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Research Article

Common Fixed Point of Two R-Weakly Commuting Mappings in b-Metric Spaces

Poom Kumam,[1,2] Wutiphol Sintunavarat,[3] Shaban Sedghi,[4] and Nabi Shobkolaei[5]

[1] *Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT), 126 Pracha Uthit Road, Bang Mod, Thrung Khru, Bangkok 10140, Thailand*
[2] *Centre of Excellence in Mathematics, CHE, Si Ayutthaya Road, Bangkok 10400, Thailand*
[3] *Department of Mathematics and Statistics, Faculty of Science and Technology, Thammasat University Rangsit Center, Pathumthani 12121, Thailand*
[4] *Department of Mathematics, Islamic Azad University, Qaemshahr Branch, Qaemshahr, Iran*
[5] *Department of Mathematics, Islamic Azad University, Babol Branch, Babol, Iran*

Correspondence should be addressed to Wutiphol Sintunavarat; wutiphol@mathstat.sci.tu.ac.th

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We prove some common fixed point results for two mappings satisfying generalized contractive condition in b-metric space. Note that b-metric of main results in this work are not necessarily continuous. So our results extend and improve several previous works. We also present one example that shows the applicability and usefulness of our results.

1. Introduction

In 1998, Czerwik [1] introduced the concept of b-metric space. Since then, several papers deal with fixed point theory for single-valued and multivalued operators in b-metric spaces (see also [1–19]). For example, Pacurar [16] proved results on sequences of almost contractions and fixed points in b-metric spaces. Also, Hussain and Shah [11] obtained results on KKM mappings in cone b-metric spaces. Furthermore, Khamsi ([12, 13]) also showed that each cone metric space has a b-metric structure.

The aim of this paper is to present some common fixed point results for two mappings under generalized contractive condition in b-metric space, where the b-metric is not necessarily continuous. Because many of the authors in their works have used the b-metric spaces in which the b-metric is continuous, the techniques used in this paper can be used for many of the results on the context of b-metric space. From this point of view the results obtained in this paper generalize and extend several ones obtained earlier in a lot of papers concerning b-metric space.

2. Preliminaries

Throughout this paper, we denote by $\mathbb{N}$, $\mathbb{R}_+$, and $\mathbb{R}$ the sets of positive integers, nonnegative real numbers, and real numbers, respectively.

Consistent with [1] and [18, page 264], the following definition and results will be needed in the sequel.

Definition 1 (see [1]). Let X be a nonempty set and let $b \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{R}_+$ is a b-metric if and only if, for all $x, y, z \in X$, the following conditions are satisfied:

(b1) $d(x, y) = 0$ if and only if $x = y$;

(b2) $d(x, y) = d(y, x)$;

(b3) $d(x, z) \leq b[d(x, y) + d(y, z)]$.

The pair (X, d) is called a b-metric space with coefficient b.

It should be noted that the class of b-metric spaces is effectively larger than that of metric spaces, since a b-metric is a metric when $b = 1$.

We present an example which shows that a b-metric on X need not be a metric on X (see also [18, page 264]).

Example 2. Let (X, d) be a metric space and $\rho(x, y) = (d(x, y))^p$, where $p > 1$ is a real number. We show that ρ is a b-metric with $b = 2^{p-1}$.

Obviously conditions (b1) and (b2) in Definition 1 are satisfied. Now we show that condition (b3) holds for ρ.

It is easy to see that if $1 < p < \infty$, then the convexity of the function $f(x) = x^p$, where $x \geq 0$, implies

$$\left(\frac{a+c}{2}\right)^p \leq \frac{1}{2}\left(a^p + c^p\right), \tag{1}$$

and hence

$$(a+c)^p \leq 2^{p-1}\left(a^p + c^p\right). \tag{2}$$

Therefore, for each $x, y, z \in X$, we obtain that

$$\begin{aligned} \rho(x, y) &= (d(x, y))^p \\ &\leq (d(x, z) + d(z, y))^p \\ &\leq 2^{p-1}\left((d(x, z))^p + (d(z, y))^p\right) \\ &= 2^{p-1}\left(\rho(x, z) + \rho(z, y)\right). \end{aligned} \tag{3}$$

So condition (b3) in Definition 1 holds and then ρ is a b-metric coefficient $s = 2^{p-1}$.

It should be noted that in Example 2, if (X, d) is a metric space, then (X, ρ) is not necessarily a metric space (see Example 3).

Example 3. For example, if $X = \mathbb{R}$ and the usual Euclidean metric $d : X \times X \to \mathbb{R}_+$ is defined by

$$d(x, y) = |x - y| \tag{4}$$

for all $x, y \in X$, then $\rho(x, y) = |x - y|^2$ is a b-metric on $\mathbb{R}$ with $b = 2$ but is not a metric on $\mathbb{R}$, because the triangle inequality does not hold.

