Research Article

Toeplitz Operators on the Weighted Bergman Space over the Two-Dimensional Unit Ball

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We extend the known results on commutative Banach algebras generated by Toeplitz operators with radial quasi-homogeneous symbols on the two-dimensional unit ball. Spherical coordinates previously used hid a possibility to detect an essentially wider class of symbols that can generate commutative Banach Toeplitz operator algebras. We characterize these new algebras describing their properties and, under a certain extra condition, construct the corresponding Gelfand theory.

1. Introduction

After a detailed study of the commutative C^* -algebras generated by Toeplitz operators acting on the weighted Bergman spaces on the unit ball [1, 2] it was quite unexpectedly observed [3] that, contrary to the one-dimensional case of the unit disk, there exist many other Banach (not C^* !) algebras generated by Toeplitz operators that are commutative on each weighted Bergman space. They were generated by Toeplitz operators with radial and the so-called quasi-homogeneous symbols, and the commutativity of the corresponding algebras was established just by an observation that the generating operators commute among themselves. Then the problem of constructing the Gelfand theory for these algebras emerged, being a tool to describe the properties, in particular spectral ones, of the operators forming the algebra. For the two-dimensional ball, the case relevant to this paper, this step was done in [4].

Note that the Introduction of [4] stated that the paper studies the *unique* commutative Toeplitz operator Banach algebra on the two-dimensional ball. This was indeed completely true for the only known by that time generating radial quasi-homogeneous symbols. As it later turns out (and this is exactly what this paper is about) the spherical coordinates used in [4] hide the possibility to detect many other commutative Banach algebras on the two-dimensional ball generated by Toeplitz operators with symbols of a more general type.

In this paper we use another representation of the points of the two-dimensional ball, which permits us to extend essentially the previous class of quasi-homogeneous symbols. Instead of a very specific function $q(\theta) = \cos \theta \sin \theta$ of [4], we are dealing here with arbitrary $q(\theta) \in L_\infty[0, \pi/2]$. Each commutative Banach algebra $\mathcal{T}_{\text{rad}, q, k}$, considered in the paper, is generated by Toeplitz operators with L_∞ radial symbols and by the Toeplitz operator with symbol $q(\theta)(t_1 \bar{t}_2)^k$. That is, the whole variety of our algebras is parametrized by $q(\theta) \in L_\infty[0, \pi/2]$ and $k \in \mathbb{N}$.

All these algebras share many common properties. We discuss, in particular, the description of the invariant subspaces, the property of being not semisimple, radical elements description, and non-uniqueness of the representation of elements in a dense subalgebra.

The tools that we use for the explicit description of both the compact set of maximal ideals of our algebras $\mathcal{T}_{\text{rad}, q, k}$ and the Gelfand transform require the continuity of $q(\theta)(t_1 \bar{t}_2)^k$ on the boundary of the unit ball. Thus here we impose an extra condition: $q(\theta) \in C[0, \pi/2]$ and $q(0) = q(\pi/2) = 0$ (which is obviously satisfied in the particular case of $q(\theta) = \cos \theta \sin \theta$ in [4]).

2. Preliminaries

2.1. Definitions and Basic Properties. Let $\mathbb{B}^2$ be the open unit ball in $\mathbb{C}^2$ and let $d\nu$ denote the standard volume form on $\mathbb{B}^2$.

For $\lambda > -1$, we introduce the one-parameter family of the standard weighted (probability) measures

$$d\nu_\lambda(z) = \frac{\Gamma(\lambda+3)}{\pi^2 \Gamma(\lambda+1)} (1 - |z|^2)^\lambda d\nu(z). \quad (1)$$

The weighted Bergman space $\mathcal{A}_\lambda^2(\mathbb{B}^2)$ is the closed subspace of $L_2(\mathbb{B}^2, d\nu_\lambda)$ that consists of all analytic functions. The orthogonal Bergman projection B_λ of $L_2(\mathbb{B}^2, d\nu_\lambda)$ onto $\mathcal{A}_\lambda^2(\mathbb{B}^2)$ has the form

$$(B_\lambda f)(z) = \int_{\mathbb{B}^2} \frac{f(\zeta) d\nu_\lambda(\zeta)}{(1 - \langle z, \zeta \rangle)^{\lambda+3}}. \quad (2)$$

The reproducing kernel of $\mathcal{A}_\lambda^2(\mathbb{B}^2)$ is defined by

$$K_z^\lambda(w) = \frac{1}{(1 - \langle w, z \rangle)^{\lambda+3}} = \sum_{|\alpha|=0}^{\infty} \frac{\Gamma(|\alpha| + \lambda + 3)}{\alpha! \Gamma(\lambda + 3)} \bar{z}^\alpha w^\alpha. \quad (3)$$

The standard orthonormal monomial basis $\{e_\alpha : \alpha \in \mathbb{Z}_+^2\}$ of $\mathcal{A}_\lambda^2(\mathbb{B}^2)$ is of the form

$$e_\alpha(z) := \sqrt{\frac{\Gamma(|\alpha| + \lambda + 3)}{\alpha! \Gamma(\lambda + 3)}} z^\alpha. \quad (4)$$

Given a function $a \in L_\infty(\mathbb{B}^2)$, the Toeplitz operator T_a with symbol a and acting on $\mathcal{A}_\lambda^2(\mathbb{B}^2)$ is defined by

$$T_a f = B_\lambda(a f), \quad f \in \mathcal{A}_\lambda^2(\mathbb{B}^2). \quad (5)$$

Recall also [4] that, given a radial function $a = a(r) \in L_\infty(0, 1)$, the Toeplitz operator T_a is diagonal with respect to the monomial basis (4) and its eigenvalue sequence is given by

$$\gamma_{a,\lambda}(\alpha) = \frac{\Gamma(|\alpha| + \lambda + 3)}{\Gamma(\lambda + 1) \Gamma(|\alpha| + 2)} \cdot \int_0^1 a(\sqrt{r}) (1 - r)^\lambda r^{|\alpha|+1} dr, \quad \alpha \in \mathbb{Z}_+^2. \quad (6)$$

In order to define our class of symbols we start with some notation.

We write a point $z \in \mathbb{B}^2$ as $z = (z_1, z_2)$, $z_j \in \mathbb{C}$. Given a multi-index $\alpha = (\alpha_1, \alpha_2)$, we use the standard notation $z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2}$.

We represent each coordinate of $z \in \mathbb{B}^2$ in the form $z_j = r_j t_j$, where

$$r_j = |z_j|, \quad t_j = \frac{z_j}{r_j} \in \mathbb{T}. \quad (7)$$

Then we pass from the Cartesian coordinates r_1, r_2 on the base $\tau(\mathbb{B}^2)$ of the unit ball $\mathbb{B}^2$ to the polar coordinates r, θ : $r_1 = r \cos \theta$, and $r_2 = r \sin \theta$, with $r \in [0, 1)$ and $\theta \in [0, \pi/2]$.

