Research Article

Weighted BMO Estimates for Toeplitz Operators on Weighted Lebesgue Spaces

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The authors establish the weighted BMO estimates for a class of Toeplitz operators related to strongly singular Calderón-Zygmund operators on weighted Lebesgue spaces. Moreover, the corresponding result for the Toeplitz operators related to classical Calderón-Zygmund operators can be deduced.

1. Introduction

The introduction of the strongly singular Calderón-Zygmund operator is motivated by a class of multiplier operators with symbols given by $e^{i|\xi|^\alpha}/|\xi|^\beta$ away from the origin, where $0 < \alpha < 1$ and $\beta > 0$. Fefferman and Stein [1] enlarged the multiplier operators onto a class of convolution operators. Coifman [2] also considered a related class of operators for $n = 1$. The strongly singular nonconvolution operator was introduced by Alvarez and Milman [3], whose properties are similar to those of the classical Calderón-Zygmund operator, but the kernel is more singular near the diagonal than that of the standard case.

Definition 1. Let $T : \mathcal{S} \rightarrow \mathcal{S}'$ be a bounded linear operator. T is called a strongly singular Calderón-Zygmund operator if the following conditions are satisfied.

- (1) T can be extended into a continuous operator from $L^2(\mathbb{R}^n)$ into itself.
- (2) There exists a function $K(x, y)$ continuous away from the diagonal $\{(x, y) : x = y\}$ such that

$$\begin{aligned} & |K(x, y) - K(x, z)| + |K(y, x) - K(z, x)| \\ & \leq C \frac{|y - z|^\delta}{|x - z|^{n+\delta/\alpha}}, \end{aligned} \quad (1)$$

if $2|y - z|^\alpha \leq |x - z|$ for some $0 < \delta \leq 1$ and $0 < \alpha < 1$.
 $\langle Tf, g \rangle = \iint K(x, y) f(y) g(x) dy dx$, for $f, g \in \mathcal{S}$ with disjoint support.

- (3) For some $n(1 - \alpha)/2 \leq \beta < n/2$, both T and its conjugate operator T^* can be extended into continuous operators from L^q to L^2 , where $1/q = 1/2 + \beta/n$.

Alvarez and Milman [3, 4] discussed the properties of the strongly singular Calderón-Zygmund operator on Lebesgue spaces. Lin [5] proved the boundedness of the strongly singular Calderón-Zygmund operator and its commutators on Morrey type spaces. The authors in [6] established the weighted BMO estimates for the commutator of the strongly singular Calderón-Zygmund operator on weighted Morrey spaces.

The commutator generated by the Calderón-Zygmund operator and a locally integrable function b can be regarded as a special case of the Toeplitz operator $T_b = \sum_{j=1}^m T_{j,1} M_b T_{j,2}$, where $T_{j,1}$ and $T_{j,2}$ are the Calderón-Zygmund operators or $\pm I$ (I is the identity operator), $M_b f(x) = b(x) f(x)$. When $b \in BMO$, Krantz and Li [7] discussed the L^p boundedness of T_b on the homogeneous spaces. The commutator generated by the fractional integral operator and a locally integrable function b can be regarded as a special case of another class of Toeplitz operators $\Theta_{\alpha_0}^b = \sum_{j=1}^m (T_{j,1} M_b I_{\alpha_0} T_{j,2} + T_{j,3} I_{\alpha_0} M_b T_{j,4})$, where $T_{j,1}$ are the Calderón-Zygmund operators or $\pm I$, $T_{j,2}$

and $T_{j,4}$ are the bounded linear operators on L^p , $T_{j,3} = \pm I$, and $I_{\alpha_0} f(x) = \int_{\mathbb{R}^n} (f(y)/|x-y|^{n-\alpha_0}) dy$ is the fractional integral operator. When $b \in BMO$, Qiu [8] obtained the boundedness of $\Theta_{\alpha_0}^b$ on the homogeneous spaces from $L^p(X)$ to $L^q(X)$, where $1/q = 1/p - \alpha_0$. Moreover, Lin and Lu [9] discussed the boundedness of the two kinds of Toeplitz operators T_b and $\Theta_{\alpha_0}^b$ related to strongly singular Calderón-Zygmund operators and BMO functions on Lebesgue spaces.

In this paper, we are interested in the Toeplitz operators $\Theta_{\alpha_0}^b$ related to strongly singular Calderón-Zygmund operators and weighted BMO functions. Actually, in this situation, $T_{j,1}$ are the strongly singular Calderón-Zygmund operators or $\pm I$, $T_{j,2}$ and $T_{j,4}$ are the bounded linear operators on weighted Lebesgue spaces, $T_{j,3} = \pm I$, and b is a weighted BMO function. And we will focus on the boundedness of this kind of Toeplitz operators on weighted Lebesgue spaces.

Before stating our main results, first we need to recall some definitions and notations as follows.

Definition 2. A nonnegative measurable function ω is said to be in the Muckenhoupt class A_p with $1 < p < \infty$ if, for every cube Q in $\mathbb{R}^n$, there exists a positive constant C independent of Q such that

$$\left(\frac{1}{|Q|} \int_Q \omega(x) dx \right) \left(\frac{1}{|Q|} \int_Q \omega(x)^{1-p'} dx \right)^{p-1} \leq C, \quad (2)$$

where Q denotes a cube in $\mathbb{R}^n$ with the side parallel to the coordinate axes and $1/p + 1/p' = 1$. When $p = 1$, a nonnegative measurable function ω is said to belong to A_1 , if there exists a constant $C > 0$ such that, for any cube Q ,

$$\frac{1}{|Q|} \int_Q \omega(y) dy \leq C \omega(x), \quad \text{a.e. } x \in Q. \quad (3)$$

Denote $A_\infty = \bigcup_{1 \leq p < \infty} A_p$. It is well known that if $\omega \in A_p$ with $1 < p < \infty$, then $\omega \in A_r$ for all $r > p$ and $\omega \in A_q$ for some $1 < q < p$.

Definition 3. A weighted function ω belongs to the reverse Hölder class RH_r if there exist two constants $r > 1$ and $C > 0$ such that the following reverse Hölder inequality

$$\left(\frac{1}{|Q|} \int_Q \omega(x)^r dx \right)^{1/r} \leq C \left(\frac{1}{|Q|} \int_Q \omega(x) dx \right) \quad (4)$$

holds for every cube Q in $\mathbb{R}^n$.

It is well known that if $\omega \in A_p$ with $1 \leq p < \infty$, then there exists a $r > 1$ such that $\omega \in RH_r$. It follows directly from Hölder's inequality that $\omega \in RH_r$ implies $\omega \in RH_s$ for all $1 < s < r$.

Definition 4. Suppose that ω is a nonnegative locally integrable function on $\mathbb{R}^n$. Define $\omega \in A_{p,q}$ ($1 < p, q < \infty$), if there exists a constant $C > 0$, such that, for any cube Q in $\mathbb{R}^n$,

$$\sup_Q \left(\frac{1}{|Q|} \int_Q \omega(x)^q dx \right)^{1/q} \left(\frac{1}{|Q|} \int_Q \omega(x)^{-p'} dx \right)^{1/p'} \leq C < \infty. \quad (5)$$

Definition 5. Let $1 \leq p < \infty$ and let ω be a weighted function. A locally integrable function b is said to be in the weighted BMO space $BMO_p(\omega)$ if

$$\begin{aligned} \|b\|_{BMO_p(\omega)} &= \sup_Q \left(\frac{1}{\omega(Q)} \int_Q |b(x) - b_Q|^p \omega(x)^{1-p} dx \right)^{1/p} \\ &< \infty, \end{aligned} \quad (6)$$

where $b_Q = (1/|Q|) \int_Q b(y) dy$ and the supremum is taken over all cubes $Q \subset \mathbb{R}^n$. Moreover, we denote $BMO_p(\omega)$ simply by $BMO(\omega)$ when $p = 1$.

Definition 6. The Hardy-Littlewood maximal operator M is defined by

$$M(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| dy. \quad (7)$$

We set $M_s(f) = M(|f|^s)^{1/s}$, where $0 < s < \infty$.

The sharp maximal operator $M^\sharp$ is defined by

$$\begin{aligned} M^\sharp(f)(x) &= \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy \\ &\sim \sup_{Q \ni x} \inf_{a \in \mathbb{C}} \frac{1}{|Q|} \int_Q |f(y) - a| dy, \end{aligned} \quad (8)$$

where $f_Q = (1/|Q|) \int_Q f(x) dx$.

