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Research Article

Convergence Theorems for Generalized Functional Sequences of Discrete-Time Normal Martingales

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The Fock transform recently introduced by the authors in a previous paper is applied to investigate convergence of generalized functional sequences of a discrete-time normal martingale M. A necessary and sufficient condition in terms of the Fock transform is obtained for such a sequence to be strongly convergent. A type of generalized martingales associated with M is introduced and their convergence theorems are established. Some applications are also shown.

1. Introduction

Hida's white noise analysis is essentially a theory of infinite dimensional calculus on generalized functionals of Brownian motion [1–4]. In 1988, Ito [5] introduced his analysis of generalized Poisson functionals, which can be viewed as a theory of infinite dimensional calculus on generalized functionals of Poisson martingale. It is known that both Brownian motion and Poisson martingale are continuous-time normal martingales. There are theories of white noise analysis for some other continuous-time processes (see, e.g., [6–10]).

Discrete-time normal martingales [11] also play an important role in many theoretical and applied fields. For example, the classical random walk (a special discrete-time normal martingale) is used to establish functional central limit theorems in probability theory [12, 13]. It would then be interesting to develop a theory of infinite dimensional calculus on generalized functionals of discrete-time normal martingales.

Let $M = (M_n)_{n\in\mathbb{N}}$ be a discrete-time normal martingale satisfying some mild conditions. In a recent paper [14], we constructed generalized functionals of M and introduced a transform, called the Fock transform, to characterize those functionals.

In this paper, we apply the Fock transform [14] to investigate generalized functional sequences of M. First, by using the Fock transform, we obtain a necessary and sufficient condition for a generalized functional sequence of M to be strongly convergent. Then, we introduce a type of generalized martingales associated with M, called M-generalized martingales, which are a special class of generalized functional sequences of M and include as a special case the classical martingales with respect to the filtration generated by M. We establish a strong-convergent criterion in terms of the Fock transform for M-generalized martingales. Some other convergence criteria are also obtained. Finally, we show some applications of our main results.

Our one interesting finding is that, for an M-generalized martingale, its strong convergence is just equivalent to its strong boundedness.

Throughout this paper, $\mathbb{N}$ designates the set of all nonnegative integers and Γ the finite power set of $\mathbb{N}$; namely,

$$\Gamma = \{\sigma \mid \sigma \subset \mathbb{N}, \#(\sigma) < \infty\}, \tag{1}$$

where $\#(\sigma)$ means the cardinality of σ as a set. In addition, we always assume that $(\Omega, \mathcal{F}, P)$ is a given probability space with $\mathbb{E}$ denoting the expectation with respect to P. We denote by $\mathcal{L}^2(\Omega, \mathcal{F}, P)$ the usual Hilbert space of square integrable complex-valued functions on $(\Omega, \mathcal{F}, P)$ and use $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ to mean its inner product and norm, respectively. By convention, $\langle\cdot,\cdot\rangle$ is conjugate-linear in its first argument and linear in its second argument.

2. Generalized Functionals

Let $M = (M_n)_{n\in\mathbb{N}}$ be a discrete-time normal martingale on $(\Omega, \mathscr{F}, P)$ that has the chaotic representation property and $Z = (Z_n)_{n\in\mathbb{N}}$ the discrete-time normal noise associated with M (see Appendix). We define the following:

$$Z_\emptyset = 1;$$
$$Z_\sigma = \prod_{i\in\sigma} Z_i, \qquad \sigma \in \Gamma,\ \sigma \neq \emptyset. \tag{2}$$

And, for brevity, we use $\mathscr{L}^2(M)$ to mean the space of square integrable functionals of M; namely,

$$\mathscr{L}^2(M) = \mathscr{L}^2(\Omega, \mathscr{F}_\infty, P), \tag{3}$$

which shares the same inner product and norm with $\mathscr{L}^2(\Omega, \mathscr{F}, P)$, namely, $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$. We note that $\{Z_\sigma \mid \sigma \in \Gamma\}$ forms a countable orthonormal basis for $\mathscr{L}^2(M)$ (see Appendix).