The Toeplitz operators with radial quasi-homogeneous symbols on the two-dimensional ball were studied in detail

in [4]. The quasi-homogeneous part of the symbols consisted of the following functions:

$$\left(\frac{r_1 r_2}{r_1^2 + r_2^2} t_1 \bar{t}_2 \right)^k, \quad k \in \mathbb{N}. \quad (8)$$

Here we extend this class of symbols to the functions of the form

$$q(\theta) (t_1 \bar{t}_2)^k, \quad q(\theta) \in L_\infty \left[0, \frac{\pi}{2} \right], \quad k \in \mathbb{N}. \quad (9)$$

Note that the old functions (8) correspond to the very particular case of (9):

$$q(\theta) = (\cos \theta \cdot \sin \theta)^k. \quad (10)$$

To start our analysis we consider the action of $T_{q(\theta)(t_1 \bar{t}_2)^k}$ on monomials:

$$\begin{aligned} & \langle T_{q(t_1 \bar{t}_2)^k} z^\alpha, z^\beta \rangle \\ &= \langle q(t_1 \bar{t}_2)^k z^\alpha, z^\beta \rangle \\ &= \frac{\Gamma(\lambda + 3)}{\pi^2 \Gamma(\lambda + 1)} \\ & \cdot \int_{\mathbb{B}^2} q(1 - r^2)^\lambda r_1^{\alpha_1 + \beta_1 + 1} r_2^{\alpha_2 + \beta_2 + 1} \\ & \cdot e^{i(\alpha_1 - \beta_1 + k)\theta_1} e^{i(\alpha_2 - \beta_2 - k)\theta_2} dr_1 dr_2 d\theta_1 d\theta_2 \\ &= \frac{\Gamma(\lambda + 3)}{\pi^2 \Gamma(\lambda + 1)} \int_{\tau(\mathbb{B}^2)} q(1 - r^2)^\lambda r_1^{\alpha_1 + \beta_1 + 1} r_2^{\alpha_2 + \beta_2 + 1} dr_1 dr_2 \\ & \cdot \int_0^{2\pi} e^{i(\alpha_1 - \beta_1 + k)\theta_1} d\theta_1 \int_0^{2\pi} e^{i(\alpha_2 - \beta_2 - k)\theta_2} d\theta_2 \\ &= \begin{cases} c_q(\alpha), & \text{if } \beta = (\alpha_1 + k, \alpha_2 - k), \\ 0, & \text{otherwise,} \end{cases} \end{aligned} \quad (11)$$

where (making the change of variables $r_1 = r \cos \theta$, $r_2 = r \sin \theta$)

$$\begin{aligned} c_q(\alpha) &= \frac{2^2 \Gamma(\lambda + 3)}{\Gamma(\lambda + 1)} \\ & \cdot \int_{\tau(\mathbb{B}^2)} q(1 - r^2)^\lambda r_1^{2\alpha_1 + k + 1} r_2^{2\alpha_2 - k + 1} dr_1 dr_2 \\ &= \frac{2^2 \Gamma(\lambda + 3)}{\Gamma(\lambda + 1)} \int_0^1 (1 - r^2)^\lambda r^{2|\alpha|+3} dr \\ & \cdot \int_0^{\pi/2} q(\theta) (\cos \theta)^{2\alpha_1 + k + 1} (\sin \theta)^{2\alpha_2 - k + 1} d\theta \end{aligned}$$

$$\begin{aligned}
&= \frac{2\Gamma(\lambda+3)}{\Gamma(\lambda+1)} \int_0^1 (1-r)^\lambda r^{|\alpha|+1} dr \\
&\quad \cdot \int_0^1 q(\arccos s) s^{2\alpha_1+k+1} (1-s^2)^{\alpha_2-k/2} ds \\
&= \frac{\Gamma(\lambda+3)}{\Gamma(\lambda+1)} \frac{\Gamma(\lambda+1)\Gamma(|\alpha|+2)}{\Gamma(|\alpha|+\lambda+3)} \\
&\quad \cdot \int_0^1 q(\arccos \sqrt{s}) s^{\alpha_1+k/2} (1-s)^{\alpha_2-k/2} ds \\
&= \frac{\Gamma(\lambda+3)\Gamma(|\alpha|+2)}{\Gamma(|\alpha|+\lambda+3)} \\
&\quad \cdot \int_0^1 q(\arccos \sqrt{s}) s^{\alpha_1+k/2} (1-s)^{\alpha_2-k/2} ds.
\end{aligned} \tag{12}$$

Taking into account that

$$\langle z^\beta, z^\beta \rangle = \frac{\Gamma(\lambda+3)\Gamma(\beta_1+1)\Gamma(\beta_2+1)}{\Gamma(|\beta|+\lambda+3)}, \tag{13}$$

we come to the following result.

Lemma 1. *Given $q = q(\theta) \in L_\infty[0, \pi/2]$, one has*

$$T_{q(t_1, \bar{t}_2)^k} z_1^{\alpha_1} z_2^{\alpha_2} = \tilde{\gamma}_q(\alpha) z_1^{\alpha_1+k} z_2^{\alpha_2-k}, \quad \alpha \in \mathbb{Z}_+^2, \quad \alpha_2 \geq k, \tag{14}$$

where

$$\begin{aligned}
\tilde{\gamma}_q(\alpha) &= \frac{\Gamma(|\alpha|+2)}{\Gamma(\alpha_1+k+1)\Gamma(\alpha_2-k+1)} \\
&\quad \cdot \int_0^1 q(\arccos \sqrt{s}) s^{\alpha_1+k/2} (1-s)^{\alpha_2-k/2} ds.
\end{aligned} \tag{15}$$

Corollary 2. *The action of $T_{q(t_1, \bar{t}_2)^k}$ does not depend on the weight parameter λ .*

Consider now some particular cases of q with $k = 1$.

(i) Let $q \equiv 1$; that is, we deal with the operator $T_{t_1, \bar{t}_2}$; then

$$\begin{aligned}
\tilde{\gamma}_1(\alpha) &= \frac{\Gamma(|\alpha|+2)}{\Gamma(\alpha_1+2)\Gamma(\alpha_2)} \frac{\Gamma(\alpha_1+(3/2))\Gamma(\alpha_2+(1/2))}{\Gamma(|\alpha|+2)} \\
&= \frac{\Gamma(\alpha_1+(3/2))\Gamma(\alpha_2+(1/2))}{\Gamma(\alpha_1+2)\Gamma(\alpha_2)}.
\end{aligned} \tag{16}$$

(ii) Let $q = \cos \theta \sin \theta$; that is, we deal with the operator $T_{\cos \theta \sin \theta, t_1, \bar{t}_2}$; then

$$\begin{aligned}
\tilde{\gamma}_q(\alpha) &= \frac{\Gamma(|\alpha|+2)}{\Gamma(\alpha_1+2)\Gamma(\alpha_2)} \int_0^1 s^{\alpha_1+1} (1-s)^{\alpha_2} ds \\
&= \frac{\Gamma(|\alpha|+2)}{\Gamma(\alpha_1+2)\Gamma(\alpha_2)} \cdot \frac{\Gamma(\alpha_1+2)\Gamma(\alpha_2+1)}{\Gamma(|\alpha|+3)} \\
&= \frac{\alpha_2}{|\alpha|+2}.
\end{aligned} \tag{17}$$

This is a particular case of (8) which was studied in [4].

