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Research Article

Jordan Type Inequalities for Hyperbolic Functions and Their Applications

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We present the best possible parameters $p, q \in (0, \infty)$ such that the double inequality $(1/3p^2)\cosh(px) + 1 - 1/3p^2 < \sinh(x)/x < (1/3q^2)\cosh(qx) + 1 - 1/3q^2$ holds for all $x \in (0, \infty)$. As applications, some new inequalities for certain special function and bivariate means are found.

1. Introduction

The well known Jordan inequality [1] is given by

$$\frac{2}{\pi}x < \sin(x) < x, \quad x \in \left(0, \frac{\pi}{2}\right). \tag{1}$$

During the past few years, the improvements, refinements, and generalizations for inequality (1) have attracted the attention of many researchers [2–13]. Recently, the hyperbolic counterpart and its generalizations have been the subject of intensive research.

Zhu [14] proved that the inequality

$$\left(\frac{\sinh(x)}{x}\right)^q > p + (1-p)\cosh(x) \tag{2}$$

holds for all $x > 0$ if and only if $q \geq 3(1-p)$ if $p \in (-\infty, 8/15] \cup (1, \infty)$.

In [3, 15], Neuman and Sándor proved that

$$\cosh^{4/3}\left(\frac{x}{2}\right) < \frac{\sinh(x)}{x} < \cosh^3(x) \tag{3}$$

for all $x > 0$.

Klén et al. [5] proved that the double inequality

$$\cosh^{1/4}(x) < \frac{\sinh(x)}{x} < \cosh^{1/2}(x) \tag{4}$$

holds for all $x \in (0, 1)$.

In [4], the authors proved that the double inequality

$$\cosh^p(x) < \frac{\sinh(x)}{x} < \cosh^q(x) \tag{5}$$

holds for all $x \in (0, 1)$ if and only if $p \leq 1/3$ and $q \geq [\log(\sinh(1))]/[\log(\cosh(1))] = 0.3721\cdots$.

Zhu [16, 17] proved that the inequalities

$$\begin{aligned}
(1-\lambda) + \lambda\cosh^p(x) &< \left(\frac{\sinh(x)}{x}\right)^p \\
&< (1-\mu) + \mu\cosh^p(x), \\
\left(\frac{\sinh(x)}{x}\right)^q &< (1-\eta) + \eta\cosh^q(x), \\
\alpha + (1-\alpha)e^{r[t\coth(t)-1]} &< \left(\frac{\sinh(x)}{x}\right)^r \\
&< \beta + (1-\beta)e^{r[t\coth(t)-1]}
\end{aligned} \tag{6}$$

hold for all $x > 0$ if and only if $\lambda \leq 0$, $\mu \geq 1/3$, $\eta \leq 1/3$, $\alpha \geq 1$, and $\beta \leq 1/2$ if $p \geq 4/5$, $q < 0$, and $r \geq 286/693$.

Very recently, Yang [18] proved that the double inequality

$$[\cosh(px)]^{1/3p^2} < \frac{\sinh(x)}{x} < [\cosh(qx)]^{1/3q^2} \tag{7}$$

holds for all $x > 0$ if and only if $p \geq \sqrt{5}/5$ and $q \leq 1/3$.

The main purpose of this paper is to find the best possible parameters $p, q \in (0, \infty)$ such that the double inequality $(1/3p^2)\cosh(px) + 1 - 1/3p^2 < \sinh(x)/x < (1/3q^2)\cosh(qx) + 1 - 1/3q^2$ holds for all $x > 0$ and present several new inequalities for certain special function and bivariate means.

2. Main Result

Theorem 1. *Let $p, q \in (0, \infty)$. Then the double inequality*

$$\frac{1}{3p^2}\cosh(px) + 1 - \frac{1}{3p^2} < \frac{\sinh(x)}{x} < \frac{1}{3q^2}\cosh(qx) + 1 - \frac{1}{3q^2} \tag{8}$$

holds for all $x > 0$ if and only if $p \le \sqrt{15}/5$ and $q \ge 1$.