Before stating and proving our results, we present some definition and proposition in b-metric space. We recall first the notions of convergence, closedness, and completeness in a b-metric space.

Definition 4 (see [6]). Let (X, d) be a b-metric space. Then a sequence $\{x_n\}$ in X is called

(i) convergent if and only if there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to +\infty$, and in this case, we write $\lim_{n\to\infty} x_n = x$,

(ii) Cauchy if and only if $d(x_n, x_m) \to 0$ as $n, m \to +\infty$.

Proposition 5 (Remark 2.1 in [6]). *In a b-metric space (X, d) the following assertions hold:*

(i) *a convergent sequence has a unique limit;*

(ii) *each convergent sequence is Cauchy;*

(iii) *in general, a b-metric is not continuous.*

Definition 6 (see [6]). The b-metric space (X, d) is complete if every Cauchy sequence in X converges.

It should be noted that in general a b-metric function d for $b > 1$ is not jointly continuous in all the two of its variables.

Since in general a b-metric is not continuous, we need the following simple lemma about the b-convergent sequences.

Lemma 7 (see [20]). *Let (X, d) be a b-metric space with coefficient $b \geq 1$, and suppose that $\{x_n\}$ and $\{y_n\}$ are b-convergent to x, y, respectively; then one has*

$$\begin{aligned} \frac{1}{b^2} d(x, y) &\leq \liminf_{n\to\infty} d(x_n, y_n) \leq \limsup_{n\to\infty} d(x_n, y_n) \\ &\leq b^2 d(x, y). \end{aligned} \tag{5}$$

In particular, if $x = y$, then one has $\lim_{n\to\infty} d(x_n, y_n) = 0$. Moreover, for each $z \in X$ one has

$$\frac{1}{b} d(x, z) \leq \liminf_{n\to\infty} d(x_n, z) \leq \limsup_{n\to\infty} d(x_n, z) \leq b d(x, z). \tag{6}$$

3. Common Fixed Point Results

Definition 8. Let f and g be mappings from a b-metric space (X, d) into itself. The mappings f and g are said to be weakly commuting if

$$d(fgx, gfx) \leq d(fx, gx) \tag{7}$$

for each x in X.

Definition 9. Let f and g be mappings from a b-metric space (X, d) into itself. The mappings f and g are said to be R-weakly commuting if there exists some positive real number R such that

$$d(fgx, gfx) \leq R d(fx, gx) \tag{8}$$

for each x in X.

Remark 10. Weak commutativity implies R-weak commutativity in b-metric space. However, R-weak commutativity implies weak commutativity only when $R \leq 1$.

Example 11. Let $X = \mathbb{R}$ and $d : X \times X \to \mathbb{R}_+$ defined as follows:

$$d(x, y) = (x - y)^2 \tag{9}$$

for all $x, y \in X$. Then (X, d) is a b-metric space. Define $fx = 2x - 1$ and $gx = x^2$. Then

$$d(fgx, gfx) = 4(x-1)^4 = 4d(fx, gx) > d(fx, gx). \tag{10}$$

Therefore, for $R = 4$, f and g are R-weakly commuting. But f and g are not weakly commuting.

Theorem 12. *Let (X, d) be a complete b-metric space and let f and g be R-weakly commuting self-mappings on X satisfying the following conditions:*

(a) $f(X) \subseteq g(X)$;

(b) *f or g is continuous;*

(c) $d(fx, fy) \leq \gamma((1/b^4)d(gx, gy))$ *for all* $x, y \in X$, *where* $\gamma : [0, \infty) \to [0, \infty)$ *is a continuous and nondecreasing function such that* $\gamma(a) < a$ *for each* $a > 0$ *and* $\gamma(0) = 0$.

Then f and g have a unique common fixed point.

Proof. Let x_0 be an arbitrary point in X. By (a), choose a point x_1 in X such that $fx_0 = gx_1$. In general choose x_{n+1} such that $fx_n = gx_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Now we observe that for each $n \in \mathbb{N}$, we have

$$\begin{aligned} d(fx_n, fx_{n+1}) &\leq \gamma\left(\frac{1}{b^4} d(gx_n, gx_{n+1})\right) \\ &= \gamma\left(\frac{1}{b^4} d(fx_{n-1}, fx_n)\right) \\ &\leq \frac{1}{b^4} d(fx_{n-1}, fx_n) \\ &\leq d(fx_{n-1}, fx_n). \end{aligned} \tag{11}$$