Let ω be a weight. The weighted maximal operator M_ω is defined by

$$M_\omega(f)(x) = \sup_{Q \ni x} \frac{1}{\omega(Q)} \int_Q |f(y)| \omega(y) dy. \quad (9)$$

We also set $M_{s,\omega}(f) = M_\omega(|f|^s)^{1/s}$, where $0 < s < \infty$. The fractional maximal operator M_{α_0} and the weighted case $M_{\alpha_0,s,\omega}$ are defined by

$$M_{\alpha_0}(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|^{1-\alpha_0/n}} \int_Q |f(y)| dy, \quad (10)$$

$$\begin{aligned} M_{\alpha_0,s,\omega}(f)(x) &= \sup_{Q \ni x} \left(\frac{1}{\omega(Q)^{1-\alpha_0s/n}} \int_Q |f(y)|^s \omega(y) dy \right)^{1/s}, \end{aligned} \quad (11)$$

$$0 < \alpha_0 < n,$$

respectively.

2. Main Results

Now we state our main results as follows.

Theorem 7. Let $T_{j,1}$ be strongly singular Calderón-Zygmund operators or $\pm I$, let α, β, δ be the same as in Definition 1, and $n(1-\alpha)/2 < \beta < n/2$. Suppose $0 < \alpha_0 < 2n\beta/(n(1-\alpha) + 2\beta)$, $n^2(1-\alpha)/\alpha_0[n(1-\alpha) + 2\beta] + 1 < p < n/\alpha_0$, $1/q = 1/p - \alpha_0/n$, $\omega^q \in A_1$, and $\omega \in RH_r$ with $r > q/p$, $b \in BMO(\omega)$, $T_{j,3} = \pm I$; $T_{j,2}$ and $T_{j,4}$ are bounded operators on $L^p(\omega)$. When $f \in C_c^\infty(\mathbb{R}^n)$, $\Theta_{\alpha_0}^1(f) = 0$. Then $\Theta_{\alpha_0}^b$ is bounded from $L^p(\omega)$ to $L^q(\omega^{q/p-q})$.

If we consider the extreme cases $\alpha \rightarrow 1$ and $\beta \rightarrow 0$ in Definition 1, then the strongly singular Calderón-Zygmund operator comes back to the classical Calderón-Zygmund operator. Thus, we can get the boundedness of the Toeplitz operators related to classical Calderón-Zygmund operators on weighted Lebesgue spaces as a corollary of Theorem 7.

Corollary 8. Let $T_{j,1}$ be classical Calderón-Zygmund operators or $\pm I$. Suppose $0 < \alpha_0 < n$, $1 < p < n/\alpha_0$, $1/q = 1/p - \alpha_0/n$, $\omega^q \in A_1$, and $\omega \in RH_r$ with $r > q/p$, $b \in BMO(\omega)$, $T_{j,3} = \pm I$; $T_{j,2}$ and $T_{j,4}$ are bounded operators on $L^p(\omega)$. When $f \in C_c^\infty(\mathbb{R}^n)$, $\Theta_{\alpha_0}^1(f) = 0$. Then $\Theta_{\alpha_0}^b$ is bounded from $L^p(\omega)$ to $L^q(\omega^{q/p-q})$.

3. Lemmas

Before giving the proof of our main results, we need some necessary lemmas.

Lemma 9. If $\omega^q \in A_1$, then $\omega \in A_1$, where $1 < q < \infty$.

Lemma 10. If $\omega \in A_1$, then $\omega^{1/q} \in A_{p,q}$, where $1 < p, q < \infty$.

The results of Lemmas 9 and 10 can be deduced directly by Definitions 2 and 4 and Hölder's inequality. We omit the details.

Lemma 11. If $\omega^q \in A_1$, then $\omega \in A_{p,q}$ and $\omega^{1/p} \in A_{p,q}$, where $1 < p, q < \infty$.

Proof. Since $\omega^q \in A_1$, then $\omega = (\omega^q)^{1/q} \in A_{p,q}$ by Lemma 10. It follows from $(\omega^{q/p})^p = \omega^q \in A_1$, $p > 1$, and Lemma 9 that $\omega^{q/p} \in A_1$. Then applying Lemma 10, we have $\omega^{1/p} = (\omega^{q/p})^{1/q} \in A_{p,q}$. $\square$

Lemma 12 (see [6]). Let $\omega \in A_1$ and let f be a function in $BMO(\omega)$. Suppose $1 \leq p < \infty$, $x \in \mathbb{R}^n$, and $r_1, r_2 > 0$. Then

$$\begin{aligned} & \left(\frac{1}{|B(x, r_1)|} \int_{B(x, r_1)} |f(y) - f_{B(x, r_2)}|^p \omega(y)^{1-p} dy \right)^{1/p} \\ & \leq C \|f\|_{BMO(\omega)} \omega(x) \left(1 + \left| \ln \frac{r_2}{r_1} \right| \right) \\ & \cdot \left(\frac{\omega(B(x, r_1))}{|B(x, r_1)|} \right)^{-1/p'}. \end{aligned} \quad (12)$$

Lemma 13 (see [6]). Given $\varepsilon > 0$, there is $\ln x \leq (1/\varepsilon)x^\varepsilon$, for all $x \geq 1$.

Lemma 14 (see [10]). The fractional integral operator I_{α_0} is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, where $1/q = 1/p - \alpha_0/n$, $1 < p < n/\alpha_0$, and $0 < \alpha_0 < n$.

Lemma 15 (see [11, 12]). Let $\omega \in A_1$. Then, for any $1 \leq p < \infty$, there exists an absolute constant $C > 0$ such that $\|b\|_{BMO_p(\omega)} \leq C\|b\|_{BMO(\omega)}$.

Lemma 16 (see [3, 4]). If T is a strongly singular Calderón-Zygmund operator, then T can be defined to be a continuous operator from L^∞ to BMO and T is of weak (L^1, L^1) type.

By Lemma 16, Definition 1, and the interpolation theory, we can get that T is bounded on L^p , $1 < p < \infty$. Besides the (L^p, L^p) -boundedness, the strongly singular Calderón-Zygmund operator T still has another kind of boundedness properties on Lebesgue spaces. If we interpolate between $(L^2, L^{q'})$ and (L^∞, BMO) , where q is given as in Definition 1 and $1/q + 1/q' = 1$, then T is bounded from L^u to L^v with $2 \leq u < \infty$ and $v = uq'/2$. It is easy to see that $0 < u/v \leq \alpha$ in this situation. Then we interpolate between $(L^2, L^{q'})$ and weak (L^1, L^1) to obtain the boundedness of T from L^u to L^v , where $1 < u \leq 2$ and $v = uq'/(2q' - uq' + 2u - 2)$. In this situation, $0 < u/v \leq \alpha$ if and only if $(n(1-\alpha) + 2\beta)/2\beta \leq u \leq 2$. In a word, the boundedness properties of the strongly singular Calderón-Zygmund operator on Lebesgue spaces can be summarized as follows.

Remark 17. The strongly singular Calderón-Zygmund operator T is bounded on L^p for $1 < p < \infty$. And T is bounded from L^u to L^v , $(n(1-\alpha) + 2\beta)/2\beta \leq u < \infty$, and $0 < u/v \leq \alpha$. In particular, if we restrict $n(1-\alpha)/2 < \beta < n/2$ in (3) of Definition 1, then T is bounded from L^u to L^v , where $(n(1-\alpha) + 2\beta)/2\beta < u < \infty$ and $0 < u/v < \alpha$.

Lemma 18 (see [13]). Let $1 < p < \infty$ and $\omega \in A_\infty$. Then $\|M_\omega(f)\|_{L^p(\omega)} \leq C\|f\|_{L^p(\omega)}$.

Lemma 19. If $1 < p < \infty$ and $\omega \in A_\infty$, then, for any $1 < s < p$, we have

$$\|M_{s,\omega}(f)\|_{L^p(\omega)} \leq C\|f\|_{L^p(\omega)}. \quad (13)$$

Proof. By Lemma 18, we have

$$\begin{aligned} \|M_{s,\omega}(f)\|_{L^p(\omega)} &= \|M_\omega(|f|^s)\|_{L^{p/s}(\omega)}^{1/s} \leq C\| |f|^s \|_{L^{p/s}(\omega)}^{1/s} \\ &= C\|f\|_{L^p(\omega)}. \end{aligned} \quad (14)$$

$\square$

Lemma 20. If $0 < s < \infty$, $r > 1$, and $\omega \in RH_r$, then $M_{s,\omega}(f)(x) \leq CM_{sr,\omega^r}(f)(x)$.