Take now any $a = a(r)$ and $q = q(\theta)$, both from L_∞ , and $k \in \mathbb{N}$; then

$$\langle T_{aq(t_1, \bar{t}_2)^k} z^\alpha, z^\beta \rangle = 0, \quad \forall \beta \neq (\alpha_1+k, \alpha_2-k). \tag{18}$$

Thus we calculate

$$\begin{aligned}
&\langle T_{aq(t_1, \bar{t}_2)^k} z^\alpha, z_1^{\alpha_1+k} z_2^{\alpha_2-k} \rangle \\
&= \frac{2^2 \Gamma(\lambda+3)}{\Gamma(\lambda+1)} \\
&\quad \cdot \int_{\tau(\mathbb{B}^2)} aq(1-r^2)^\lambda r_1^{2\alpha_1+k+1} r_2^{2\alpha_2-k+1} dr_1 dr_2 \\
&= \frac{2^2 \Gamma(\lambda+3)}{\Gamma(\lambda+1)} \int_0^1 a(r) (1-r^2)^\lambda r^{2|\alpha|+3} dr \\
&\quad \cdot \int_0^{\pi/2} q(\theta) (\cos \theta)^{2\alpha_1+k+1} (\sin \theta)^{2\alpha_2-k+1} d\theta \\
&= \frac{2\Gamma(\lambda+3)}{\Gamma(\lambda+1)} \int_0^1 a(\sqrt{r}) (1-r)^\lambda r^{|\alpha|+1} dr \\
&\quad \cdot \int_0^1 q(\arccos s) s^{2\alpha_1+k+1} (1-s^2)^{\alpha_2-k/2} ds \\
&= \frac{\Gamma(|\alpha|+\lambda+3)}{\Gamma(\lambda+1)\Gamma(|\alpha|+2)} \int_0^1 a(\sqrt{r}) (1-r)^\lambda r^{|\alpha|+1} dr \\
&\quad \cdot \frac{\Gamma(\lambda+3)\Gamma(|\alpha|+2)}{\Gamma(|\alpha|+\lambda+3)} \\
&\quad \cdot \int_0^1 q(\arccos \sqrt{s}) s^{\alpha_1+k/2} (1-s)^{\alpha_2-k/2} ds.
\end{aligned} \tag{19}$$

That is,

$$T_{aq(t_1, \bar{t}_2)^k} z^\alpha = c_{a,q}(\alpha) z_1^{\alpha_1+k} z_2^{\alpha_2-k}, \tag{20}$$

where

$$\begin{aligned}
c_{a,q}(\alpha) &= \frac{\Gamma(|\alpha|+\lambda+3)}{\Gamma(\lambda+3)\Gamma(\alpha_1+k+1)\Gamma(\alpha_2-k+1)}
\end{aligned}$$

$$\begin{aligned}
& \cdot \frac{\Gamma(|\alpha| + \lambda + 3)}{\Gamma(\lambda + 1) \Gamma(|\alpha| + 2)} \int_0^1 a(\sqrt{r}) (1-r)^\lambda r^{|\alpha|+1} dr \\
& \cdot \frac{\Gamma(\lambda + 3) \Gamma(|\alpha| + 2)}{\Gamma(|\alpha| + \lambda + 3)} \\
& \cdot \int_0^1 q(\arccos \sqrt{s}) s^{\alpha_1+k/2} (1-s)^{\alpha_2-k/2} ds \\
& = \frac{\Gamma(|\alpha| + \lambda + 3)}{\Gamma(\lambda + 1) \Gamma(|\alpha| + 2)} \int_0^1 a(\sqrt{r}) (1-r)^\lambda r^{|\alpha|+1} dr \\
& \cdot \frac{\Gamma(|\alpha| + 2)}{\Gamma(\alpha_1 + k + 1) \Gamma(\alpha_2 - k + 1)} \\
& \cdot \int_0^1 q(\arccos \sqrt{s}) s^{\alpha_1+k/2} (1-s)^{\alpha_2-k/2} ds \\
& = \gamma_{a,\lambda}(|\alpha|) \cdot \tilde{\gamma}_q(\alpha),
\end{aligned} \tag{21}$$

or

$$T_{aq(t_1 \bar{t}_2)^k} z^\alpha = \gamma_{a,\lambda}(|\alpha|) \cdot \tilde{\gamma}_q(\alpha) z_1^{\alpha_1+k} z_2^{\alpha_2-k}. \tag{22}$$

Note that the last formula implies the next result.

Lemma 3. *Given any $a = a(r)$ and any $q = q(\theta)$, both from L_∞ , one has*

$$T_{aq(t_1 \bar{t}_2)^k} = T_a \cdot T_{q(t_1 \bar{t}_2)^k} = T_{q(t_1 \bar{t}_2)^k} \cdot T_a. \tag{23}$$

In what follows, to simplify the notation, we introduce the pairs $\kappa = (k, -k)$, $k \in \mathbb{N}$, and $t = (t_1, t_2) \in \mathbb{T}^2$. Then

$$(t_1 \bar{t}_2)^k = t_1^k t_2^{-k} = t^\kappa, \tag{24}$$

so that $q(\theta)(t_1 \bar{t}_2)^k = q t^\kappa$.

2.2. Comparison of the Toeplitz Operators with Symbols (8) and (9). Let $q_1, q_2 \in L_\infty[0, \pi/2]$, $k_1, k_2 \in \mathbb{N}$, and $\kappa_1 = (k_1, -k_1)$, $\kappa_2 = (k_2, -k_2)$. Consider the symbols $q_1 t^{\kappa_1}$ and $q_2 t^{\kappa_2}$, and calculate the action on the monomials of the products of the corresponding Toeplitz operators:

$$\begin{aligned}
& T_{q_1 t^{\kappa_1}} T_{q_2 t^{\kappa_2}} z^\alpha \\
& = \frac{\Gamma(|\alpha| + 2)}{\Gamma(\alpha_1 + k_2 + k_1 + 1) \Gamma(\alpha_2 - k_2 - k_1 + 1)} \\
& \cdot \int_0^1 q_1(\arccos \sqrt{s}) s^{\alpha_1+k_2+k_1/2} (1-s)^{\alpha_2-k_2-k_1/2} ds \\
& \cdot \frac{\Gamma(|\alpha| + 2)}{\Gamma(\alpha_1 + k_2 + 1) \Gamma(\alpha_2 - k_2 + 1)} \\
& \cdot \int_0^1 q_2(\arccos \sqrt{s}) s^{\alpha_1+k_2/2} (1-s)^{\alpha_2-k_2/2} ds \\
& \cdot z^{(\alpha_1+k_2+k_1, \alpha_2-k_2-k_1)},
\end{aligned}$$

$$\begin{aligned}
& T_{q_2 t^{\kappa_2}} T_{q_1 t^{\kappa_1}} z^\alpha \\
& = \frac{\Gamma(|\alpha| + 2)}{\Gamma(\alpha_1 + k_1 + k_2 + 1) \Gamma(\alpha_2 - k_1 - k_2 + 1)} \\
& \cdot \int_0^1 q_2(\arccos \sqrt{s}) s^{\alpha_1+k_1+k_2/2} (1-s)^{\alpha_2-k_1-k_2/2} ds \\
& \cdot \frac{\Gamma(|\alpha| + 2)}{\Gamma(\alpha_1 + k_1 + 1) \Gamma(\alpha_2 - k_1 + 1)} \\
& \cdot \int_0^1 q_1(\arccos \sqrt{s}) s^{\alpha_1+k_1/2} (1-s)^{\alpha_2-k_1/2} ds \\
& \cdot z^{(\alpha_1+k_1+k_2, \alpha_2-k_1-k_2)}.
\end{aligned} \tag{25}$$

That is, the Toeplitz operators $T_{q_1 t^{\kappa_1}}$ and $T_{q_2 t^{\kappa_2}}$ do not commute, in general.