Lemma 1 (see [15]). *Let $\sigma \mapsto \lambda_\sigma$ be the $\mathbb{N}$-valued function on Γ given by*

$$\lambda_\sigma = \begin{cases} \prod_{k\in\sigma}(k+1), & \sigma \neq \emptyset,\ \sigma \in \Gamma; \\ 1, & \sigma = \emptyset,\ \sigma \in \Gamma. \end{cases} \tag{4}$$

Then, for $p > 1$, the positive term series $\sum_{\sigma\in\Gamma}\lambda_\sigma^{-p}$ converges and, moreover,

$$\sum_{\sigma\in\Gamma}\lambda_\sigma^{-p} \le \exp\left[\sum_{k=1}^{\infty} k^{-p}\right] < \infty. \tag{5}$$

Using the $\mathbb{N}$-valued function defined by (4), we can construct a chain of Hilbert spaces consisting of functionals of M as follows. For $p \ge 0$, we define a norm $\|\cdot\|_p$ on $\mathscr{L}^2(M)$ through

$$\|\xi\|_p^2 = \sum_{\sigma\in\Gamma}\lambda_\sigma^{2p}\left|\langle Z_\sigma, \xi\rangle\right|^2, \quad \xi \in \mathscr{L}^2(M) \tag{6}$$

and put

$$\mathscr{S}_p(M) = \left\{\xi \in \mathscr{L}^2(M) \mid \|\xi\|_p < \infty\right\}. \tag{7}$$

It is not hard to check that $\|\cdot\|_p$ is a Hilbert norm and $\mathscr{S}_p(M)$ becomes a Hilbert space with $\|\cdot\|_p$. Moreover, the inner product corresponding to $\|\cdot\|_p$ is given by

$$\langle\xi,\eta\rangle_p = \sum_{\sigma\in\Gamma}\lambda_\sigma^{2p}\overline{\langle Z_\sigma,\xi\rangle}\langle Z_\sigma,\eta\rangle, \quad \xi,\eta \in \mathscr{S}_p(M). \tag{8}$$

Here, $\overline{\langle Z_\sigma,\xi\rangle}$ means the complex conjugate of $\langle Z_\sigma,\xi\rangle$.

Lemma 2 (see [14]). *For each $p \ge 0$, one has $\{Z_\sigma \mid \sigma \in \Gamma\} \subset \mathscr{S}_p(M)$ and moreover the system $\{\lambda_\sigma^{-p}Z_\sigma \mid \sigma \in \Gamma\}$ forms an orthonormal basis for $\mathscr{S}_p(M)$.*

It is easy to see that $\lambda_\sigma \ge 1$, for all $\sigma \in \Gamma$. This implies that $\|\cdot\|_p \le \|\cdot\|_q$ and $\mathscr{S}_q(M) \subset \mathscr{S}_p(M)$ whenever $0 \le p \le q$. Thus, we actually get a chain of Hilbert spaces of functionals of M:

$$\begin{aligned} \cdots \subset \mathscr{S}_{p+1}(M) \subset \mathscr{S}_p(M) \subset \cdots \subset \mathscr{S}_1(M) \subset \mathscr{S}_0(M) \\ = \mathscr{L}^2(M). \end{aligned} \tag{9}$$

We now put

$$\mathscr{S}(M) = \bigcap_{p=0}^{\infty}\mathscr{S}_p(M) \tag{10}$$

and endow it with the topology generated by the norm sequence $\{\|\cdot\|_p\}_{p\ge0}$. Note that, for each $p \ge 0$, $\mathscr{S}_p(M)$ is just the completion of $\mathscr{S}(M)$ with respect to $\|\cdot\|_p$. Thus, $\mathscr{S}(M)$ is a countably Hilbert space [16, 17]. The next lemma, however, shows that $\mathscr{S}(M)$ even has a much better property.

Lemma 3 (see [14]). *The space $\mathscr{S}(M)$ is a nuclear space; namely, for any $p \ge 0$, there exists $q > p$ such that the inclusion mapping $i_{pq} : \mathscr{S}_q(M) \to \mathscr{S}_p(M)$ defined by $i_{pq}(\xi) = \xi$ is a Hilbert-Schmidt operator.*

For $p \ge 0$, we denote by $\mathscr{S}_p^*(M)$ the dual of $\mathscr{S}_p(M)$ and by $\|\cdot\|_{-p}$ the norm of $\mathscr{S}_p^*(M)$. Then, $\mathscr{S}_p^*(M) \subset \mathscr{S}_q^*(M)$ and $\|\cdot\|_{-p} \ge \|\cdot\|_{-q}$ whenever $0 \le p \le q$. The lemma below is then an immediate consequence of the general theory of countably Hilbert spaces (see, e.g., [16] or [17]).

Lemma 4 (see [14]). *Let $\mathscr{S}^*(M)$ be the dual of $\mathscr{S}(M)$ and endow it with the strong topology. Then,*

$$\mathscr{S}^*(M) = \bigcup_{p=0}^{\infty}\mathscr{S}_p^*(M) \tag{11}$$

and moreover the inductive limit topology on $\mathscr{S}^*(M)$ given by space sequence $\{\mathscr{S}_p^*(M)\}_{p\ge0}$ coincides with the strong topology.