Proof. For any ball $B \subset R^n$ which contains x , we have

$$\begin{aligned} & \left(\frac{1}{\omega(B)} \int_B |f(y)|^s \omega(y) dy \right)^{1/s} \\ & \leq \frac{1}{\omega(B)^{1/s}} \left(\int_B |f(y)|^{sr} \omega^r(y) dy \right)^{1/sr} |B|^{1/s-1/sr} \quad (15) \\ & \leq M_{sr, \omega^r}(f)(x) \frac{[\omega^r(B)]^{1/sr}}{\omega(B)^{1/s}} |B|^{1/s-1/sr}. \end{aligned}$$

Since

$$\begin{aligned} \omega^r(B) &= \int_B \omega^r(y) dy \leq C \left(\frac{1}{|B|} \int_B \omega(y) dy \right)^r |B| \\ &= C \omega(B)^r |B|^{1-r}, \end{aligned} \quad (16)$$

then

$$\begin{aligned} & \left(\frac{1}{\omega(B)} \int_B |f(y)|^s \omega(y) dy \right)^{1/s} \\ & \leq C M_{sr, \omega^r}(f)(x) \frac{\omega(B)^{1/s} |B|^{1/sr-1/s}}{\omega(B)^{1/s}} |B|^{1/s-1/sr} \quad (17) \\ & = C M_{sr, \omega^r}(f)(x). \end{aligned}$$

Thus we can get the desired conclusion by taking over all balls B which contain x . $\square$

Lemma 21 (see [13]). Suppose that $1 \leq s < p < n/\alpha_0$, $1/q = 1/p - \alpha_0/n$, $0 < \alpha_0 < n$, and $\omega \in A_{p,q}$. Then $\|M_{\alpha_0, s, \omega}(f)\|_{L^q(\omega)} \leq C \|f\|_{L^p(\omega)}$.

Lemma 22 (see [14]). Suppose that $0 < \alpha_0 < n$, $1 < p < n/\alpha_0$, $1/q = 1/p - \alpha_0/n$, and $\omega \in A_{p,q}$. Then

$$\begin{aligned} & \left(\int_{R^n} |I_{\alpha_0} f(x) \omega(x)|^q dx \right)^{1/q} \\ & \leq C \left(\int_{R^n} |f(x) \omega(x)|^p dx \right)^{1/p}. \end{aligned} \quad (18)$$

Lemma 23 (see [14]). Suppose that $0 < \alpha_0 < n$, $1 < p < n/\alpha_0$, and $1/q = 1/p - \alpha_0/n$. If $\omega \in A_{p,q}$, then

$$\begin{aligned} & \left(\int_{R^n} [M_{\alpha_0} f(x) \omega(x)]^q dx \right)^{1/q} \\ & \leq C \left(\int_{R^n} |f(x) \omega(x)|^p dx \right)^{1/p}, \end{aligned} \quad (19)$$

where C is independent of f .

Lemma 24 (see [14]). Let $\omega \in A_p$, $1 \leq p_0 \leq p < \infty$. If f is such that $Mf \in L^{p_0}(\omega)$, then

$$\int_{R^n} |Mf(x)|^p \omega(x) dx \leq C \int_{R^n} |M^\sharp f(x)|^p \omega(x) dx. \quad (20)$$

Lemma 25. Let $T_{j,1}$ be strongly singular Calderón-Zygmund operators or $\pm I$, let α, β, δ be the same as in Definition 1, and $n(1-\alpha)/2 < \beta < n/2$. Suppose $0 < \alpha_0 < 2n\beta/(n(1-\alpha) + 2\beta)$, $(n(1-\alpha) + 2\beta)/2\beta < s < n/\alpha_0$, and $\omega \in A_1 \cap RH_r$ with $r > [n(1-\alpha) + 2\beta](s-1)/(2\beta s - [n(1-\alpha) + 2\beta])$, $b \in BMO(\omega)$, and $T_{j,3} = \pm I$. Suppose also $\Theta_{\alpha_0}^1(f) = 0$, when $f \in C_c^\infty(R^n)$. Then for a.e. $x \in R^n$ we have

$$\begin{aligned} M^\sharp(\Theta_{\alpha_0}^b(f))(x) &\leq C \sum_{j=1}^m \|T_{j,1}\| \|b\|_{BMO(\omega)} \omega(x) \\ &\quad \cdot M_{s, \omega}(I_{\alpha_0} T_{j,2} f)(x) + C \sum_{j=1}^m \|b\|_{BMO(\omega)} \omega(x) \\ &\quad \cdot M_{s, \omega}(I_{\alpha_0} T_{j,2} f)(x) + C \sum_{j=1}^m \|I_{\alpha_0}\| \|b\|_{BMO(\omega)} \\ &\quad \cdot \omega(x)^{1-\alpha_0/n} M_{\alpha_0, s, \omega}(T_{j,4} f)(x) + C \sum_{j=1}^m \|b\|_{BMO(\omega)} \\ &\quad \cdot \omega(x)^{1-\alpha_0/n} M_{\alpha_0, s, \omega}(T_{j,4} f)(x) + C \sum_{j=1}^m \|b\|_{BMO(\omega)} \\ &\quad \cdot \omega(x) M_{\alpha_0}(T_{j,4} f)(x). \end{aligned} \quad (21)$$

Proof. Denote $\Theta_{\alpha_0}^b = A_{\alpha_0}^b + B_{\alpha_0}^b$, where

$$\begin{aligned} A_{\alpha_0}^b &= \sum_{j=1}^m T_{j,1} M_b I_{\alpha_0} T_{j,2}, \\ B_{\alpha_0}^b &= \sum_{j=1}^m T_{j,3} I_{\alpha_0} M_b T_{j,4}. \end{aligned} \quad (22)$$

For $x \in R^n$, let B be a ball centered at x with radius r_B . Then

$$\Theta_{\alpha_0}^b(f) = \Theta_{\alpha_0}^{b-b_{2B}}(f) = A_{\alpha_0}^{b-b_{2B}}(f) + B_{\alpha_0}^{b-b_{2B}}(f). \quad (23)$$

We consider $B_{\alpha_0}^{b-b_{2B}}(f)$ first. Write

$$\begin{aligned} B_{\alpha_0}^{b-b_{2B}}(f) &= \sum_{j=1}^m T_{j,3} I_{\alpha_0} M_{(b-b_{2B})\chi_{2B}} T_{j,4}(f) \\ &\quad + \sum_{j=1}^m T_{j,3} I_{\alpha_0} M_{(b-b_{2B})\chi_{(2B)^c}} T_{j,4}(f) \\ &=: f_{11} + f_{12}. \end{aligned} \quad (24)$$

Then

$$\begin{aligned} & \frac{1}{|B|} \int_B |B_{\alpha_0}^{b-b_{2B}}(f)(y) - f_{12}(x)| dy \\ & \leq \frac{1}{|B|} \int_B |f_{11}(y)| dy \\ & \quad + \frac{1}{|B|} \int_B |f_{12}(y) - f_{12}(x)| dy =: I_1 + I_2. \end{aligned} \quad (25)$$

Since $(n(1 - \alpha) + 2\beta)/2\beta < s < n/\alpha_0$ and $r > [n(1 - \alpha) + 2\beta](s - 1)/(2\beta s - [n(1 - \alpha) + 2\beta])$, then $(n(1 - \alpha) + 2\beta)/2\beta < rs/(r + s - 1)$. There is l_1 such that $(n(1 - \alpha) + 2\beta)/2\beta < l_1 < rs/(r + s - 1)$. Denote $1/l_2 = 1/l_1 - \alpha_0/n$. By Lemma 14 we can get that I_{α_0} is bounded from L^{l_1} to L^{l_2} . Then

$$\begin{aligned} I_1 &\leq \sum_{j=1}^m \left(\frac{1}{|B|} \int_B |I_{\alpha_0} M_{(b-b_{2B})\chi_{2B}} T_{j,4} f(y)|^{l_2} dy \right)^{1/l_2} \\ &\leq \sum_{j=1}^m \|I_{\alpha_0}\|_{(l_1, l_2)} |B|^{-1/l_2} \left(\int_{2B} |b(y) - b_{2B}|^{l_1} \right. \\ &\quad \cdot \omega(y)^{l_1/s - l_1/s} |T_{j,4} f(y)|^{l_1} dy \Big)^{1/l_1}. \end{aligned} \quad (26)$$