Consider now a more specific case, close to (8):

$$\begin{aligned}
q_1(\theta) &= \cos^{m_1} \theta \sin^{m_2} \theta, \\
q_2(\theta) &= \cos^{\ell_1} \theta \sin^{\ell_2} \theta.
\end{aligned} \tag{26}$$

Then

$$\begin{aligned}
& T_{q_1 t^{\kappa_1}} T_{q_2 t^{\kappa_2}} z^\alpha \\
& = \tilde{\gamma}_{q_1}(\alpha_1 + k_2, \alpha_2 - k_2) \\
& \cdot \tilde{\gamma}_{q_2}(\alpha_1, \alpha_2) z_1^{\alpha_1+k_2+k_1} z_2^{\alpha_2-k_2-k_1} \\
& = \frac{\Gamma(|\alpha| + 2)}{\Gamma(\alpha_1 + k_2 + k_1 + 1) \Gamma(\alpha_2 - k_2 - k_1 + 1)} \\
& \cdot \int_0^1 s^{\alpha_1+k_2+k_1/2+m_1/2} (1-s)^{\alpha_2-k_2-k_1/2+m_2/2} ds \\
& \cdot \frac{\Gamma(|\alpha| + 2)}{\Gamma(\alpha_1 + k_2 + 1) \Gamma(\alpha_2 - k_2 + 1)} \\
& \cdot \int_0^1 s^{\alpha_1+k_2/2+\ell_1/2} (1-s)^{\alpha_2-k_2/2+\ell_2/2} ds \\
& \cdot z_1^{\alpha_1+k_2+k_1} z_2^{\alpha_2-k_2-k_1} \\
& = \Gamma(|\alpha| + 2) \Gamma\left(\alpha_1 + k_2 + \frac{k_1}{2} + \frac{m_1}{2} + 1\right) \\
& \cdot \Gamma\left(\alpha_2 - k_2 - \frac{k_1}{2} + \frac{m_2}{2} + 1\right) \\
& \cdot \left(\Gamma(\alpha_1 + k_2 + k_1 + 1) \Gamma(\alpha_2 - k_2 - k_1 + 1)\right) \\
& \cdot \Gamma\left(|\alpha| + \frac{m_1 + m_2}{2} + 2\right)^{-1} \\
& \cdot \Gamma\left(\alpha_1 + \frac{k_2}{2} + \frac{\ell_1}{2} + 1\right) \Gamma\left(\alpha_2 - \frac{k_2}{2} + \frac{\ell_2}{2} + 1\right)
\end{aligned}$$

$$\begin{aligned}
& \cdot \left(\Gamma(\alpha_1 + k_2 + 1) \Gamma(\alpha_2 - k_2 + 1) \right. \\
& \quad \cdot \Gamma\left(|\alpha| + \frac{\ell_1 + \ell_2}{2} + 2\right) \Big)^{-1} \\
& \cdot z_1^{\alpha_1 + k_2 + k_1} z_2^{\alpha_2 - k_2 - k_1}, \\
& T_{q_2 t^{\kappa_2}} T_{q_1 t^{\kappa_1}} z^\alpha \\
& = \tilde{\gamma}_{q_2}(\alpha_1 + k_1, \alpha_2 - k_1) \\
& \quad \cdot \tilde{\gamma}_{q_1}(\alpha_1, \alpha_2) z_1^{\alpha_1 + k_1 + k_2} z_2^{\alpha_2 - k_1 - k_2} \\
& = \frac{\Gamma(|\alpha| + 2)}{\Gamma(\alpha_1 + k_1 + k_2 + 1) \Gamma(\alpha_2 - k_1 - k_2 + 1)} \\
& \quad \cdot \int_0^1 s^{\alpha_1 + k_1 + k_2/2 + \ell_1/2} (1-s)^{\alpha_2 - k_1 - k_2/2 + \ell_2/2} ds \\
& \quad \cdot \frac{\Gamma(|\alpha| + 2)}{\Gamma(\alpha_1 + k_1 + 1) \Gamma(\alpha_2 - k_1 + 1)} \\
& \quad \cdot \int_0^1 s^{\alpha_1 + k_1/2 + m_1/2} (1-s)^{\alpha_2 - k_1/2 + m_2/2} ds \\
& \quad \cdot z_1^{\alpha_1 + k_1 + k_2} z_2^{\alpha_2 - k_1 - k_2} \\
& = \Gamma(|\alpha| + 2) \Gamma\left(\alpha_1 + k_1 + \frac{k_2}{2} + \frac{\ell_1}{2} + 1\right) \\
& \quad \cdot \Gamma\left(\alpha_2 - k_1 - \frac{k_2}{2} + \frac{\ell_2}{2} + 1\right) \\
& \quad \cdot \left(\Gamma(\alpha_1 + k_1 + k_2 + 1) \Gamma(\alpha_2 - k_1 - k_2 + 1) \right. \\
& \quad \cdot \Gamma\left(|\alpha| + \frac{\ell_1 + \ell_2}{2} + 2\right) \Big)^{-1} \\
& \quad \cdot \Gamma\left(\alpha_1 + \frac{k_1}{2} + \frac{m_1}{2} + 1\right) \\
& \quad \cdot \Gamma\left(\alpha_2 - \frac{k_1}{2} + \frac{m_2}{2} + 1\right) \\
& \quad \cdot \left(\Gamma(\alpha_1 + k_1 + 1) \Gamma(\alpha_2 - k_1 + 1) \right. \\
& \quad \cdot \Gamma\left(|\alpha| + \frac{m_1 + m_2}{2} + 2\right) \Big)^{-1} \\
& \quad \cdot z_1^{\alpha_1 + k_1 + k_2} z_2^{\alpha_2 - k_1 - k_2}.
\end{aligned} \tag{27}$$

That is, $T_{q_1 t^{\kappa_1}}$ commutes with $T_{q_2 t^{\kappa_2}}$ if and only if

$$\begin{aligned}
& \Gamma\left(\alpha_1 + k_2 + \frac{k_1}{2} + \frac{m_1}{2} + 1\right) \Gamma\left(\alpha_2 - k_2 - \frac{k_1}{2} + \frac{m_2}{2} + 1\right) \\
& \quad \cdot \Gamma\left(\alpha_1 + \frac{k_2}{2} + \frac{\ell_1}{2} + 1\right) \Gamma\left(\alpha_2 - \frac{k_2}{2} + \frac{\ell_2}{2} + 1\right)
\end{aligned}$$

$$\begin{aligned}
& \cdot \left(\Gamma(\alpha_1 + k_2 + 1) \Gamma(\alpha_2 - k_2 + 1) \right)^{-1} \\
& = \Gamma\left(\alpha_1 + k_1 + \frac{k_2}{2} + \frac{\ell_1}{2} + 1\right) \Gamma\left(\alpha_2 - k_1 - \frac{k_2}{2} + \frac{\ell_2}{2} + 1\right) \\
& \quad \cdot \Gamma\left(\alpha_1 + \frac{k_1}{2} + \frac{m_1}{2} + 1\right) \Gamma\left(\alpha_2 - \frac{k_1}{2} + \frac{m_2}{2} + 1\right) \\
& \quad \cdot \left(\Gamma(\alpha_1 + k_1 + 1) \Gamma(\alpha_2 - k_1 + 1) \right)^{-1}.
\end{aligned} \tag{28}$$

The last equality holds if and only if

$$m_1 = m_2 = k_1, \quad \ell_1 = \ell_2 = k_2. \tag{29}$$

This brings us back to the case analyzed in [4], showing that in our more general case of $q \in L_\infty[0, \pi/2]$ we have fewer properties as compared with the specific case of [4]. In particular, we lose the commutativity of generating operators in [4], extending at the same time the class of generating symbols.