Proof. Let $\lambda > 0$ and let the function f_λ be defined on $(0, \infty)$ by

$$f_\lambda(x) = \frac{\sinh(x)}{x} - \left(\frac{1}{3\lambda^2}\cosh(\lambda x) + 1 - \frac{1}{3\lambda^2}\right). \tag{9}$$

Then making use of power series expansions and (9) we get

$$\begin{aligned} f_\lambda(x) &= \sum_{n=0}^{\infty}\frac{x^{2n}}{(2n+1)!} - \left(\frac{1}{3\lambda^2}\sum_{n=0}^{\infty}\frac{(\lambda x)^{2n}}{(2n)!} + 1 - \frac{1}{3\lambda^2}\right) \\ &= \sum_{n=2}^{\infty}\frac{3 - (2n+1)\lambda^{2n-2}}{3(2n+1)!}x^{2n}. \end{aligned} \tag{10}$$

Let

$$a_n(\lambda) = 3 - (2n+1)\lambda^{2n-2}. \tag{11}$$

Then

$$a_n(1) = 3 - (2n+1) = -2(n-1) < 0 \tag{12}$$

for all $n \ge 2$. Consider

$$\begin{aligned} &a_2\left(\frac{\sqrt{15}}{5}\right) = 0, \qquad a_3\left(\frac{\sqrt{15}}{5}\right) = \frac{12}{25} > 0, \\ &a_{n+1}\left(\frac{\sqrt{15}}{5}\right) - a_n\left(\frac{\sqrt{15}}{5}\right) = \frac{4\times 3^{n-1}}{5^n}(n-1) > 0 \end{aligned} \tag{13}$$

for all $n \ge 2$.

It follows from (13) that

$$a_2\left(\frac{\sqrt{15}}{5}\right) = 0, \qquad a_n\left(\frac{\sqrt{15}}{5}\right) > 0 \tag{14}$$

for all $n \ge 3$.

Therefore, inequality (8) holds for all $x > 0$ with $p = \sqrt{15}/5$ and $q = 1$ follows from (9)–(12) and (14).

Next, we prove that $p \le \sqrt{15}/5$ and $q \ge 1$ if inequality (8) holds for all $x > 0$.

If the first inequality of (8) holds for all $x > 0$, then from (9) and (10) we have

$$\lim_{x\to 0^+}\frac{f_p(x)}{x^4} = \frac{3 - 5p^2}{360} \ge 0 \tag{15}$$

and $p \le \sqrt{15}/5$.

If the second inequality of (8) holds for all $x > 0$, then it follows from (9) that

$$\begin{aligned} \lim_{x\to +\infty}\frac{f_q(x)}{e^{qx}} &= \frac{1 - e^{-2x}}{2x}e^{(1-q)x} - \frac{1 + e^{-2qx}}{6q^2} \\ &\quad - \left(1 - \frac{1}{3q^2}\right)e^{-qx} \le 0. \end{aligned} \tag{16}$$

We clearly see that $\lim_{x\to+\infty}(f_q(x)/e^{qx}) = +\infty$ if $q < 1$. Therefore, $q \ge 1$ follows from (16). □

Remark 2. It is not difficult to verify that the bound

$$g_p(x) = \frac{1}{3p^2}\cosh(px) + 1 - \frac{1}{3p^2} \tag{17}$$

given in Theorem 1 is strictly increasing with respect to p on $(0, \infty)$ for fixed $x \in (0, \infty)$.

Remark 3. Let $p = \sqrt{15}/5 > 3/4 > \sqrt{2}/2 > 2/3 > \sqrt{3}/3$ and $q = 1 < 2\sqrt{3}/3$. Then Theorem 1 and Remark 2 lead to

$$\begin{aligned} \cosh\left(\frac{\sqrt{3}x}{3}\right) &< \frac{3}{4}\cosh\left(\frac{2x}{3}\right) + \frac{1}{4} < \frac{2}{3}\cosh\left(\frac{\sqrt{2}x}{2}\right) + \frac{1}{3} \\ &< \frac{16}{27}\cosh\left(\frac{3x}{4}\right) + \frac{11}{27} < \frac{5}{9}\cosh\left(\frac{\sqrt{15}x}{5}\right) \\ &\quad + \frac{4}{9} < \frac{\sinh(x)}{x} \\ &< \frac{1}{3}\cosh(x) + \frac{2}{3} < \frac{1}{2}\cosh^2\left(\frac{\sqrt{3}x}{3}\right) + \frac{1}{2} \end{aligned} \tag{18}$$

for all $x > 0$.