Since $(n(1 - \alpha) + 2\beta)/2\beta < l_1 < rs/(r + s - 1) < s$, there is an l ($1 < l < \infty$) such that $1/l_1 = 1/s + 1/l$. Let $p_0 = (r-1)(s-l_1)/((r-1)(s-l_1) - s(l_1-1))$. Since $l_1 < rs/(r + s - 1)$, we have $1 < p_0 < \infty$, $p'_0 = (r-1)(s-l_1)/s(l_1-1)$, $1 < p'_0 < \infty$, and $r = lp'_0/s' - p'_0/p_0$. By Hölder's inequality, we get

$$\begin{aligned} I_1 &\leq \sum_{j=1}^m \|I_{\alpha_0}\|_{(l_1, l_2)} |B|^{-1/l_2} \\ &\quad \cdot \left(\int_{2B} |b(y) - b_{2B}|^l \omega(y)^{-l/s} dy \right)^{1/l} \\ &\quad \cdot \left(\int_{2B} |T_{j,4} f(y)|^s \omega(y) dy \right)^{1/s} \leq \sum_{j=1}^m \|I_{\alpha_0}\|_{(l_1, l_2)} \\ &\quad \cdot |B|^{-1/l_2} \left(\int_{2B} |b(y) - b_{2B}|^l \omega(y)^{-l/s} dy \right)^{1/l} \\ &\quad \cdot \omega(2B)^{1/s - \alpha_0/n} \left(\frac{1}{\omega(2B)^{1 - \alpha_0 s/n}} \right. \\ &\quad \cdot \left. \int_{2B} |T_{j,4} f(y)|^s \omega(y) dy \right)^{1/s} \leq \sum_{j=1}^m \|I_{\alpha_0}\|_{(l_1, l_2)} \\ &\quad \cdot |B|^{-1/l_2} \omega(2B)^{1/s - \alpha_0/n} M_{\alpha_0, s, \omega}(T_{j,4} f)(x) \\ &\quad \cdot \left(\int_{2B} |b(y) - b_{2B}|^l \omega(y)^{1/p_0 - l/l' - 1/p_0} dy \right)^{1/l} \\ &\leq \sum_{j=1}^m \|I_{\alpha_0}\|_{(l_1, l_2)} |B|^{-1/l_2} \omega(2B)^{1/s - \alpha_0/n} M_{\alpha_0, s, \omega}(T_{j,4} f) \\ &\quad \cdot (x) \left(\int_{2B} |b(y) - b_{2B}|^{lp_0} \omega(y)^{1 - lp_0} dy \right)^{1/lp_0} \\ &\quad \cdot \left(\int_{2B} \omega(y)^{lp'_0/s' - p'_0/p_0} dy \right)^{1/lp'_0} \leq \sum_{j=1}^m \|I_{\alpha_0}\|_{(l_1, l_2)} \\ &\quad \cdot |B|^{-1/l_2} \omega(2B)^{1/s - \alpha_0/n} M_{\alpha_0, s, \omega}(T_{j,4} f)(x) \|b\|_{BMO(\omega)} \\ &\quad \cdot \omega(2B)^{1/lp_0} |2B|^{1/lp'_0} \left(\frac{1}{|2B|} \int_{2B} \omega(y)^r dy \right)^{1/lp'_0} \end{aligned}$$

$$\begin{aligned} &\leq C \sum_{j=1}^m \|I_{\alpha_0}\|_{(l_1, l_2)} M_{\alpha_0, s, \omega}(T_{j,4} f)(x) \|b\|_{BMO(\omega)} \\ &\quad \cdot \left(\frac{\omega(2B)}{|2B|} \right)^{1 - \alpha_0/n} \leq C \sum_{j=1}^m \|I_{\alpha_0}\|_{(l_1, l_2)} \|b\|_{BMO(\omega)} \\ &\quad \cdot \omega(x)^{1 - \alpha_0/n} M_{\alpha_0, s, \omega}(T_{j,4} f)(x). \end{aligned} \quad (27)$$

Note that $\omega \in A_1$; a direct calculation shows that

$$|b_{2^{k+1}B} - b_{2B}| \leq C \|b\|_{BMO(\omega)} k \omega(x). \quad (28)$$

Applying Hölder's inequality to I_2 , we can get

$$\begin{aligned} I_2 &\leq \sum_{j=1}^m \frac{1}{|B|} \int_B \int_{(2B)^c} |b(z) - b_{2B}| |T_{j,4} f(z)| \left| \frac{1}{|y - z|^{n - \alpha_0}} \right. \\ &\quad \cdot \left. \frac{1}{|x - z|^{n - \alpha_0}} \right| dz dy \\ &\leq C \sum_{j=1}^m \sum_{k=1}^{\infty} \int_{2^{k+1}B \setminus 2^k B} |b(z) - b_{2B}| |T_{j,4} f(z)| \\ &\quad \cdot \frac{r_B}{|x - z|^{n - \alpha_0 + 1}} dz \leq C \sum_{j=1}^m \sum_{k=1}^{\infty} \frac{r_B}{(2^k r_B)^{n - \alpha_0 + 1}} \\ &\quad \cdot \left(\int_{2^{k+1}B} |b(z) - b_{2^{k+1}B}| |T_{j,4} f(z)| dz + |b_{2^{k+1}B} - b_{2B}| \int_{2^{k+1}B} |T_{j,4} f(z)| dz \right) \\ &\leq C \sum_{j=1}^m \sum_{k=1}^{\infty} \frac{r_B}{(2^k r_B)^{n - \alpha_0 + 1}} \frac{r_B}{(2^k r_B)^{n - \alpha_0 + 1}} \\ &\quad \cdot \left(\int_{2^{k+1}B} |b(z) - b_{2^{k+1}B}|^{s'} \omega(z)^{1 - s'} dz \right)^{1/s'} \\ &\quad \cdot \left(\int_{2^{k+1}B} |T_{j,4} f(z)|^s \omega(z) dz \right)^{1/s} \\ &\quad + C \sum_{j=1}^m \sum_{k=1}^{\infty} \frac{r_B}{(2^k r_B)^{n - \alpha_0 + 1}} k \|b\|_{BMO(\omega)} \omega(x) \\ &\quad \cdot \int_{2^{k+1}B} |T_{j,4} f(z)| dz \\ &\leq C \sum_{j=1}^m \sum_{k=1}^{\infty} \frac{r_B}{(2^k r_B)^{n - \alpha_0 + 1}} \left(\frac{1}{\omega(2^{k+1}B)^{1 - \alpha_0 s/n}} \right. \\ &\quad \cdot \left. \int_{2^{k+1}B} |T_{j,4} f(z)|^s \omega(z) dz \right)^{1/s} \\ &\quad \cdot \omega(2^{k+1}B)^{1/s - \alpha_0/n + 1/s'} \left(\frac{1}{\omega(2^{k+1}B)} \int_{2^{k+1}B} |b(z) - b_{2^{k+1}B}|^{s'} \omega(z)^{1 - s'} dz \right)^{1/s'} \end{aligned}$$

$$\begin{aligned}
& + C \sum_{j=1}^m \sum_{k=1}^{\infty} \frac{r_B}{(2^k r_B)^{n-\alpha_0+1}} k \|b\|_{BMO(\omega)} \omega(x) \\
& \cdot |2^{k+1} B|^{1-\alpha_0/n} \left(\frac{1}{|2^{k+1} B|^{1-\alpha_0/n}} \int_{2^{k+1} B} |T_{j,4} f(z)| dz \right) \\
& \leq C \sum_{j=1}^m \sum_{k=1}^{\infty} \frac{r_B}{(2^k r_B)^{n-\alpha_0+1}} \omega(2^{k+1} B)^{1-\alpha_0/n} \|b\|_{BMO(\omega)} \\
& \cdot M_{\alpha_0, s, \omega}(T_{j,4} f)(x) \\
& + C \sum_{j=1}^m \sum_{k=1}^{\infty} \frac{r_B}{(2^k r_B)^{n-\alpha_0+1}} k \|b\|_{BMO(\omega)} \omega(x) \\
& \cdot (2^{k+1} r_B)^{n-\alpha_0} M_{\alpha_0}(T_{j,4} f)(x) \\
& \leq C \sum_{j=1}^m \sum_{k=1}^{\infty} 2^{-k} \|b\|_{BMO(\omega)} \omega(x)^{1-\alpha_0/n} M_{\alpha_0, s, \omega}(T_{j,4} f) \\
& \cdot (x) + C \sum_{j=1}^m \sum_{k=1}^{\infty} 2^{-k} k \|b\|_{BMO(\omega)} \omega(x) M_{\alpha_0}(T_{j,4} f) \\
& \cdot (x) \leq C \sum_{j=1}^m \|b\|_{BMO(\omega)} \omega(x)^{1-\alpha_0/n} M_{\alpha_0, s, \omega}(T_{j,4} f) \\
& \cdot (x) + C \sum_{j=1}^m \|b\|_{BMO(\omega)} \omega(x) M_{\alpha_0}(T_{j,4} f)(x).
\end{aligned} \tag{29}$$

