \\
& - T_k(u)) dx = \delta(\varepsilon, n).
\end{aligned} \tag{73}$$

Since $\{|u_n| \leq k\} \subset \{|u_n| \leq k + \varepsilon\}$, we have

$$\begin{aligned}
& \int_{\Omega} [a(x, T_k(u_n), \nabla T_k(u_n)) \\
& - a(x, T_k(u_n), \nabla T_k(u))] \cdot \nabla(T_k(u_n) \\
& - T_k(u)) dx = \int_{\{|u_n| \leq k\}} [a(x, u_n, \nabla u_n) \\
& - a(x, u_n, \nabla T_k(u))] \cdot \nabla(u_n - T_k(u)) dx \\
& \leq \int_{\{|u_n| \leq k+\varepsilon\}} [a(x, u_n, \nabla u_n) - a(x, u_n, \nabla T_k(u))] \\
& \cdot \nabla(u_n - T_k(u)) dx \\
& = \int_{\Omega} [a(x, T_{k+\varepsilon}(u_n), \nabla T_{k+\varepsilon}(u_n)) \\
& - a(x, T_{k+\varepsilon}(u_n), \nabla T_k(u))] \cdot \nabla(T_{k+\varepsilon}(u_n) \\
& - T_k(u)) dx = \delta(\varepsilon, n).
\end{aligned} \tag{74}$$

Together with Lemma 5 and the assumptions on a , we obtain

$$T_k(u_n) \longrightarrow T_k(u) \quad \text{strongly in } W_0^{1,p}(\Omega, \omega), \tag{75}$$

which in turn implies

$$\nabla u_n \longrightarrow \nabla u \quad \text{a.e. in } \Omega. \tag{76}$$

For any measurable set E of Ω , we have

$$\begin{aligned}
\int_E \sum_{i=1}^N \left| \frac{\partial u_n}{\partial x_i} \right|^p \omega_i dx & = \int_{E \cap \{|u_n| < k\}} \sum_{i=1}^N \left| \frac{\partial u_n}{\partial x_i} \right|^p \omega_i dx \\
& + \int_{E \cap \{|u_n| \geq k\}} \sum_{i=1}^N \left| \frac{\partial u_n}{\partial x_i} \right|^p \omega_i dx.
\end{aligned} \tag{77}$$

Let $\varepsilon > 0$. Thanks to

$$\int_{E \cap \{|u_n| \geq k\}} \sum_{i=1}^N \left| \frac{\partial u_n}{\partial x_i} \right|^p \omega_i dx \leq \int_{\{|u_n| \geq k\}} \sum_{i=1}^N \left| \frac{\partial u_n}{\partial x_i} \right|^p \omega_i dx, \tag{78}$$

by (39), there exists $k \geq h$ such that

$$\int_{E \cap \{|u_n| \geq k\}} \sum_{i=1}^N \left| \frac{\partial u_n}{\partial x_i} \right|^p \omega_i dx \leq \frac{\varepsilon}{2}, \quad n \in N^*. \tag{79}$$

While k is fixed, we get

$$\int_{E \cap \{|u_n| < k\}} \sum_{i=1}^N \left| \frac{\partial u_n}{\partial x_i} \right|^p \omega_i dx \leq \int_E \sum_{i=1}^N \left| \frac{\partial T_k(u_n)}{\partial x_i} \right|^p \omega_i dx. \tag{80}$$

Owing to the strong compactness of $\{T_k(u_n)\}$ in $W_0^{1,p}(\Omega, \omega)$, there exists $\delta' > 0$ such that if $\text{meas}(E) < \delta'$, then

$$\int_{E \cap \{|u_n| < k\}} \sum_{i=1}^N \left| \frac{\partial u_n}{\partial x_i} \right|^p \omega_i dx \leq \frac{\varepsilon}{2}, \quad n \in N^*. \tag{81}$$

Hence,

$$\int_E \sum_{i=1}^N \left| \frac{\partial u_n}{\partial x_i} \right|^p \omega_i dx \leq \varepsilon, \quad n \in N^*. \quad (82)$$

Thus, the sequence $\{|\nabla u_n|^p\}$ is equi-integrable. Thanks to Vitali theorem, the equi-integrability together with (76) implies that u_n converges strongly to u in $W_0^{1,p}(\Omega, \omega)$.

Step 4 (the strong convergence $g_n(x, u_n, \nabla u_n) \rightarrow g(x, u, \nabla u)$ in $L^1(\Omega)$). Note that (76) implies that

$$g_n(x, u_n, \nabla u_n) \rightarrow g(x, u, \nabla u) \text{ a.e. in } \Omega. \quad (83)$$

On the other hand, for any measurable set E of Ω , we have

$$\begin{aligned} & \int_E |g_n(x, u_n, \nabla u_n)| dx \\ &= \int_{E \cap \{|u_n| < k\}} |g_n(x, u_n, \nabla u_n)| dx \\ &+ \int_{E \cap \{|u_n| \geq k\}} |g_n(x, u_n, \nabla u_n)| dx. \end{aligned} \quad (84)$$

Let $\varepsilon > 0$ be fixed. We have

$$\begin{aligned} & \int_{E \cap \{|u_n| \geq k\}} |g_n(x, u_n, \nabla u_n)| dx \\ & \leq \int_{\{|u_n| \geq k\}} |g_n(x, u_n, \nabla u_n)| dx. \end{aligned} \quad (85)$$

Choose $k \geq h$ in (46) such that

$$\int_{E \cap \{|u_n| \geq k\}} |g_n(x, u_n, \nabla u_n)| dx \leq \frac{\varepsilon}{2}, \quad n \in N^*. \quad (86)$$

By (10), we have

$$\begin{aligned} & \int_{E \cap \{|u_n| < k\}} |g_n(x, T_k(u_n), \nabla T_k(u_n))| dx \\ & \leq b(k) \int_E \left| \frac{\partial T_k(u_n)}{\partial x_i} \right|^p \omega_i(x) dx + d(x). \end{aligned} \quad (87)$$

Since $d(x)$ belongs to $L^1(\Omega)$ and $T_k(u_n)$ is compact in $W_0^{1,p}(\Omega, \omega)$, there exists $\delta'' > 0$ such that if $\text{meas}(E) < \delta''$, then

$$\begin{aligned} & \int_{E \cap \{|u_n| < k\}} |g_n(x, T_k(u_n), \nabla T_k(u_n))| dx \leq \frac{\varepsilon}{2}, \\ & n \in N^*. \end{aligned} \quad (88)$$

Thus, we have proved that $\{g_n(x, u_n, \nabla u_n)\}$ is equi-integrable. Invoking (86) and (88) and by Vitali theorem,

$$g_n(x, u_n, \nabla u_n) \rightarrow g(x, u, \nabla u) \text{ strongly in } L^1(\Omega). \quad (89)$$

Step 5 (passing to the limit). Now, by passing to the limit in (27), we obtain

$$\begin{aligned} & \int_{\Omega} a(x, u, \nabla u) \cdot \nabla v dx + \int_{\Omega} g(x, u, \nabla u) v dx \\ &= \int_{\Omega} f v dx + \int_{\Omega} F \cdot \nabla v dx; \end{aligned} \quad (90)$$

that is, u is a weak solution to problem (1). The proof is complete. $\square$

Conflict of Interests

The authors declare that they have no competing interests.

Authors' Contribution

All authors contributed equally to the paper and read and approved its final version.

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