3. The Algebra $\mathcal{T}_{\text{rad}, q, k}$

Denote by $\mathcal{T}_{\text{rad}}$ the C^* -algebra generated by all Toeplitz operators T_a 's with radial symbols $a = a(r) \in L_\infty[0, 1]$. We fix then a function $q = q(\theta) \in L_\infty[0, \pi/2]$ and $k \in \mathbb{N}$ and denote by $\mathcal{T}_{q, k}$ the unital Banach algebra generated by Toeplitz operator T_{q^k} . Since the generators of both these algebras commute (Lemma 3), the Banach algebra $\mathcal{T}_{\text{rad}, q, k}$ generated by elements of $\mathcal{T}_{\text{rad}}$ and $\mathcal{T}_{q, k}$ is commutative.

In this section we study the algebra $\mathcal{T}_{\text{rad}, q, k}$ as well its generating subalgebras. The analysis of the algebra $\mathcal{T}_{\text{rad}}$ is the same as in [4, Section 3.1]; thus we recall here just the most important facts.

3.1. Toeplitz Operators with Radial Symbols. Given a sequence $\gamma = \{\gamma(|\alpha|)\}_{|\alpha| \in \mathbb{Z}_+} \in \ell_\infty$, we denote by D_γ the diagonal operator defined on $\mathcal{A}_\lambda^2(\mathbb{B}^2)$ as follows:

$$D_\gamma z^\alpha = \gamma(|\alpha|) z^\alpha, \quad \alpha \in \mathbb{Z}_+^2. \tag{30}$$

If T_a is a Toeplitz operator with radial symbol we have obviously

$$T_a = D_{\gamma_{a, \lambda}}. \tag{31}$$

Recall [5, Section 5] that the sequence $\gamma_{a, \lambda}$ of a Toeplitz operator T_a belongs to the C^* -algebra $\text{SO}(\mathbb{Z}_+)$, where $\text{SO}(\mathbb{Z}_+)$ consists of all bounded sequences that slowly oscillate in the sense of Schmidt [6]; that is,

$$\text{SO}(\mathbb{Z}_+) = \left\{ x \in \ell_\infty : \lim_{(j+1)/(i+1) \rightarrow 1} |x_j - x_i| = 0 \right\}. \tag{32}$$

Moreover [5, Section 5], the C^* -algebra $\mathcal{T}_{\text{rad}}$ is isomorphic and isometric to the algebra $\text{SO}(\mathbb{Z}_+)$, via identification of a diagonal operator with its eigenvalue sequence.

Corollary 4. Let γ be a convergent sequence; then $D_\gamma \in \mathcal{T}_{\text{rad}}$. In particular, for all $n \in \mathbb{Z}_+$ the orthogonal projection P_n of $\mathcal{A}_\lambda^2(\mathbb{B}^2)$ onto $\text{span}\{z^\alpha : |\alpha| = n\}$ belongs to the algebra $\mathcal{T}_{\text{rad}}$.

Recall [4] that the compact set $M(\mathcal{T}_{\text{rad}})$ of maximal ideals of the algebra $\mathcal{T}_{\text{rad}}$ has the form

$$M(\mathcal{T}_{\text{rad}}) = \mathbb{Z}_+ \cup M_{\infty}. \quad (33)$$

The fiber M_{∞} is the set of all multiplicative functionals φ such that $\varphi(D_{\gamma}) = 0$ whenever D_{γ} is a compact operator, or $\gamma \in c_0$, where c_0 denotes the set of all sequences converging to zero. And $\mathbb{Z}_+$ can be considered a part of $M(\mathcal{T}_{\text{rad}})$ since each $\ell \in \mathbb{Z}_+$ defines the multiplicative evaluation functional $\varphi_{\ell} : D_{\gamma} \mapsto \gamma(\ell)$.

Moreover, by [7, Chapter I, Theorem 8.2], the set $\mathbb{Z}_+$ is densely and homeomorphically embedded into $M(\mathcal{T}_{\text{rad}})$, and, by [8], the fiber M_{∞} is connected.

3.2. Toeplitz Operators with Symbol $q(\theta)(t_1 \overline{t_2})^k$ and Its Spectrum. Recall that the spectrum of the Toeplitz operator T_{q^k} is independent of the weight parameter λ , and note that

$$\begin{aligned} \|T_{q^k}\| &\leq \|q(\theta)(t_1 \overline{t_2})^k\|_{\infty} \\ &= \sup_{\theta \in [0, \pi/2], t_1, t_2 \in \mathbb{T}} |q(\theta)(t_1 \overline{t_2})^k| = \sup_{\theta \in [0, \pi/2]} |q(\theta)| \\ &= \|q\|_{\infty}. \end{aligned} \quad (34)$$

That is, the spectral radius of T_{q^k} is at most $\|q\|_{\infty}$. And (Lemma 1) since the operator T_{q^k} is not invertible, we have that

$$0 \in \text{sp} T_{q^k} \subset \overline{D}(0, \|q\|_{\infty}). \quad (35)$$

We denote by $\Delta(q, k)$ the set $\text{sp} T_{q^k}$. Then the maximal ideal space $M(\mathcal{T}_{q,k})$ of the Banach algebra $\mathcal{T}_{q,k}$ coincides with the spectrum of its generator; that is,

$$M(\mathcal{T}_{q,k}) = \Delta(q, k). \quad (36)$$

3.3. Invariant Subspaces. Given $p \in \mathbb{Z}_+$, we denote by H_p the finite dimensional subspace

$$H_p = \text{span} \{e_{\alpha} : \alpha \in \mathbb{Z}_+, |\alpha| = p\}. \quad (37)$$

It is obvious that

$$\mathcal{A}_{\lambda}^2(\mathbb{B}^2) = \bigoplus_{p=0}^{\infty} H_p. \quad (38)$$

Observe now that each space H_p is invariant for all operators from $\mathcal{T}_{\text{rad}, q, k}$. Moreover, by Corollary 4, each orthogonal projection P_p onto H_p is a diagonal operator from $\mathcal{T}_{\text{rad}}$.