Thus

$$\begin{aligned}
& \frac{1}{|B|} \int_B |B_{\alpha_0}^{b-b_{2B}}(f)(y) - f_{12}(x)| dy \\
& \leq C \sum_{j=1}^m \|I_{\alpha_0}\| \|b\|_{BMO(\omega)} \omega(x)^{1-\alpha_0/n} M_{\alpha_0, s, \omega}(T_{j,4} f)(x) \\
& + C \sum_{j=1}^m \|b\|_{BMO(\omega)} \omega(x)^{1-\alpha_0/n} M_{\alpha_0, s, \omega}(T_{j,4} f)(x) \\
& + C \sum_{j=1}^m \|b\|_{BMO(\omega)} \omega(x) M_{\alpha_0}(T_{j,4} f)(x).
\end{aligned} \tag{30}$$

Secondly, we consider $A_{\alpha_0}^{b-b_{2B}}(f)$ in the following two cases. $\square$

Case 1 ($0 < r_B \leq 1$). Since $(n(1-\alpha) + 2\beta)/2\beta < s < n/\alpha_0$ and $r > [n(1-\alpha) + 2\beta](s-1)/(2\beta s - [n(1-\alpha) + 2\beta])$, then $(n(1-\alpha) + 2\beta)/2\beta < rs/(r+s-1)$ and there is u satisfying $(n(1-\alpha) + 2\beta)/2\beta < u < rs/(r+s-1)$. By Remark 17, we can find v such that the strongly singular Calderón-Zygmund operator T is bounded from L^u to L^v and $0 < u/v < \alpha$. So

there is θ such that $0 < u/v < \theta < \alpha$. Let $\tilde{B}$ be a ball centered at x whose radius is r_B^θ . Consider

$$\begin{aligned}
A_{\alpha_0}^{b-b_{2B}}(f) & = \sum_{j=1}^m T_{j,1} M_{(b-b_{2B})\chi_{2\tilde{B}}} I_{\alpha_0} T_{j,2}(f) \\
& + \sum_{j=1}^m T_{j,1} M_{(b-b_{2B})\chi_{(2\tilde{B})^c}} I_{\alpha_0} T_{j,2}(f) \\
& =: f_{21} + f_{22}.
\end{aligned} \tag{31}$$

Then

$$\begin{aligned}
& \frac{1}{|B|} \int_B |A_{\alpha_0}^{b-b_{2B}}(f)(y) - f_{22}(x)| dy \\
& \leq \frac{1}{|B|} \int_B |f_{21}(y)| dy \\
& + \frac{1}{|B|} \int_B |f_{22}(y) - f_{22}(x)| dy =: II_1 + II_2.
\end{aligned} \tag{32}$$

Write

$$\begin{aligned}
II_1 & \leq \sum_{j=1}^m \frac{1}{|B|} \int_B |T_{j,1} M_{(b-b_{2B})\chi_{2\tilde{B}}} I_{\alpha_0} T_{j,2}(f)(y)| dy \\
& =: \sum_{j=1}^m II_{1,j}.
\end{aligned} \tag{33}$$

If $T_{j,1}$ is a strongly singular Calderón-Zygmund operator, then

$$\begin{aligned}
II_{1,j} & \leq \left(\frac{1}{|B|} \int_B |T_{j,1} M_{(b-b_{2B})\chi_{2\tilde{B}}} I_{\alpha_0} T_{j,2}(f)(y)|^v dy \right)^{1/v} \\
& \leq \|T_{j,1}\|_{(u,v)} |B|^{-1/v} \\
& \cdot \left(\int_{2\tilde{B}} |b(y) - b_{2B}|^u |I_{\alpha_0} T_{j,2} f(y)|^u dy \right)^{1/u}.
\end{aligned} \tag{34}$$

Since $(n(1-\alpha) + 2\beta)/2\beta < u < rs/(r+s-1) < s$, there is s_1 ($1 < s_1 < \infty$) such that $1/u = 1/s + 1/s_1$. Let $p_1 = (r-1)(s-u)/((r-1)(s-u) - s(u-1))$; then $r = s_1 p_1' / s' - p_1' / p_1$. Applying Lemma 12, we have

$$\begin{aligned}
II_{1,j} & \leq \|T_{j,1}\|_{(u,v)} |B|^{-1/v} \\
& \cdot \left(\int_{2\tilde{B}} |b(y) - b_{2B}|^{s_1} \omega(y)^{-s_1/s} dy \right)^{1/s_1} \\
& \cdot \left(\int_{2\tilde{B}} |I_{\alpha_0} T_{j,2} f(y)|^s \omega(y) dy \right)^{1/s} \leq \|T_{j,1}\|_{(u,v)} \\
& \cdot |B|^{-1/v} \omega(2\tilde{B})^{1/s} M_{s,\omega}(I_{\alpha_0} T_{j,2} f)(x) \\
& \cdot \left(\int_{2\tilde{B}} |b(y) - b_{2B}|^{s_1} \omega(y)^{1/p_1 - s_1/s' - 1/p_1} dy \right)^{1/s_1} \\
& \leq \|T_{j,1}\|_{(u,v)} |B|^{-1/v} \omega(2\tilde{B})^{1/s} M_{s,\omega}(I_{\alpha_0} T_{j,2} f)(x)
\end{aligned}$$

$$\begin{aligned}
& \cdot \left(\int_{2\bar{B}} \omega(y)^r dy \right)^{1/s_1 p_1'} \\
& \cdot \left(\int_{2\bar{B}} |b(y) - b_{2B}|^{s_1 p_1} \omega(y)^{1-s_1 p_1} dy \right)^{1/(s_1 p_1)} \\
& \leq \|T_{j,1}\|_{(u,v)} |B|^{-1/\nu} \omega(2\bar{B})^{1/s} M_{s,\omega}(I_{\alpha_0} T_{j,2} f)(x) \\
& \cdot |2\bar{B}|^{1/(s_1 p_1)} \|b\|_{BMO(\omega)} \omega(x) \left(\frac{\omega(2\bar{B})}{|2\bar{B}|} \right)^{-1/(s_1 p_1)'} \\
& \cdot \left(1 + \left| \ln \frac{2r_B}{2r_B^\theta} \right| \right) |2\bar{B}|^{1/(s_1 p_1')} \\
& \cdot \left[\left(\frac{1}{|2\bar{B}|} \int_{2\bar{B}} \omega(y)^r dy \right)^{1/r} \right]^{r/(s_1 p_1')} \leq C \|T_{j,1}\|_{(u,v)} \\
& \cdot r_B^{n(\theta/u-1/\nu)} M_{s,\omega}(I_{\alpha_0} T_{j,2} f)(x) \|b\|_{BMO(\omega)} \omega(x) \\
& \cdot \left(1 + (1-\theta) \ln \frac{1}{r_B} \right). \tag{35}
\end{aligned}$$