This implies that

$$\{c_n\} := \{d(fx_n, fx_{n+1})\} \tag{12}$$

is nonincreasing sequence in $[0, \infty)$. Therefore, it tends to a limit $a \geq 0$. Next, we claim that $a = 0$. Suppose that $a > 0$. Making $n \to \infty$ in the inequality (11), we get

$$a \leq \gamma\left(\frac{1}{b^4} a\right) < \frac{1}{b^4} a \leq a, \tag{13}$$

which is a contradiction. Hence $a = 0$; that is,

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} d(fx_n, fx_{n+1}) = 0. \tag{14}$$

Now, we prove that $\{fx_n\}$ is a Cauchy sequence in $f(X)$. Suppose that $\{fx_n\}$ is not a Cauchy sequence in $f(X)$. For convenience, let $y_n = fx_n$ for $n = 1, 2, 3, \ldots$. Then there is an $\varepsilon > 0$ such that, for each integer k, there exist integers $m(k)$ and $n(k)$ with $m(k) > n(k) \geq k$ such that

$$d_k := d(y_{n(k)}, y_{m(k)}) \geq \varepsilon \quad \text{for } k = 1, 2, \ldots. \tag{15}$$

We may assume that

$$d(y_{n(k)}, y_{m(k)-1}) < \varepsilon, \tag{16}$$

by choosing $m(k)$ as the smallest number exceeding $n(k)$ for which (15) holds. Using (11), we have

$$\begin{aligned} \varepsilon \leq d_k &\leq b\left[d(y_{n(k)}, y_{m(k)-1}) + d(y_{m(k)-1}, y_{m(k)})\right] \\ &< b\varepsilon + bc_{m(k)-1} \leq b\varepsilon + bc_k. \end{aligned} \tag{17}$$

Hence, $\varepsilon \leq \lim_{k\to\infty} d_k \leq b\varepsilon$. Also notice

$$\begin{aligned} d_k &= d(y_{n(k)}, y_{m(k)}) \\ &\leq bd(y_{n(k)}, y_{n(k)+1}) \\ &\quad + b^2\left[d(y_{n(k)+1}, y_{m(k)+1}) + d(y_{m(k)+1}, y_{m(k)})\right] \\ &\leq bc_k + b^2\gamma\left(\frac{1}{b^4} d(y_{n(k)}, y_{m(k)})\right) + b^2 c_k \\ &= bc_k + b^2\gamma\left(\frac{1}{b^4} d_k\right) + b^2 c_k \end{aligned} \tag{18}$$

for each $k \in \mathbb{N}$. Thus, as $k \to \infty$ in the above inequality we have

$$\varepsilon \leq \lim_{k\to\infty} d_k \leq b \lim_{k\to\infty} c_k + b^2\gamma\left(\frac{1}{b^4} \lim_{k\to\infty} d_k\right) + b^2 \lim_{k\to\infty} c_k, \tag{19}$$

and thus

$$\varepsilon \leq b^2\gamma\left(\frac{1}{b^3}\varepsilon\right) < \frac{1}{b}\varepsilon, \tag{20}$$

which is a contradiction. Thus, $\{fx_n\}$ is Cauchy sequence in X and by the completeness of X, $\{fx_n\}$ converges to z in X. Also $\{gx_n\}$ converges to z in X. Let us suppose that the mapping f is continuous. Then $\lim_{n\to\infty} ffx_n = fz$ and $\lim_{n\to\infty} fgx_n = fz$. Further we have that since f and g are R-weakly commuting

$$d(fgx_n, gfx_n) \leq Rd(fx_n, gx_n) \tag{21}$$

for all $n \in \mathbb{N}$. Taking the upper limit as $n \to \infty$ in the above inequality, we get

$$\begin{aligned} \frac{1}{b^2} d\left(fz, \limsup_{n\to\infty} gfx_n\right) &\leq \limsup_{n\to\infty} d(fgx_n, gfx_n) \\ &\leq R\limsup_{n\to\infty} d(fx_n, gx_n) \\ &\leq Rb^2 d(z, z) = 0. \end{aligned} \tag{22}$$