Each diagonal operator D_{γ} restricted to H_p is the scalar operator $\gamma(p)I$, while the operator T_{q^k} acts on H_p as a weighted shift operator. It is also nilpotent on each H_p , since

$$\left(T_{q^k}|_{H_p}\right)^{\lfloor p/k \rfloor + 1} = 0. \quad (39)$$

The last implies that

$$\bigoplus_{s=0}^{(p-1)k} H_s \subset \ker T_{q^k}^p, \quad \forall p \in \mathbb{N}. \quad (40)$$

3.4. Radical

Lemma 5. *The algebra $\mathcal{T}_{\text{rad}, q, k}$ is not semisimple. Its radical $\text{Rad} \mathcal{T}_{\text{rad}, q, k}$ contains, in particular, the operators of the form $D_{\gamma} T_{q^k}$, where $\gamma \in c_0$.*

Proof. Following the proof of [4, Lemma 3.7], we only need to prove that the operator $D_{\gamma} T_{q^k}$ is topologically nilpotent; that is,

$$\lim_{p \rightarrow \infty} \|(D_{\gamma} T_{q^k})^p\|^{1/p} = 0. \quad (41)$$

Note that

$$\begin{aligned} (D_{\gamma} T_{q^k})^p &= D_{\gamma}^p T_{q^k}^p \\ &= D_{\gamma}^p T_{q^k}^p (I - (P_0 + \dots + P_{(p-1)k})) \\ &= [D_{\gamma} (I - (P_0 + \dots + P_{(p-1)k}))]^p T_{q^k}^p. \end{aligned} \quad (42)$$

That is, since $\gamma \in c_0$,

$$\begin{aligned} \|(D_{\gamma} T_{q^k})^p\|^{1/p} &\leq \|D_{\gamma} (I - (P_0 + \dots + P_{(p-1)k}))\| \cdot \|T_{q^k}\| \\ &= \sup_{\ell > pk} |\gamma(\ell)| \cdot \|T_{q^k}\| \xrightarrow{p \rightarrow \infty} 0. \end{aligned} \quad (43)$$

□

3.5. Dense Subalgebra in $\mathcal{T}_{\text{rad}, q, k}$. The set of all operators of the form

$$A = \sum_{p=0}^m D_{\gamma_p} T_{q^k}^p, \quad (44)$$

where $D_{\gamma_p} \in \mathcal{T}_{\text{rad}}$, constitutes the dense (nonclosed) subalgebra $\mathcal{D} = \mathcal{D}_{\text{rad}, q, k}$ of the algebra $\mathcal{T}_{\text{rad}, q, k}$. At the same time the representation of the operators from $\mathcal{D}$ in the above form is not unique.

To describe the source of such non-uniqueness, we denote by $K(p)$ the set of finite dimensional diagonal operators with eigenvalue sequences

$$\gamma = \{\gamma(0), \gamma(1), \dots, \gamma((p-1)k), 0, 0, \dots\}, \quad (45)$$

and $K(0) = \{0\}$ for $p = 0$.

Lemma 6. *One has*

$$\sum_{p=0}^m D_{\gamma_p} T_{q^k}^p = 0 \quad (46)$$

if and only if $D_{\gamma_p} \in K(p)$, for each $p = 0, 1, \dots, m$.

Proof. The proof follows almost literally the arguments of proof [4, Lemma 3.8], given for a particular case of $q(\theta) = \cos \theta \sin \theta$ and $k = 1$. $\square$

The next lemma is a special case of the previous one.

Lemma 7. *Let*

$$A = \sum_{j=1}^m \left(\prod_{\ell=1}^{m_j} T_{a_{j,\ell}} T_{qt^k}^{p_j} \right) = 0, \quad (47)$$

where all p_j are different. Then, for each k at least one radial symbol $a_{j,\ell}$ is identically zero.

Proof. The previous lemma implies that if

$$A = \sum_{j=1}^m \left(\prod_{\ell=1}^{m_j} T_{a_{j,\ell}} T_{qt^k}^{p_j} \right) = 0, \quad (48)$$

then each diagonal operator $\prod_{\ell=1}^{m_j} T_{a_{j,\ell}}$ is finite dimensional. The rest of the proof follows the arguments of the proof of [4, Theorem 3.9]. $\square$

As an application to the so-called zero-product problem we give the next corollary.

Corollary 8. *For the operator*

$$A = \prod_{\ell=1}^m T_{a_\ell} T_{qt^k}^p \quad (49)$$

the following statements are equivalent:

- (1) $A = 0$,
- (2) A is finite dimensional,
- (3) At least one radial symbol a_ℓ is identically zero.

3.6. A Special Case of $q=q(\theta)$. In what follows we will assume that a function $q = q(\theta) \in L_\infty[0, \pi/2]$ satisfies the extra condition:

$$q = q(\theta) \in C \left[0, \frac{\pi}{2} \right], \quad q(0) = 0 = q\left(\frac{\pi}{2}\right). \quad (50)$$

We note that this is the (only) case when the homogeneous of order zero function (our generating symbol) $qt^k = q(\theta)(t_1 \bar{t}_2)^k$ is continuous on the boundary S^3 of the unit ball $\mathbb{B}^2$.

We note also that the quasi-homogeneous symbols (8) of [4] satisfy the above conditions.

Condition (50) permits us to make the statements of Section 3.2 more precise.

First of all (50), together with the results of [9], implies that

$$\begin{aligned} \text{ess-sp} T_{qt^k} &= \text{Im} \left(q(\theta)(t_1 \bar{t}_2)^k \right) \Big|_{\theta \in [0, \pi/2], t_1, t_2 \in \mathbb{T}} \\ &= \overline{D}(0, \|q\|_\infty). \end{aligned} \quad (51)$$

Therefore

$$\text{sp} T_{qt^k} = \Delta(q, k) = \overline{D}(0, \|q\|_\infty), \quad (52)$$

as

$$\overline{D}(0, \|q\|_\infty) = \text{ess-sp} T_{qt^k} \subset \text{sp} T_{qt^k} \subset \overline{D}(0, \|q\|_\infty), \quad (53)$$

where $D(0, r)$ is the disk of radius r centered at the origin.

That is, now we have that

$$M(\mathcal{T}_{q,k}) = \overline{D}(0, \|q\|_\infty). \quad (54)$$

Theorem 9. *The Banach algebra $\mathcal{T}_{q,k}$ is isomorphic via the Gelfand transform to the algebra $C_a(\overline{D}(0, \|q\|_\infty))$ of all functions analytic in $D(0, \|q\|_\infty)$ and continuous on $\overline{D}(0, \|q\|_\infty)$.*

Proof. It practically literally follows the proof of [4, Theorem 3.6]. $\square$

4. Gelfand Theory of $\mathcal{T}_{\text{rad},q,k}$

4.1. Finitely Generated Subalgebras of $\mathcal{T}_{\text{rad},q,k}$. We present here relevant material from [4], slightly modified when the more general symbols (9) will be involved.

We start from a finite number of diagonal operators $D_{\gamma_1}, \dots, D_{\gamma_n}$ on $\mathcal{A}_\lambda^2(\mathbb{B}^2)$ from the algebra $\mathcal{T}_{\text{rad}}$. They act on the standard monomial basis (4) as

$$D_{\gamma_j} e_\alpha = \gamma_j(|\alpha|) e_\alpha, \quad \text{for } \alpha \in \mathbb{Z}_+^2. \quad (55)$$

Denote by

$$\mathcal{A}_{\mathbf{D}}^* := \mathcal{A}(D_{\gamma_1}, \dots, D_{\gamma_n}) \subset L(\mathcal{A}_\lambda^2(\mathbb{B}^2)) \quad (56)$$

the unital C^* -algebra generated by elements of $\mathbf{D} = (D_{\gamma_1}, \dots, D_{\gamma_n})$.