The fact $u/\nu < \theta$ implies that $\varepsilon_1 := n(\theta/u - 1/\nu) > 0$. According to Lemma 13, we get

$$\begin{aligned}
II_{1,j} & \leq C \|T_{j,1}\|_{(u,v)} M_{s,\omega}(I_{\alpha_0} T_{j,2} f)(x) \|b\|_{BMO(\omega)} \omega(x) \\
& \cdot r_B^{n(\theta/\mu-1/\nu)} \left(1 + \frac{1}{\varepsilon_1} r_B^{-\varepsilon_1} \right) \leq C \|T_{j,1}\|_{(u,v)} \\
& \cdot M_{s,\omega}(I_{\alpha_0} T_{j,2} f)(x) \|b\|_{BMO(\omega)} \omega(x) r_B^{n(\theta/\mu-1/\nu)-\varepsilon_1} \\
& = C \|T_{j,1}\|_{(u,v)} \|b\|_{BMO(\omega)} \omega(x) M_{s,\omega}(I_{\alpha_0} T_{j,2} f)(x). \tag{36}
\end{aligned}$$

If $T_{j,1} = \pm I$, then

$$\begin{aligned}
II_{1,j} & = \frac{1}{|B|} \int_B |b(y) - b_{2B}| |I_{\alpha_0} T_{j,2} f(y)| dy \\
& \leq \frac{1}{|B|} \left(\int_B |b(y) - b_{2B}|^{s'} \omega(y)^{1-s'} dy \right)^{1/s'} \\
& \cdot \left(\int_B |I_{\alpha_0} T_{j,2} f(y)|^s \omega(y) dy \right)^{1/s} \leq |B|^{1/s'-1} \\
& \cdot \|b\|_{BMO(\omega)} \omega(x) \left(1 + \left| \ln \frac{2r_B}{r_B} \right| \right) \left(\frac{\omega(B)}{|B|} \right)^{-1/s} \\
& \cdot \omega(B)^{1/s} M_{s,\omega}(I_{\alpha_0} T_{j,2} f)(x) \leq C \|b\|_{BMO(\omega)} \omega(x) \\
& \cdot M_{s,\omega}(I_{\alpha_0} T_{j,2} f)(x). \tag{37}
\end{aligned}$$

By (36) and (37), we have

$$II_1 \leq C \sum_{j=1}^m \|T_{j,1}\| \|b\|_{BMO(\omega)} \omega(x) M_{s,\omega}(I_{\alpha_0} T_{j,2} f)(x). \tag{38}$$

Write

$$\begin{aligned}
II_2 & \leq \sum_{j=1}^m \frac{1}{|B|} \int_B |T_{j,1} M_{(b-b_{2B})\chi_{(2\bar{B})^c}} I_{\alpha_0} T_{j,2} f(y) \\
& - T_{j,1} M_{(b-b_{2B})\chi_{(2\bar{B})^c}} I_{\alpha_0} T_{j,2} f(x)| dy =: \sum_{j=1}^m II_{2,j}. \tag{39}
\end{aligned}$$

If $T_{j,1}$ is a strongly singular Calderón-Zygmund operator, then

$$\begin{aligned}
II_{2,j} & \leq \frac{1}{|B|} \int_B \int_{(2\bar{B})^c} |K(y, z) - K(x, z)| |b(z) - b_{2B}| \\
& \cdot |I_{\alpha_0} T_{j,2} f(z)| dz dy \leq C \frac{1}{|B|} \\
& \cdot \int_B \sum_{k=1}^{\infty} \int_{2^{k+1}\bar{B} \setminus 2^k\bar{B}} \frac{|y-x|^\delta}{|x-z|^{n+\delta/\alpha}} |b(z) - b_{2B}| |I_{\alpha_0} T_{j,2} f(z)| dz dy \\
& \leq C \frac{1}{|B|} \int_B \sum_{k=1}^{\infty} \frac{r_B^\delta}{(2^k r_B^\theta)^{n+\delta/\alpha}} \int_{2^{k+1}\bar{B}} |b(z) - b_{2B}| \\
& \cdot |I_{\alpha_0} T_{j,2} f(z)| dz dy \leq C \sum_{k=1}^{\infty} \frac{r_B^\delta}{(2^k r_B^\theta)^{n+\delta/\alpha}} \\
& \cdot \left(\int_{2^{k+1}\bar{B}} |b(z) - b_{2B}|^{s'} \omega(z)^{1-s'} dz \right)^{1/s'} \\
& \cdot \left(\int_{2^{k+1}\bar{B}} |I_{\alpha_0} T_{j,2} f(z)|^s \omega(z) dz \right)^{1/s} \\
& \leq C \sum_{k=1}^{\infty} \frac{r_B^\delta}{(2^k r_B^\theta)^{n+\delta/\alpha}} |2^{k+1}\bar{B}|^{1/s'} \|b\|_{BMO(\omega)} \omega(x) \left(1 \right. \\
& \left. + \left| \ln \frac{2r_B}{2^{k+1}r_B^\theta} \right| \right) \left(\frac{\omega(2^{k+1}\bar{B})}{|2^{k+1}\bar{B}|} \right)^{-1/s} \omega(2^{k+1}\bar{B})^{1/s} \\
& \cdot M_{s,\omega}(I_{\alpha_0} T_{j,2} f)(x) \leq C \sum_{k=1}^{\infty} \frac{r_B^\delta}{(2^k r_B^\theta)^{\delta/\alpha}} \|b\|_{BMO(\omega)} \omega(x) \\
& \cdot k \left(1 + (1-\theta) \ln \frac{1}{r_B} \right) M_{s,\omega}(I_{\alpha_0} T_{j,2} f)(x). \tag{40}
\end{aligned}$$

The fact $\theta < \alpha$ implies that $\varepsilon_2 := \delta - \theta\delta/\alpha = \delta(1-\theta/\alpha) > 0$. According to Lemma 13, we get

$$\begin{aligned}
II_{2,j} & \leq C r_B^{\delta-\theta\delta/\alpha} \|b\|_{BMO(\omega)} \omega(x) M_{s,\omega}(I_{\alpha_0} T_{j,2} f)(x) \\
& \cdot \left(1 + \frac{1}{\varepsilon_2} r_B^{-\varepsilon_2} \right) \leq C r_B^{\delta-\theta\delta/\alpha-\varepsilon_2} \|b\|_{BMO(\omega)} \omega(x) \\
& \cdot M_{s,\omega}(I_{\alpha_0} T_{j,2} f)(x) = C \|b\|_{BMO(\omega)} \omega(x) \\
& \cdot M_{s,\omega}(I_{\alpha_0} T_{j,2} f)(x). \tag{41}
\end{aligned}$$

Note that $II_{2,j} = 0$; if $T_{j,1} = \pm I$, then

$$II_2 \leq C \sum_{j=1}^m \|b\|_{BMO(\omega)} \omega(x) M_{s,\omega}(I_{\alpha_0} T_{j,2} f)(x). \tag{42}$$

Thus,

$$\begin{aligned} & \frac{1}{|B|} \int_B |A_{\alpha_0}^{b-b_{2B}}(f)(y) - f_{22}(x)| dy \\ & \leq C \sum_{j=1}^m \|T_{j,1}\| \|b\|_{BMO(\omega)} \omega(x) M_{s,\omega}(I_{\alpha_0} T_{j,2} f)(x) \\ & \quad + C \sum_{j=1}^m \|b\|_{BMO(\omega)} \omega(x) M_{s,\omega}(I_{\alpha_0} T_{j,2} f)(x). \end{aligned} \quad (43)$$

So

$$\begin{aligned} & \frac{1}{|B|} \int_B |\Theta_{\alpha_0}^b(f)(y) - [f_{12}(x) + f_{22}(x)]| dy \leq \frac{1}{|B|} \\ & \cdot \int_B |A_{\alpha_0}^{b-b_{2B}}(f)(y) - f_{22}(x)| dy + \frac{1}{|B|} \\ & \cdot \int_B |B_{\alpha_0}^{b-b_{2B}}(f)(y) - f_{12}(x)| dy \leq C \sum_{j=1}^m \|T_{j,1}\| \\ & \cdot \|b\|_{BMO(\omega)} \omega(x) M_{s,\omega}(I_{\alpha_0} T_{j,2} f)(x) \\ & + C \sum_{j=1}^m \|b\|_{BMO(\omega)} \omega(x) M_{s,\omega}(I_{\alpha_0} T_{j,2} f)(x) \\ & + C \sum_{j=1}^m \|I_{\alpha_0}\| \|b\|_{BMO(\omega)} \omega(x)^{1-\alpha_0/n} M_{\alpha_0,s,\omega}(T_{j,4} f) \\ & \cdot (x) + C \sum_{j=1}^m \|b\|_{BMO(\omega)} \omega(x)^{1-\alpha_0/n} M_{\alpha_0,s,\omega}(T_{j,4} f) \\ & \cdot (x) + C \sum_{j=1}^m \|b\|_{BMO(\omega)} \omega(x) M_{\alpha_0}(T_{j,4} f)(x). \end{aligned} \quad (44)$$