Similarly,

$$\frac{1}{b^2} d\left(fz, \liminf_{n\to\infty} gfx_n\right) = 0, \tag{23}$$

and hence we get $\lim_{n\to\infty} gfx_n = fz$. We now prove that $z = fz$. Suppose that $z \neq fz$ and then $d(z, fz) > 0$. By (c), we have

$$\begin{aligned} \frac{1}{b} d(fz, fx_m) &\leq \limsup_{n\to\infty} d(ffx_n, fx_m) \\ &\leq \gamma\left(\limsup_{n\to\infty} \frac{1}{b^4} d(gfx_n, gx_m)\right) \\ &\leq \gamma\left(\frac{1}{b^3} d(fz, gx_m)\right) \end{aligned} \tag{24}$$

for each $m \in \mathbb{N}$. On taking the upper limit as $m \to \infty$ in the above inequality we get

$$\begin{aligned}\frac{1}{b^2}d(fz,z) &\le \limsup_{m\to\infty}\frac{1}{b}d(fz,fx_m)\\ &\le \gamma\left(\limsup_{m\to\infty}\frac{1}{b^3}d(fz,gx_m)\right)\\ &\le \gamma\left(\frac{1}{b^2}d(fz,z)\right) < \frac{1}{b^2}d(fz,z),\end{aligned} \tag{25}$$

which is a contradiction. Therefore, $z = fz$. Since $f(X) \subseteq g(X)$ we can find z_1 in X such that $z = fz = gz_1$. Now, we have

$$d(ffx_n, fz_1) \le \gamma\left(\frac{1}{b^4}d(gfx_n, gz_1)\right) \tag{26}$$

for all $n \in \mathbb{N}$. Taking limit sup as $n \to \infty$ we get

$$\begin{aligned}\frac{1}{b}d(fz,fz_1) &\le \limsup_{n\to\infty} d(ffx_n, fz)\\ &\le \gamma\left(\limsup_{n\to\infty}\frac{1}{b^4}d(gfx_n, gz_1)\right)\\ &\le \gamma\left(\frac{1}{b^3}d(fz,gz_1)\right)\\ &= 0\end{aligned} \tag{27}$$

since $\gamma(0) = 0$, which implies that $fz = fz_1$; that is, $z = fz = fz_1 = gz_1$. Also,

$$d(fz,gz) = d(fgz_1, gfz_1) \le Rd(fz_1, gz_1) = 0 \tag{28}$$

which again implies that $fz = gz$. Thus z is a common fixed point of f and g.

Now to prove uniqueness let if possible $z' \neq z$ be another common fixed point of f and g. Then $d(z,z') > 0$ and so

$$\begin{aligned}d(z,z') = d(fz,fz') &\le \gamma\left(\frac{1}{b^4}d(gz,gz')\right)\\ &= \gamma\left(\frac{1}{b^4}d(z,z')\right) < \frac{1}{b^4}d(z,z') \le d(z,z'),\end{aligned} \tag{29}$$

which is a contradiction. Therefore, $z = z'$, that is, z is a unique common fixed point of f and g. This completes the proof. □

Now we give an example to support Theorem 12.

Example 13. Let $X = \mathbb{R}$ and $d : X \times X \to \mathbb{R}_+$ defined by

$$d(x,y) = (x-y)^2 \tag{30}$$

for all $x, y \in X$. Then (X,d) is a b-metric space for $b = 2$. Define $fx = 1$ and $gx = 2x-1$ on X. It is evident that $f(X) \subseteq g(X)$ and f is continuous. Now we observe that

$$d(fx,fy) \le \gamma\left(\frac{1}{16}d(gx,gy)\right) \tag{31}$$

for all x, y in X and $\gamma : (0,\infty) \to (0,\infty)$ defined by $\gamma(t) = kt$ for $0 < k < 1$. Moreover, it is easy to see that f and g are R-weakly commuting. Thus all the conditions of Theorem 12 are satisfied and 1 is a common fixed point of f and g.

Corollary 14. *Let (X,d) be a complete b-metric space and let f be a self-mapping on X satisfying the following condition:*

$$d(fx,fy) \le \gamma\left(\frac{1}{b^4}d(x,y)\right) \tag{32}$$

for all $x, y \in X$, where $\gamma : [0,\infty) \to [0,\infty)$ is a continuous and nondecreasing function such that $\gamma(a) < a$ for each $a > 0$ and $\gamma(0) = 0$. Then f has a unique fixed point.

Proof. If we take g as identity mapping on X, then Theorem 12 follows that f has a unique fixed point. □

Corollary 15. *Let (X,d) be a complete metric space and let f and g be R-weakly commuting self-mappings of X satisfying the following conditions:*

(a) *$f(X) \subseteq g(X)$;*

(b) *f or g is continuous;*

(c) *$d(fx,fy) \le \gamma(d(gx,gy))$ for all $x, y \in X$, where $\gamma : [0,\infty) \to [0,\infty)$ is a continuous and nondecreasing function such that $\gamma(a) < a$ for each $a > 0$ and $\gamma(0) = 0$.*

Then f and g have a unique common fixed point.

Proof. If we take $b = 1$, then Theorem 12 follows that f and g have a unique common fixed point. □

Conflict of Interests

The authors declare that they have no conflict of interests.

Authors' Contribution

All authors contributed equally and significantly in writing this paper. All authors read and approved the paper.

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