We mention that, by identifying $\mathscr{L}^2(M)$ with its dual, one comes to a Gel'fand triple:

$$\mathscr{S}(M) \subset \mathscr{L}^2(M) \subset \mathscr{S}^*(M), \tag{12}$$

which we refer to as the Gel'fand triple associated with M.

Lemma 5 (see [14]). *The system $\{Z_\sigma \mid \sigma \in \Gamma\}$ is contained in $\mathscr{S}(M)$ and moreover it serves as a basis in $\mathscr{S}(M)$ in the sense that*

$$\xi = \sum_{\sigma\in\Gamma}\langle Z_\sigma, \xi\rangle Z_\sigma, \quad \xi \in \mathscr{S}(M), \tag{13}$$

where $\langle\cdot,\cdot\rangle$ is the inner product of $\mathscr{L}^2(M)$ and the series converges in the topology of $\mathscr{S}(M)$.

Definition 6 (see [14]). Elements of $\mathscr{S}^*(M)$ are called generalized functionals of M, while elements of $\mathscr{S}(M)$ are called testing functionals of M.

Denote by $\langle\langle\cdot,\cdot\rangle\rangle$ the canonical bilinear form on $\mathcal{S}^*(M)\times\mathcal{S}(M)$; namely,

$$\langle\langle\Phi,\xi\rangle\rangle=\Phi(\xi),\quad \Phi\in\mathcal{S}^*(M),\ \xi\in\mathcal{S}(M),\tag{14}$$

where $\Phi(\xi)$ means Φ acting on ξ as usual. Note that $\langle\cdot,\cdot\rangle$ denotes the inner product of $\mathcal{L}^2(M)$, which is different from $\langle\langle\cdot,\cdot\rangle\rangle$.

Definition 7 (see [14]). For $\Phi\in\mathcal{S}^*(M)$, its Fock transform is the function $\widehat{\Phi}$ on Γ given by

$$\widehat{\Phi}(\sigma)=\langle\langle\Phi,Z_\sigma\rangle\rangle,\quad \sigma\in\Gamma,\tag{15}$$

where $\langle\langle\cdot,\cdot\rangle\rangle$ is the canonical bilinear form.

It is easy to verify that, for $\Phi,\Psi\in\mathcal{S}^*(M)$, $\Phi=\Psi$ if and only if $\widehat{\Phi}=\widehat{\Psi}$. Thus, a generalized functional of M is completely determined by its Fock transform. The following theorem characterizes generalized functionals of M through their Fock transforms.

Lemma 8 (see [14]). *Let F be a function on Γ. Then, F is the Fock transform of an element Φ of $\mathcal{S}^*(M)$ if and only if it satisfies*

$$|F(\sigma)|\le C\lambda_\sigma^p,\quad \sigma\in\Gamma\tag{16}$$

for some constants $C\ge 0$ and $p\ge 0$. In that case, for $q>p+1/2$, one has

$$\|\Phi\|_{-q}\le C\left[\sum_{\sigma\in\Gamma}\lambda_\sigma^{-2(q-p)}\right]^{1/2}\tag{17}$$

and in particular $\Phi\in\mathcal{S}_q^*(M)$.

3. Convergence Theorems for Generalized Functional Sequences

Let $M=(M_n)_{n\in\mathbb{N}}$ be the same discrete-time normal martingale as described in Section 2. In the present section, we apply the Fock transform (see Definition 7) to establish convergence theorems for generalized functionals of M.

In order to prove our main results in a convenient way, we first give a preliminary proposition, which is an immediate consequence of the general theory of countably normed spaces, especially nuclear spaces [16–18], since $\mathcal{S}(M)$ is a nuclear space (see Lemma 3).

Proposition 9. *Let $\Phi,\Phi_n\in\mathcal{S}^*(M)$, $n\ge 1$, be generalized functionals of M. Then, the following conditions are equivalent:*

(i) *The sequence (Φ_n) converges weakly to Φ in $\mathcal{S}^*(M)$.*

(ii) *The sequence (Φ_n) converges strongly to Φ in $\mathcal{S}^*(M)$.*

(iii) *There exists a constant $p\ge 0$ such that $\Phi,\Phi_n\in\mathcal{S}_p^*(M)$, $n\ge 1$, and the sequence (Φ_n) converges to Φ in the norm of $\mathcal{S}_p^*(M)$.*

Here, we mention that "(Φ_n) converges strongly (resp., weakly) to Φ" meaning that (Φ_n) converges to Φ in the strong (resp., weak) topology of $\mathcal{S}^*(M)$. For details about various topologies on the dual of a countably normed space, we refer to [16, 18].