Case 2 ($r_B > 1$). Consider

$$\begin{aligned} A_{\alpha_0}^{b-b_{2B}}(f) &= \sum_{j=1}^m T_{j,1} M_{(b-b_{2B})\chi_{2B}} I_{\alpha_0} T_{j,2}(f) \\ & \quad + \sum_{j=1}^m T_{j,1} M_{(b-b_{2B})\chi_{(2B)^c}} I_{\alpha_0} T_{j,2}(f) \\ & =: f_{31} + f_{32}. \end{aligned} \quad (45)$$

Then

$$\begin{aligned} & \frac{1}{|B|} \int_B |A_{\alpha_0}^{b-b_{2B}}(f)(y) - f_{32}(x)| dy \\ & \leq \frac{1}{|B|} \int_B |f_{31}(y)| dy \\ & \quad + \frac{1}{|B|} \int_B |f_{32}(y) - f_{32}(x)| dy =: III_1 + III_2. \end{aligned} \quad (46)$$

Since $(n(1-\alpha) + 2\beta)/2\beta < rs/(r+s-1)$, there is u_1 satisfying $n(1-\alpha) + 2\beta/2\beta < u_1 < rs/(r+s-1) < s$. Then the

strongly singular Calderón-Zygmund operator T is bounded from L^{u_1} to L^{u_1} . And there is s_2 ($1 < s_2 < \infty$) such that $1/u_1 = 1/s + 1/s_2$. Let $p_2 = (r-1)(s-u_1)/((r-1)(s-u_1) - s(u_1-1))$. We have $1 < p_2 < \infty$, $p_2' = (r-1)(s-u_1)/s(u_1-1)$, $1 < p_2' < \infty$, and $r = s_2 p_2'/s' - p_2'/p_2$. Then

$$\begin{aligned} III_1 &\leq \sum_{j=1}^m \left(\frac{1}{|B|} \right. \\ & \cdot \left. \int_B |T_{j,1} M_{(b-b_{2B})\chi_{2B}} I_{\alpha_0} T_{j,2} f(y)|^{u_1} dy \right)^{1/u_1} \\ &\leq \sum_{j=1}^m \|T_{j,1}\|_{(u_1, u_1)} |B|^{-1/u_1} \\ & \cdot \left(\int_{2B} |b(y) - b_{2B}|^{u_1} |I_{\alpha_0} T_{j,2} f(y)|^{u_1} dy \right)^{1/u_1} \\ &\leq \sum_{j=1}^m \|T_{j,1}\|_{(u_1, u_1)} |B|^{-1/u_1} \\ & \cdot \left(\int_{2B} |b(y) - b_{2B}|^{s_2} \omega(y)^{-s_2/s} dy \right)^{1/s_2} \\ & \cdot \left(\int_{2B} |I_{\alpha_0} T_{j,2} f(y)|^s \omega(y) dy \right)^{1/s} \leq \sum_{j=1}^m \|T_{j,1}\|_{(u_1, u_1)} \\ & \cdot |B|^{-1/u_1} \omega(2B)^{1/s} M_{s,\omega}(I_{\alpha_0} T_{j,2} f)(x) \\ & \cdot \left(\int_{2B} |b(y) - b_{2B}|^{s_2 p_2} \omega(y)^{1-s_2 p_2} dy \right)^{1/(s_2 p_2)} \\ & \cdot \left(\int_{2B} \omega(y)^r dy \right)^{1/(s_2 p_2')} \leq C \sum_{j=1}^m \|T_{j,1}\|_{(u_1, u_1)} |B|^{-1/u_1} \\ & \cdot \omega(2B)^{1/s} M_{s,\omega}(I_{\alpha_0} T_{j,2} f)(x) \omega(2B)^{1/(s_2 p_2)} \\ & \cdot \|b\|_{BMO(\omega)} |2B|^{1/(s_2 p_2')} \left(\frac{\omega(2B)}{|2B|} \right)^{r/(s_2 p_2')} \\ & \leq C \sum_{j=1}^m \|T_{j,1}\|_{(u_1, u_1)} \|b\|_{BMO(\omega)} \omega(x) M_{s,\omega}(I_{\alpha_0} T_{j,2} f) \\ & \cdot (x). \end{aligned} \quad (47)$$

Write

$$\begin{aligned} III_2 &\leq \sum_{j=1}^m \frac{1}{|B|} \int_B |T_{j,1} M_{(b-b_{2B})\chi_{(2B)^c}} I_{\alpha_0} T_{j,2} f(y) \\ & \quad - T_{j,1} M_{(b-b_{2B})\chi_{(2B)^c}} I_{\alpha_0} T_{j,2} f(x)| dy =: \sum_{j=1}^m III_{2,j}. \end{aligned} \quad (48)$$

If $T_{j,1}$ is a strongly singular Calderón-Zygmund operator, then

$$\begin{aligned} III_{2,j} &\leq \frac{1}{|B|} \int_B \int_{(2B)^c} |K(y, z) - K(x, z)| |b(z) - b_{2B}| \\ & \cdot |I_{\alpha_0} T_{j,2} f(z)| dz dy \leq C \frac{1}{|B|} \end{aligned}$$

$$\begin{aligned}
& \cdot \int_B \sum_{k=1}^{\infty} \int_{2^{k+1}B \setminus 2^k B} \frac{|y-z|^\delta}{|x-z|^{n+\delta/\alpha}} |b(z) - b_{2B}| |I_{\alpha_0} T_{j,2} f(z)| dz dy \\
& \leq C \sum_{k=1}^{\infty} \frac{r_B^\delta}{(2^k r_B)^{n+\delta/\alpha}} \left(\int_{2^{k+1}B} |b(z) - b_{2B}|^{s'} \omega(z)^{1-s'} dz \right)^{1/s'} \\
& \cdot \left(\int_{2^{k+1}B} |I_{\alpha_0} T_{j,2} f(z)|^s \omega(z) dz \right)^{1/s} \leq C \sum_{k=1}^{\infty} \frac{r_B^\delta}{(2^k r_B)^{n+\delta/\alpha}} \\
& \cdot |2^{k+1}B|^{1/s'} \|b\|_{BMO(\omega)} \omega(x) \left(1 + \left| \ln \frac{2r_B}{2^{k+1}r_B} \right| \right) \\
& \cdot \left(\frac{\omega(2^{k+1}B)}{|2^{k+1}B|} \right)^{-1/s} \omega(2^{k+1}B)^{1/s} M_{s,\omega}(I_{\alpha_0} T_{j,2} f)(x) \\
& \leq C \sum_{k=1}^{\infty} k 2^{-k(\delta/\alpha)} r_B^{\delta-\delta/\alpha} \|b\|_{BMO(\omega)} \omega(x) M_{s,\omega}(I_{\alpha_0} T_{j,2} f)(x) \\
& \leq C \|b\|_{BMO(\omega)} \omega(x) M_{s,\omega}(I_{\alpha_0} T_{j,2} f)(x). \tag{49}
\end{aligned}$$

Since $III_{2,j} = 0$ when $T_{j,1} = \pm I$, then

$$III_2 \leq C \sum_{j=1}^m \|b\|_{BMO(\omega)} \omega(x) M_{s,\omega}(I_{\alpha_0} T_{j,2} f)(x). \tag{50}$$

Thus,

$$\begin{aligned}
& \frac{1}{|B|} \int_B |A_{\alpha_0}^{b-b_{2B}}(f)(y) - f_{32}(x)| dy \\
& \leq C \sum_{j=1}^m \|T_{j,1}\| \|b\|_{BMO(\omega)} \omega(x) M_{s,\omega}(I_{\alpha_0} T_{j,2} f)(x) \\
& + C \sum_{j=1}^m \|b\|_{BMO(\omega)} \omega(x) M_{s,\omega}(I_{\alpha_0} T_{j,2} f)(x). \tag{51}
\end{aligned}$$