Recall [10] that the joint spectrum $\sigma(D_{\gamma_1}, \dots, D_{\gamma_n})$ of the operators $D_{\gamma_1}, \dots, D_{\gamma_n}$ can be identified with the maximal ideal space $M(\mathcal{A}_{\mathbf{D}}^*)$ of the C^* -algebra $\mathcal{A}_{\mathbf{D}}^*$ and it has the form

$$\begin{aligned} \sigma(D_{\gamma_1}, \dots, D_{\gamma_n}) &= \{(\mu_1, \dots, \mu_n) \in \mathbb{C}^n \\ &: J(D_{\gamma_1} - \mu_1 I, \dots, D_{\gamma_n} - \mu_n I) \neq \mathcal{A}_{\mathbf{D}}^*\}, \end{aligned} \quad (57)$$

where $J(D_{\gamma_1} - \mu_1 I, \dots, D_{\gamma_n} - \mu_n I)$ denotes the smallest ideal in the algebra $\mathcal{A}_{\mathbf{D}}^*$ containing the elements $D_{\gamma_j} - \mu_j I$, for $j = 1, \dots, n$.

Lemma 10 (see [4, Corollary 4.11]). *Let $(\mu_1, \dots, \mu_n) \in \sigma(D_{\gamma_1}, \dots, D_{\gamma_n})$. Then either there is $k \in \mathbb{Z}_+$ such that $\gamma_j(k) = \mu_j$ for all $j = 1, \dots, n$ or there is a sequence $\{m_\ell\}_\ell \in \mathbb{Z}_+$ such that for all $j = 1, \dots, n$*

$$\lim_{\ell \rightarrow \infty} \gamma_j(m_\ell) = \mu_j. \quad (58)$$

Let $(\mu_1, \dots, \mu_n) \in \sigma(D_{\gamma_1}, \dots, D_{\gamma_n})$. Assuming the first option of the above lemma, we define a multiplicative functional $\psi_{(k)}$ on $\mathcal{A}_{\mathbf{D}}^*$ by

$$\psi_{(k)}(D) := \langle De_{(k,0)}, e_{(k,0)} \rangle \quad (59)$$

for all $D \in \mathcal{A}_{\mathbf{D}}^*$. Note that $\psi_{(k)}(D_{\gamma_j}) = \mu_j$, for all $j = 1, \dots, n$.

Alternatively, if the second option of Lemma 10 holds, then we define a multiplicative functional $\psi_{\{m_\ell\}}$ on $\mathcal{A}_{\mathbf{D}}^*$ by

$$\psi_{\{m_\ell\}}(D) := \lim_{\ell \rightarrow \infty} \langle De_{(m_\ell,0)}, e_{(m_\ell,0)} \rangle. \quad (60)$$

Lemma 11 (see [4, Lemma 4.12]). *The limit (60) exists for all $D \in \mathcal{A}_{\mathbf{D}}^*$. The functional $\psi_{\{m_\ell\}}$ is multiplicative with $\psi_{\{m_\ell\}}(D_{\gamma_j}) = \mu_j$, for all $j = 1, \dots, n$.*

Following [4] we give another formula defining the functional $\psi_{\{m_\ell\}}$ that permits an extension of $\psi_{\{m_\ell\}}$ to a larger subalgebra of $\mathcal{T}_{\text{rad},q,k}$.

We start with the set $S = \{m_\ell : \ell \in \mathbb{N}\}$ and define the Hilbert space

$$H_S = \overline{\text{span}} \{e_\alpha : \alpha \in \mathbb{Z}_+^2, |\alpha| \in S\} \subset \mathcal{A}_\lambda^2(\mathbb{B}^2). \quad (61)$$

The reproducing kernel K_S of H_S has the form

$$K_S(z, w) = \sum_{|\alpha| \in S} e_\alpha(z) \overline{e_\alpha(w)}. \quad (62)$$

Given $\zeta \in \overline{D}(0, \|q\|_\infty)$, let $\theta \in [0, \pi/2]$, $t_1, t_2 \in \mathbb{T}$ be such that $\zeta = q(\theta)(t_1 t_2)^k$, and denote by $v = v(\theta, t_1, t_2)$ the corresponding point on $\partial\mathbb{B}^2$. Consider then a sequence $\{w_h\}_{h \in \mathbb{N}} \subset \mathbb{B}^2$ defined by $w_h = ((h-1)/h)v$, and define the sequence $\{f_h\}_h$ of unit vectors in H_S by

$$f_h = \frac{K_S(\cdot, w_h)}{\|K_S(\cdot, w_h)\|}. \quad (63)$$

Then, by [4, Lemma 4.13], the functional (60) can be also defined as

$$\psi_{\{m_\ell\}}(D_\gamma) = \lim_{h \rightarrow \infty} \langle D_\gamma f_h, f_h \rangle. \quad (64)$$

In order to extend the multiplicative functional (64) to a larger algebra we proceed as follows. Let $\zeta \in \overline{D}(0, \|q\|_\infty)$ and let $\theta \in [0, \pi/2]$, $t_1, t_2 \in \mathbb{T}$ be such that $\zeta = q(\theta)(t_1 t_2)^k$. Then we take a sequence $\{w_h\}_h$ convergent to $v = v(\theta, t_1, t_2) \in \partial\mathbb{B}^2$ and $\{f_h\}_h$ as above.

Lemma 12 (see [4, Lemma 4.14]). *The functional $\psi_{\{m_\ell\}}$ extends to the functional $\psi = (\psi_{\{m_\ell\}}, \zeta)$ on the algebra generated by elements of $\mathcal{A}_{\mathbf{D}}^*$ and T_{q^k} via*

$$\psi(DT_{q^k}^j) = \lim_{h \rightarrow \infty} \langle DT_{q^k}^j f_h, f_h \rangle \quad (65)$$

with $j \in \mathbb{Z}_+$ and $D \in \mathcal{A}_{\mathbf{D}}^*$. Moreover, for elements of the form $\sum_{j=0}^m D_j T_{q^k}^j$ one has

$$\psi\left(\sum_{j=0}^m D_j T_{q^k}^j\right) = \sum_{j=0}^m \psi_{\{m_\ell\}}(D_j) \zeta^j. \quad (66)$$

4.2. Gelfand Theory. Recall that the commutative Banach algebra $\mathcal{T}_{\text{rad},q,k}$ is generated by its two unital subalgebras $\mathcal{T}_{\text{rad}}$ and $\mathcal{T}_{q,k}$.

Recall also the following general fact: let $\mathcal{A}$ be a unital commutative Banach algebra generated by its two subalgebras $\mathcal{A}_1, \mathcal{A}_2$ sharing the same identity, and let $M(\mathcal{A}), M(\mathcal{A}_1)$, and $M(\mathcal{A}_2)$ be their respective sets of maximal ideals. Then we have a natural continuous mapping

$$\iota : \psi \in M(\mathcal{A}) \mapsto (\psi_1, \psi_2) \in M(\mathcal{A}_1) \times M(\mathcal{A}_2), \quad (67)$$

defined by the restrictions ψ_1 and ψ_2 of the functional ψ onto the subalgebras $\mathcal{A}_1, \mathcal{A}_2$.

The mapping ι is injective identifying thus its range with $M(\mathcal{A})$.

That is, by the results of Sections 3.1 and 3.2 ι identifies $M(\mathcal{T}_{\text{rad},q,k})$ with a subset of $(\mathbb{Z}_+ \cup M_\infty) \times \Delta(q, k)$.