The next theorem is one of our main results, which offers a criterion in terms of the Fock transform for checking whether or not a sequence in $\mathcal{S}^*(M)$ is strongly convergent.

Theorem 10. *Let $\Phi,\Phi_n\in\mathcal{S}^*(M)$, $n\ge 1$, be generalized functionals of M. Then, the sequence (Φ_n) converges strongly to Φ in $\mathcal{S}^*(M)$ if and only if it satisfies the following:*

(1) *$\widehat{\Phi_n}(\sigma)\to\widehat{\Phi}(\sigma)$, for all $\sigma\in\Gamma$.*

(2) *There are constants $C\ge 0$ and $p\ge 0$ such that*

$$\sup_{n\ge 1}\left|\widehat{\Phi_n}(\sigma)\right|\le C\lambda_\sigma^p,\quad \sigma\in\Gamma.\tag{18}$$

Proof. Regarding the "only if" part, let (Φ_n) converge strongly to Φ in $\mathcal{S}^*(M)$. Then, we obviously have

$$\widehat{\Phi_n}(\sigma)=\langle\langle\Phi_n,Z_\sigma\rangle\rangle\longrightarrow\langle\langle\Phi,Z_\sigma\rangle\rangle=\widehat{\Phi}(\sigma),\quad \sigma\in\Gamma,\tag{19}$$

because $\{Z_\sigma\mid\sigma\in\Gamma\}\subset\mathcal{S}(M)$ and (Φ_n) also converges weakly to Φ. On the other hand, by Proposition 9, we know that there exists $p\ge 0$ such that $\Phi,\Phi_n\in\mathcal{S}_p^*(M)$, $n\ge 1$, and (Φ_n) converges to Φ in the norm of $\mathcal{S}_p^*(M)$, which implies that $C\equiv\sup_{n\ge 1}\|\Phi_n\|_{-p}<\infty$. Therefore,

$$\begin{aligned}\sup_{n\ge 1}\left|\widehat{\Phi_n}(\sigma)\right|&=\sup_{n\ge 1}\left|\langle\langle\Phi_n,Z_\sigma\rangle\rangle\right|\le\sup_{n\ge 1}\|\Phi_n\|_{-p}\|Z_\sigma\|_p\\&=C\lambda_\sigma^p,\quad \sigma\in\Gamma.\end{aligned}\tag{20}$$

Regarding the "if" part, let (Φ_n) satisfy conditions (1) and (2). Then, by taking $q>p+1/2$ and using Lemma 8, we get

$$\sup_{n\ge 1}\|\Phi_n\|_{-q}\le C\left[\sum_{\sigma\in\Gamma}\lambda_\sigma^{-2(q-p)}\right]^{1/2};\tag{21}$$

in particular, $\Phi_n\in\mathcal{S}_q^*(M)$, $n\ge 1$. On the other hand, $\{Z_\sigma\mid\sigma\in\Gamma\}$ is total in $\mathcal{S}_q(M)$, which, together with (21) as well as the property

$$\langle\langle\Phi_n,Z_\sigma\rangle\rangle=\widehat{\Phi_n}(\sigma)\longrightarrow\widehat{\Phi}(\sigma)=\langle\langle\Phi,Z_\sigma\rangle\rangle,\quad \sigma\in\Gamma,\tag{22}$$

implies that $\Phi\in\mathcal{S}_q^*(M)$ and

$$\langle\langle\Phi_n,\xi\rangle\rangle\longrightarrow\langle\langle\Phi,\xi\rangle\rangle,\quad \forall\xi\in\mathcal{S}_q(M).\tag{23}$$

Thus, (Φ_n) converges weakly to Φ in $\mathcal{S}^*(M)$, which together with Proposition 9 implies that (Φ_n) converges strongly to Φ in $\mathcal{S}^*(M)$. □

In a similar way, we can prove the following theorem, which is slightly different from Theorem 10, but more convenient to use.

Theorem 11. *Let $(\Phi_n) \subset \mathcal{S}^*(M)$ be a sequence of generalized functionals of M. Suppose $(\widehat{\Phi_n}(\sigma))$ converges, for all $\sigma \in \Gamma$, and moreover there are constants $C \geq 0$ and $p \geq 0$ such that*

$$\sup_{n\geq 1} \left|\widehat{\Phi_n}(\sigma)\right| \leq C\lambda_\sigma^p, \quad \sigma \in \Gamma. \tag{24}$$

Then, there exists a generalized functional $\Phi \in \mathcal{S}^*(M)$ such that (Φ_n) converges strongly to Φ.