So

$$\begin{aligned}
& \frac{1}{|B|} \int_B |\Theta_{\alpha_0}^b(f)(y) - [f_{12}(x) + f_{32}(x)]| dy \leq \frac{1}{|B|} \\
& \cdot \int_B |A_{\alpha_0}^{b-b_{2B}}(f)(y) - f_{32}(x)| dy + \frac{1}{|B|} \\
& \cdot \int_B |B_{\alpha_0}^{b-b_{2B}}(f)(y) - f_{12}(x)| dy \leq C \sum_{j=1}^m \|T_{j,1}\| \\
& \cdot \|b\|_{BMO(\omega)} \omega(x) M_{s,\omega}(I_{\alpha_0} T_{j,2} f)(x) \\
& + C \sum_{j=1}^m \|b\|_{BMO(\omega)} \omega(x) M_{s,\omega}(I_{\alpha_0} T_{j,2} f)(x) \\
& + C \sum_{j=1}^m \|I_{\alpha_0}\| \|b\|_{BMO(\omega)} \omega(x)^{1-\alpha_0/n} M_{\alpha_0,s,\omega}(T_{j,4} f)
\end{aligned}$$

$$\begin{aligned}
& \cdot (x) + C \sum_{j=1}^m \|b\|_{BMO(\omega)} \omega(x)^{1-\alpha_0/n} M_{\alpha_0,s,\omega}(T_{j,4} f) \\
& \cdot (x) + C \sum_{j=1}^m \|b\|_{BMO(\omega)} \omega(x) M_{\alpha_0}(T_{j,4} f)(x). \tag{52}
\end{aligned}$$

In conclusion, combining the two cases, we can get the desired estimate for $M^\sharp(\Theta_{\alpha_0}^b(f))(x)$.

4. Proof of Main Result

Now we are able to prove our main result.

Proof of Theorem 7. Since $n^2(1-\alpha)/\alpha_0[n(1-\alpha)+2\beta]+1 < p$, we have $q/p = n/(n-p\alpha_0) > [n(1-\alpha)+2\beta](p-1)/(2\beta p - [n(1-\alpha)+2\beta])$. So $r > q/p > [n(1-\alpha)+2\beta](p-1)/(2\beta p - [n(1-\alpha)+2\beta])$; that is, $p > [n(1-\alpha)+2\beta](r-1)/(2\beta r - [n(1-\alpha)+2\beta])$. There exists s such that $p > s > [n(1-\alpha)+2\beta](r-1)/(2\beta r - [n(1-\alpha)+2\beta]) > (n(1-\alpha)+2\beta)/2\beta$. Since $s > [n(1-\alpha)+2\beta](r-1)/(2\beta r - [n(1-\alpha)+2\beta])$, then $r > [n(1-\alpha)+2\beta](s-1)/(2\beta s - [n(1-\alpha)+2\beta])$. Applying Lemmas 24 and 25, we have

$$\begin{aligned}
& \|\Theta_{\alpha_0}^b(f)\|_{L^q(\omega^{q/p-q})} \leq \|M(\Theta_{\alpha_0}^b(f))\|_{L^q(\omega^{q/p-q})} \\
& \leq C \|M^\sharp(\Theta_{\alpha_0}^b(f))\|_{L^q(\omega^{q/p-q})} \leq C \sum_{j=1}^m \|T_{j,1}\| \\
& \cdot \|b\|_{BMO(\omega)} \|\omega M_{s,\omega}(I_{\alpha_0} T_{j,2} f)\|_{L^q(\omega^{q/p-q})} \\
& + C \sum_{j=1}^m \|b\|_{BMO(\omega)} \|\omega M_{s,\omega}(I_{\alpha_0} T_{j,2} f)\|_{L^q(\omega^{q/p-q})} \\
& + C \sum_{j=1}^m \|I_{\alpha_0}\| \|b\|_{BMO(\omega)} \\
& \cdot \|\omega^{1-\alpha_0/n} M_{\alpha_0,s,\omega}(T_{j,4} f)\|_{L^q(\omega^{q/p-q})} \\
& + C \sum_{j=1}^m \|b\|_{BMO(\omega)} \|\omega^{1-\alpha_0/n} M_{\alpha_0,s,\omega}(T_{j,4} f)\|_{L^q(\omega^{q/p-q})} \\
& + C \sum_{j=1}^m \|b\|_{BMO(\omega)} \|\omega M_{\alpha_0}(T_{j,4} f)\|_{L^q(\omega^{q/p-q})}. \tag{53}
\end{aligned}$$

Since $\omega \in RH_r$ and $r > q/p$, then $\omega \in RH_{q/p}$. It follows from $p > s$ and $\omega^q \in A_1$ that $1 < s(q/p) < q$ and $\omega^{q/p} \in A_1$. Then $\omega^{q/p} \in A_\infty$ and $\omega^{1/p} \in A_{p,q}$ by Lemma 11. Applying Lemmas 19, 20, and 22, we have

$$\begin{aligned}
I_1 & =: \|\omega M_{s,\omega}(I_{\alpha_0} T_{j,2} f)\|_{L^q(\omega^{q/p-q})} \\
& = \|M_{s,\omega}(I_{\alpha_0} T_{j,2} f)\|_{L^q(\omega^{q/p})}
\end{aligned}$$

$$\begin{aligned}
&\leq C \|M_{s(q/p), \omega^{q/p}} (I_{\alpha_0} T_{j,2} f)\|_{L^q(\omega^{q/p})} \\
&\leq C \|I_{\alpha_0} (T_{j,2} f)\|_{L^q(\omega^{q/p})} \leq C \|T_{j,2} f\|_{L^p(\omega)} \\
&\leq C \|T_{j,2}\| \|f\|_{L^p(\omega)}.
\end{aligned} \tag{54}$$

Since $\omega^q \in A_1$, $\omega \in A_{p,q}$ by Lemma 11. Applying Lemma 21, we have

$$\begin{aligned}
I_2 &=: \|\omega^{1-\alpha_0/n} M_{\alpha_0, s, \omega} (T_{j,4} f)\|_{L^q(\omega^{q/p-q})} \\
&= \|M_{\alpha_0, s, \omega} (T_{j,4} f)\|_{L^q(\omega)} \leq C \|T_{j,4} f\|_{L^p(\omega)} \\
&\leq C \|T_{j,4}\| \|f\|_{L^p(\omega)}.
\end{aligned} \tag{55}$$

It follows from $\omega^{1/p} \in A_{p,q}$ and Lemma 23 that

$$\begin{aligned}
I_3 &=: \|\omega M_{\alpha_0} (T_{j,4} f)\|_{L^q(\omega^{q/p-q})} \\
&= \left(\int_{\mathbb{R}^n} |M_{\alpha_0} (T_{j,4} f)(x)|^q \omega(x)^{1/p} dx \right)^{1/q} \\
&\leq C \|T_{j,4} f\|_{L^p(\omega)} \leq C \|T_{j,4}\| \|f\|_{L^p(\omega)}.
\end{aligned} \tag{56}$$

Therefore,

$$\begin{aligned}
\|\Theta_{\alpha_0}^b(f)\|_{L^q(\omega^{q/p-q})} &\leq C \sum_{j=1}^m \|T_{j,1}\| \|b\|_{BMO(\omega)} \|T_{j,2}\| \\
&\quad \cdot \|f\|_{L^p(\omega)} + C \sum_{j=1}^m \|b\|_{BMO(\omega)} \|T_{j,2}\| \|f\|_{L^p(\omega)} \\
&\quad + C \sum_{j=1}^m \|I_{\alpha_0}\| \|b\|_{BMO(\omega)} \|T_{j,4}\| \|f\|_{L^p(\omega)} \\
&\quad + C \sum_{j=1}^m \|b\|_{BMO(\omega)} \|T_{j,4}\| \|f\|_{L^p(\omega)} + C \sum_{j=1}^m \|b\|_{BMO(\omega)} \\
&\quad \cdot \|T_{j,4}\| \|f\|_{L^p(\omega)} \leq C \|b\|_{BMO(\omega)} \\
&\quad \cdot \sum_{j=1}^m (\|T_{j,1}\| \|T_{j,2}\| + \|T_{j,2}\| + \|I_{\alpha_0}\| \|T_{j,4}\| + \|T_{j,4}\|) \\
&\quad \cdot \|f\|_{L^p(\omega)}.
\end{aligned} \tag{57}$$

This completes the proof of Theorem 7. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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