Lemma 13. *None of the points of the set $\mathbb{Z}_+ \times (\Delta(q, k) \setminus \{0\})$ belongs to $M(\mathcal{T}_{\text{rad},q,k})$.*

Proof. Suppose that $\psi = (m, \zeta) \in \mathbb{Z}_+ \times (\Delta(q, k) \setminus \{0\})$ is an element of $M(\mathcal{T}_{\text{rad},q,k})$ and let $A = P_m T_{q^k} \in \mathcal{T}_{\text{rad},q,k}$ where P_m is the orthogonal projection onto H_m ; then $\psi(A) = 1 \cdot \zeta \neq 0$ and, on the other hand, the operator A belongs to the radical of $\mathcal{T}_{\text{rad},q,k}$; therefore $\psi(A) = 0$ and we have a contradiction. $\square$

Note that the result of this lemma is independent of an extra condition (50) and thus from the concrete form of the spectrum $\Delta(q, k)$.

For the next result recall that the function q satisfies condition (50), and thus $\Delta(q, k) = \overline{D}(0, \|q\|_\infty)$.

Lemma 14. *One has that*

- (i) *the set $\mathbb{Z}_+ \times \{0\}$ belongs to $M(\mathcal{T}_{\text{rad},q,k})$,*
- (ii) *the set $M_\infty \times \overline{D}(0, \|q\|_\infty)$ belongs to $M(\mathcal{T}_{\text{rad},q,k})$.*

Proof. (i) Let $\psi = (m, 0) \in \mathbb{Z}_+ \times \{0\}$. Let $\psi = (m, 0) = (\psi_m, 0)$ be a functional on a dense subalgebra $\mathcal{D}$ of $\mathcal{T}_{\text{rad},q,k}$ defined, for any $A = \sum_{p=0}^n D_p T_{q^k}^p \in \mathcal{D}$ where $D_p \in \mathcal{T}_{\text{rad}}$, by

$$\begin{aligned} \psi(A) &:= \langle Ae_{(m,0)}, e_{(m,0)} \rangle \\ &= \psi_{(m)}(D_0) \\ &= \gamma_{D_0}(m), \end{aligned} \quad (68)$$

where $\psi_{(m)}$ is the multiplicative functional on $\mathcal{T}_{\text{rad}}$ given in (59).

The functional ψ is well defined, since $\sum_{p=0}^n D_p T_{qt^k}^p = 0$ implies $D_0 = 0$ by Lemma 6; moreover, ψ is continuous and extends to a multiplicative functional on $\mathcal{T}_{\text{rad},q,k}$ since

$$\left| \psi \left(\sum_{p=0}^n D_p T_{qt^k}^p \right) \right| = \left| \left\langle \sum_{p=0}^n D_p T_{qt^k}^p e_{(m,0)}, e_{(m,0)} \right\rangle \right| \leq \left\| \sum_{p=0}^n D_p T_{qt^k}^p \right\|. \quad (69)$$

(ii) Let $\psi = (\mu, \zeta) \in M_\infty \times \overline{D}(0, \|q\|_\infty)$ and define a multiplicative functional on a dense subalgebra of $\mathcal{T}_{\text{rad},q,k}$ as

$$\psi \left(\sum_{p=0}^n D_p T_{qt^k}^p \right) := \sum_{p=0}^n \mu(D_p) \zeta^p = \sum_{p=0}^n \gamma_{D_p}(\mu) \zeta^p. \quad (70)$$

The functional ψ is well defined:

$$A = \sum_{p=0}^n D_p T_{qt^k}^p = 0 \quad (71)$$

implies that D_p is compact for all $p = 0, \dots, n$ and since $\mu \in M_\infty$, $\mu(D_p) = 0$ for all $p = 0, \dots, n$ we have that $\psi(A) = 0$.

The functional ψ is continuous: let $A = \sum_{p=0}^n D_p T_{qt^k}^p \in \mathcal{D}$ be a fixed element. Consider the unital C^* -algebra $\mathcal{A}_D^*$ generated by $D_0, \dots, D_n$. The restriction $\hat{\psi}$ of ψ to $\mathcal{A}_D^*$ defines obviously a multiplicative functional on $\mathcal{A}_D^*$. We note that

$$(\mu_0, \dots, \mu_n) = (\psi(D_0), \dots, \psi(D_n)) \in \sigma(D_0, \dots, D_n). \quad (72)$$

Since $\hat{\psi}$ maps compact operators to zero, by (64), it has the form

$$\hat{\psi}(D_\gamma) = \lim_{h \rightarrow \infty} \langle D_\gamma f_h, f_h \rangle, \quad D_\gamma \in \mathcal{A}_D^*, \quad (73)$$

where $\{m_h\}_h \subset \mathbb{Z}_+$ is a suitable sequence induced by $(\mu_0, \dots, \mu_n)$ as in Section 4.1 and f_h is given by (63).

By Lemma 12 we have

$$\left| \psi \left(\sum_{p=0}^n D_p T_{qt^k}^p \right) \right| = \lim_{h \rightarrow \infty} \left| \left\langle \sum_{p=0}^n D_p T_{qt^k}^p f_h, f_h \right\rangle \right| \leq \left\| \sum_{p=0}^n D_p T_{qt^k}^p \right\|; \quad (74)$$

therefore, ψ is continuous and thus extends to a multiplicative functional on $\mathcal{T}_{\text{rad},q,k}$. $\square$

Finally, Lemmas 13 and 14, Theorem 9, and properties of the injective tensor product imply the description of the set of maximal ideals and the Gelfand transform of Banach algebra $\mathcal{T}_{\text{rad},q,k}$.

Theorem 15. *The compact set $M(\mathcal{T}_{\text{rad},q,k})$ of maximal ideals of the algebra $\mathcal{T}_{\text{rad},q,k}$ has the form*

$$M(\mathcal{T}_{\text{rad},q,k}) = \mathbb{Z}_+ \times \{0\} \cup M_\infty \times \overline{D}(0, \|q\|_\infty). \quad (75)$$

(i) *The Gelfand image of the algebra $\mathcal{T}_{\text{rad},q,k}$ is isomorphic to $\mathcal{T}_{\text{rad},q,k}/\text{Rad}\mathcal{T}_{\text{rad},q,k}$ and coincides with the algebra*

$$SO(\mathbb{Z}_+) \cup [C(M_\infty) \hat{\otimes}_e C_a(\overline{D}(0, \|q\|_\infty))], \quad (76)$$

which is identified with the set of all pairs

$$(\gamma, f) \in SO(\mathbb{Z}_+) \times [C(M_\infty) \hat{\otimes}_e C_a(\overline{D}(0, \|q\|_\infty))] \quad (77)$$

satisfying the following condition $\gamma(\mu) = f(\mu, 0)$ for all $\mu \in M_\infty$, where $\gamma(\mu)$ is identified with the value $\mu \in M_\infty$ on the element $\gamma \in SO(\mathbb{Z}_+)$.

The symbol $\hat{\otimes}_e$ denotes the injective tensor product.

(ii) *The Gelfand transform is generated by the following mapping of elements of $\mathcal{D}$:*

$$\sum_{j=0}^n D_{\gamma_j} T_{qt^k}^j \mapsto \begin{cases} \gamma_0(m), & \text{if } (m, 0) \in \mathbb{Z}_+ \times \{0\}, \\ \sum_{j=0}^n \gamma_j(\mu) \zeta^j, & \text{if } (\mu, \zeta) \in M_\infty \times \overline{D}(0, \|q\|_\infty). \end{cases} \quad (78)$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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