4. M-Generalized Martingales and Their Convergence Theorems

In this section, we first introduce a type of generalized martingales associated with M, which we call M-generalized martingales, and then we use the Fock transform to give necessary and sufficient condition for such a generalized martingale to be strongly convergent. Some other convergence results are also obtained.

For a nonnegative integer $n \geq 0$, we denote by Γ_n the power set of $\{0, 1, \ldots, n\}$; namely,

$$\Gamma_n = \{\sigma \mid \sigma \subset \{0, 1, \ldots, n\}\}. \tag{25}$$

Clearly $\Gamma_n \subset \Gamma$. We use $\mathbf{I}_n$ to mean the indicator of Γ_n, which is a function on Γ given by

$$\mathbf{I}_n(\sigma) = \begin{cases} 1, & \sigma \in \Gamma_n; \\ 0, & \sigma \notin \Gamma_n. \end{cases} \tag{26}$$

Definition 12. A sequence $(\Phi_n)_{n\geq 0} \subset \mathcal{S}^*(M)$ is called an M-generalized martingale if it satisfies that

$$\widehat{\Phi_n}(\sigma) = \mathbf{I}_n(\sigma)\widehat{\Phi_{n+1}}(\sigma), \quad \sigma \in \Gamma, \ n \geq 0, \tag{27}$$

where $\mathbf{I}_n$ mean the indicator of Γ_n as defined by (26).

Let $\mathfrak{F} = (\mathcal{F}_n)_{n\geq 0}$ be the filtration on $(\Omega, \mathcal{F}, P)$ generated by $Z = (Z_n)_{n\geq 0}$; namely,

$$\mathcal{F}_n = \sigma\{Z_k \mid 0 \leq k \leq n\}, \quad n \geq 0. \tag{28}$$

The following theorem justifies Definition 12.

Theorem 13. *Suppose $(\xi_n)_{n\geq 1} \subset \mathcal{L}^2(M)$ is a martingale with respect to filtration $\mathfrak{F} = (\mathcal{F}_n)_{n\geq 0}$. Then, $(\xi_n)_{n\geq 1}$ is an M-generalized martingale.*

Proof. By the assumptions, $(\xi_n)_{n\geq 1}$ satisfies the following conditions:

$$\xi_n = \mathbb{E}[\xi_{n+1} \mid \mathcal{F}_n], \quad n \geq 0, \tag{29}$$

where $\mathbb{E}[\cdot \mid \mathcal{F}_n]$ means the conditional expectation given σ-algebra $\mathcal{F}_n$. Note that

$$\mathbb{E}[Z_\tau \mid \mathcal{F}_n] = \mathbf{I}_n(\tau) Z_\tau, \quad \tau \in \Gamma, \tag{30}$$

which, together with (29) and the expansion $\xi_{n+1} = \sum_{\tau\in\Gamma}\langle Z_\tau, \xi_{n+1}\rangle Z_\tau$, gives

$$\begin{aligned} \xi_n &= \mathbb{E}[\xi_{n+1} \mid \mathcal{F}_n] = \sum_{\tau\in\Gamma} \langle Z_\tau, \xi_{n+1}\rangle \mathbb{E}[Z_\tau \mid \mathcal{F}_n] \\ &= \sum_{\tau\in\Gamma} \langle Z_\tau, \xi_{n+1}\rangle \mathbf{I}_n(\tau) Z_\tau. \end{aligned} \tag{31}$$

Taking Fock transforms yields

$$\begin{aligned} \widehat{\xi_n}(\sigma) &= \sum_{\tau\in\Gamma} \langle \xi_{n+1}, Z_\tau\rangle \mathbf{I}_n(\tau) \widehat{Z_\tau}(\sigma) \\ &= \langle \xi_{n+1}, Z_\sigma\rangle \mathbf{I}_n(\sigma) = \mathbf{I}_n(\sigma)\widehat{\xi_{n+1}}(\sigma), \end{aligned} \tag{32}$$

where $\sigma \in \Gamma$. Thus, $(\xi_n)_{n\geq 1}$ is an M-generalized martingale. □

The next theorem gives a necessary and sufficient condition in terms of the Fock transform for an M-generalized martingale to be strongly convergent.

Theorem 14. *Let $(\Phi_n)_{n\geq 1} \subset \mathcal{S}^*(M)$ be an M-generalized martingale. Then, the following two conditions are equivalent:*

(1) *$(\Phi_n)_{n\geq 1}$ is strongly convergent in $\mathcal{S}^*(M)$.*

(2) *There are constants $C \geq 0$ and $p \geq 0$ such that*

$$\sup_{n\geq 1} \left|\widehat{\Phi_n}(\sigma)\right| \leq C\lambda_\sigma^p, \quad \sigma \in \Gamma. \tag{33}$$

Proof. By Theorem 10, we need only to prove "(2) $\Rightarrow$ (1)". Let $\sigma \in \Gamma$ be taken. Then, by the definition of M-generalized martingales (see Definition 12), we have

$$\widehat{\Phi_m}(\sigma) = \mathbf{I}_m(\sigma)\widehat{\Phi_{m+k}}(\sigma), \quad m, k \geq 0. \tag{34}$$

Now take $n_0 \geq 0$ such that $\sigma \in \Gamma_{n_0}$. Then, $\mathbf{I}_{n_0}(\sigma) = 1$ and moreover

$$\widehat{\Phi_{n_0}}(\sigma) = \mathbf{I}_{n_0}(\sigma)\widehat{\Phi_n}(\sigma) = \widehat{\Phi_n}(\sigma), \quad n > n_0, \tag{35}$$

which implies that $(\widehat{\Phi_n}(\sigma))$ converges. Thus, by Theorem 11, $(\Phi_n)_{n\geq 1}$ is strongly convergent in $\mathcal{S}^*(M)$. □

Theorem 15. *Let D be a subset of $\mathcal{S}^*(M)$. Then, the following two conditions are equivalent:*

(1) *There is a constant $p \geq 0$ such that D is contained and bounded in $\mathcal{S}_p^*(M)$.*

(2) *There are constants $C \geq 0$ and $p \geq 0$ such that*

$$\sup_{\Phi\in D} \left|\widehat{\Phi}(\sigma)\right| \leq C\lambda_\sigma^p, \quad \sigma \in \Gamma. \tag{36}$$

Proof. Obviously, condition (1) implies condition (2). We now verify the inverse implication relation. In fact, under condition (2), by using Lemma 8. we have

$$\sup_{\Phi\in D} \|\Phi\|_{-q} \leq C\left[\sum_{\sigma\in\Gamma} \lambda_\sigma^{-2(q-p)}\right]^{1/2}, \tag{37}$$

where $q > p + 1/2$, which clearly implies condition (1). □

The next theorem shows that, for an M-generalized martingale, its strong (weak) convergence is just equivalent to its strong (weak) boundedness.

Theorem 16. *Let $(\Phi_n)_{n\geq 1} \subset \mathcal{S}^*(M)$ be an M-generalized martingale. Then, the following conditions are equivalent:*

(1) *$(\Phi_n)_{n\geq 1}$ is strongly convergent in $\mathcal{S}^*(M)$.*

(2) *$(\Phi_n)_{n\geq 1}$ is weakly bounded in $\mathcal{S}^*(M)$.*

(3) *$(\Phi_n)_{n\geq 1}$ is strongly bounded in $\mathcal{S}^*(M)$.*

(4) *$(\Phi_n)_{n\geq 1}$ is bounded in $\mathcal{S}_p^*(M)$ for some $p \geq 0$.*

Proof. Clearly, conditions (2), (3), and (4) are equivalent to each other because $\mathcal{S}(M)$ is a nuclear space (see Lemma 3). Using Theorems 14 and 15, we immediately know that conditions (1) and (4) are also equivalent. □

5. Applications

In the last section, we show some applications of our main results.

Recall that the system $\{Z_\sigma \mid \sigma \in \Gamma\}$ is an orthonormal basis of $\mathcal{L}^2(M)$. Now, if we write

$$\Psi_n^{(0)} = \sum_{\tau\in\Gamma_n} Z_\tau, \quad n \geq 0, \tag{38}$$

then $(\Psi_n^{(0)})_{n\geq 0} \subset \mathcal{L}^2(M)$, and moreover $(\Psi_n^{(0)})_{n\geq 0}$ is a martingale with respect to filtration $\mathfrak{F} = (\mathcal{F}_n)_{n\geq 0}$. However, $(\Psi_n^{(0)})_{n\geq 0}$ is not convergent in $\mathcal{L}^2(M)$ since

$$\left\|\Psi_n^{(0)}\right\| = \sqrt{\#(\Gamma_n)} = 2^{(n+1)/2} \longrightarrow \infty \quad (\text{as } n \longrightarrow \infty), \tag{39}$$

where $\#(\Gamma_n)$ means the cardinality of Γ_n as a set and $\|\cdot\|$ is the norm in $\mathcal{L}^2(M)$.

Proposition 17. *The sequence $(\Psi_n^{(0)})_{n\geq 0}$ defined above is an M-generalized martingale, and moreover it is strongly convergent in $\mathcal{S}^*(M)$.*

Proof. According to Theorem 13, $(\Psi_n^{(0)})_{n\geq 0}$ is certainly an M-generalized martingale. On the other hand, in viewing the relation between the canonical bilinear form on $\mathcal{S}^*(M) \times \mathcal{S}(M)$ and the inner product in $\mathcal{L}^2(M)$, we have

$$\widehat{\Psi_n^{(0)}}(\sigma) = \left\langle\left\langle \Psi_n^{(0)}, Z_\sigma\right\rangle\right\rangle = \left\langle \Psi_n^{(0)}, Z_\sigma\right\rangle = \mathbf{I}_n(\sigma), \quad \sigma \in \Gamma,\ n \geq 0, \tag{40}$$

which implies that

$$\sup_{n\geq 0}\left|\widehat{\Psi_n^{(0)}}(\sigma)\right| \leq C\lambda_\sigma^p, \quad \sigma \in \Gamma \tag{41}$$

with $C = 1$ and $p = 0$. It then follows from Theorem 14 that $(\Psi_n^{(0)})_{n\geq 0}$ is strongly convergent in $\mathcal{S}^*(M)$. □

Recall that [14], for two generalized functionals $\Phi_1, \Phi_2 \in \mathcal{S}^*(M)$, their convolution $\Phi_1 * \Phi_2$ is defined by

$$\widehat{\Phi_1 * \Phi_2}(\sigma) = \widehat{\Phi_1}(\sigma)\widehat{\Phi_2}(\sigma), \quad \sigma \in \Gamma. \tag{42}$$

The next theorem provides a method to construct an M-generalized martingale through the M-generalized martingale $(\Psi_n^{(0)})_{n\geq 0}$ defined in (38).

Theorem 18. *Let $\Phi \in \mathcal{S}^*(M)$ be a generalized functional and define*

$$\Phi_n = \Psi_n^{(0)} * \Phi, \quad n \geq 0. \tag{43}$$

Then, $(\Phi_n)_{n\geq 0}$ is an M-generalized martingale, and moreover it converges strongly to Φ in $\mathcal{S}^*(M)$.

Proof. By Lemma 8, there exist some constants $C \geq 0$ and $p \geq 0$ such that

$$\left|\widehat{\Phi}(\sigma)\right| \leq C\lambda_\sigma^p, \quad \sigma \in \Gamma. \tag{44}$$

On the other hand, by using (40), we get

$$\widehat{\Phi_n}(\sigma) = \widehat{\Psi_n^{(0)}}(\sigma)\widehat{\Phi}(\sigma) = \mathbf{I}_n(\sigma)\widehat{\Phi}(\sigma), \quad \sigma \in \Gamma,\ n \geq 0, \tag{45}$$

which, together with the fact that $\mathbf{I}_n(\sigma)\mathbf{I}_{n+1}(\sigma) = \mathbf{I}_n(\sigma)$, gives

$$\widehat{\Phi_n}(\sigma) = \mathbf{I}_n(\sigma)\widehat{\Phi_{n+1}}(\sigma), \quad \sigma \in \Gamma,\ n \geq 0. \tag{46}$$

Thus, $(\Phi_n)_{n\geq 0}$ is an M-generalized martingale. Additionally, it easily follows from (44) and (45) that $\widehat{\Phi_n}(\sigma) \to \widehat{\Phi}(\sigma)$, for each $\sigma \in \Gamma$, and

$$\sup_{n\geq 0}\left|\widehat{\Phi_n}(\sigma)\right| = \sup_{n\geq 0}\left[\mathbf{I}_n(\sigma)\right]\left|\widehat{\Phi}(\sigma)\right| \leq C\lambda_\sigma^p, \quad \sigma \in \Gamma. \tag{47}$$

Therefore, by Theorem 11, we finally find $(\Phi_n)_{n\geq 0}$ converges strongly to Φ. □

Appendix

In this appendix, we provide some basic notions and facts about discrete-time normal martingales. For details, we refer to [11, 19].

Let $(\Omega, \mathcal{F}, P)$ be a given probability space with $\mathbb{E}$ denoting the expectation with respect to P. We denote by $\mathcal{L}^2(\Omega, \mathcal{F}, P)$ the usual Hilbert space of square integrable complex-valued functions on $(\Omega, \mathcal{F}, P)$ and use $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ to mean its inner product and norm, respectively.

Definition A.1. A stochastic process $M = (M_n)_{n\in\mathbb{N}}$ on $(\Omega, \mathcal{F}, P)$ is called a discrete-time normal martingale if it is square integrable and satisfies the following:

(i) $\mathbb{E}[M_0 \mid \mathcal{F}_{-1}] = 0$ and $\mathbb{E}[M_n \mid \mathcal{F}_{n-1}] = M_{n-1}$, for $n \geq 1$.

(ii) $\mathbb{E}[M_0^2 \mid \mathcal{F}_{-1}] = 1$ and $\mathbb{E}[M_n^2 \mid \mathcal{F}_{n-1}] = M_{n-1}^2 + 1$, for $n \geq 1$,

where $\mathscr{F}_{-1} = \{\emptyset, \Omega\}$, $\mathscr{F}_n = \sigma(M_k;\ 0 \le k \le n)$, for $n \in \mathbb{N}$, and $\mathbb{E}[\cdot \mid \mathscr{F}_k]$ means the conditional expectation.

Let $M = (M_n)_{n\in\mathbb{N}}$ be a discrete-time normal martingale on $(\Omega, \mathscr{F}, P)$. Then, one can construct from M a process $Z = (Z_n)_{n\in\mathbb{N}}$ as follows:

$$\begin{aligned} Z_0 &= M_0, \\ Z_n &= M_n - M_{n-1}, \quad n \ge 1. \end{aligned} \tag{A.1}$$

It can be verified that Z admits the following properties:

$$\begin{aligned} &\mathbb{E}\left[Z_n \mid \mathscr{F}_{n-1}\right] = 0, \\ &\mathbb{E}\left[Z_n^2 \mid \mathscr{F}_{n-1}\right] = 1, \\ &\qquad n \in \mathbb{N}. \end{aligned} \tag{A.2}$$

Thus, it can be viewed as a discrete-time noise.

Definition A.2. Let $M = (M_n)_{n\in\mathbb{N}}$ be a discrete-time normal martingale. Then, the process Z defined by (A.2) is called the discrete-time normal noise associated with M.

The next lemma shows that, from the discrete-time normal noise Z, one can get an orthonormal system in $\mathscr{L}^2(\Omega, \mathscr{F}, P)$, which is indexed by $\sigma \in \Gamma$.

Lemma A.3. *Let $M = (M_n)_{n\in\mathbb{N}}$ be a discrete-time normal martingale and $Z = (Z_n)_{n\in\mathbb{N}}$ be the discrete-time normal noise associated with M. Define $Z_\emptyset = 1$, where $\emptyset$ denotes the empty set, and*

$$Z_\sigma = \prod_{i\in\sigma} Z_i, \quad \sigma \in \Gamma,\ \sigma \ne \emptyset. \tag{A.3}$$

Then, $\{Z_\sigma \mid \sigma \in \Gamma\}$ forms a countable orthonormal system in $\mathscr{L}^2(\Omega, \mathscr{F}, P)$.

Let $\mathscr{F}_\infty = \sigma(M_n;\ n \in \mathbb{N})$, the σ-field over Ω generated by M. In the literature, $\mathscr{F}_\infty$-measurable functions on Ω are also known as functionals of M. Thus, elements of $\mathscr{L}^2(\Omega, \mathscr{F}_\infty, P)$ can be called square integrable functionals of M. For brevity, we usually denote by $\mathscr{L}^2(M)$ the space of square integrable functionals of M; namely,

$$\mathscr{L}^2(M) = \mathscr{L}^2\left(\Omega, \mathscr{F}_\infty, P\right). \tag{A.4}$$

Definition A.4. The discrete-time normal martingale M is said to have the chaotic representation property if the system $\{Z_\sigma \mid \sigma \in \Gamma\}$ defined by (A.3) is total in $\mathscr{L}^2(M)$.

So, if the discrete-time normal martingale M has the chaotic representation property, then the system $\{Z_\sigma \mid \sigma \in \Gamma\}$ is actually an orthonormal basis for $\mathscr{L}^2(M)$, which is a closed subspace of $\mathscr{L}^2(\Omega, \mathscr{F}, P)$ as is known.

Remark A.5. Émery [20] called a $\mathbb{Z}$-indexed process $X = (X_n)_{n\in\mathbb{Z}}$ satisfying (A.2) a novation and introduced the notion of the chaotic representation property for such a process.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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