3. Applications

It is well known that

$$\int_0^\infty \frac{x}{\sinh(x)} = \frac{1}{2}\psi'\left(\frac{1}{2}\right), \tag{19}$$

where ψ' is the trigamma function defined by

$$\psi'(x) = \int_0^\infty \frac{te^{-xt}}{1 - e^{-t}}dt. \tag{20}$$

Let

$$\mathrm{Sh}(x) = \int_0^x \frac{t}{\sinh(t)}dt. \tag{21}$$

Then Remark 3 leads to

$$\int_0^x \frac{3}{\cosh(t)+2}dt < \mathrm{Sh}(x) < \int_0^x \frac{9}{5\cosh\left(\sqrt{15}t/5\right)+4}dt \tag{22}$$

for all $x > 0$.

From (19) and (22) we get the following.

Remark 4. For all $x > 0$ one has

$$\begin{aligned}\sqrt{3}\log\left(\frac{e^x+2-\sqrt{3}}{e^x+2+\sqrt{3}}\right)+\sqrt{3}\log\left(2+\sqrt{3}\right)\\ < \mathrm{Sh}(x) < 2\sqrt{15}\\ \times\left[\arctan\left(\frac{5}{3}e^{\sqrt{15}x/5}+\frac{4}{3}\right)-\arctan(3)\right].\end{aligned} \tag{23}$$

In particular, we have

$$\begin{aligned}4.5620\cdots &= 2\sqrt{3}\log\left(2+\sqrt{3}\right) < \psi'\left(\frac{1}{2}\right)\\ &< 2\sqrt{15}\left[\pi-2\arctan(3)\right] = 4.9845\cdots.\end{aligned} \tag{24}$$

For $a, b > 0$, the Schwab-Borchardt mean $\mathrm{SB}(a,b)$ [19–21] is given by

$$\mathrm{SB}(a,b) = \frac{\sqrt{b^2-a^2}}{\arccos(a/b)}\ (a<b), \tag{25}$$

$$\mathrm{SB}(a,b) = a\ (a=b),$$

$$\mathrm{SB}(a,b) = \frac{\sqrt{a^2-b^2}}{\cosh^{-1}(a/b)}\ (a>b). \tag{26}$$

Let $a > b$ and let $x = \cosh^{-1}(a/b)$. Then $\cosh(x) = a/b$ and $\sqrt{a^2-b^2}/\cosh^{-1}(a/b) = b\sinh(x)/x$. It follows from Remark 3 and (26) that

$$\left[\frac{16}{27}\cosh\left(\frac{3x}{4}\right)+\frac{11}{27}\right]b < \mathrm{SB}(a,b) < \left[\frac{1}{3}\cosh(x)+\frac{2}{3}\right]b. \tag{27}$$

Note that

$$\begin{aligned}\left[\frac{1}{3}\cosh(x)+\frac{2}{3}\right]b &= \frac{a+2b}{3},\\ \left[\frac{16}{27}\cosh\left(\frac{3x}{4}\right)+\frac{11}{27}\right]b &= \frac{8b^{1/4}}{27}\left(\sqrt{2(a+b)(2a-b)^2}+2b^{3/2}\right)^{1/2}+\frac{11b}{27}.\end{aligned} \tag{28}$$

From (27) and (28) we get the following.

Remark 5. Let $a > b > 0$; then the Schwab-Borchardt mean $\mathrm{SB}(a,b)$ satisfies the double inequality

$$\begin{aligned}\frac{8b^{1/4}}{27}\left(\sqrt{2(a+b)(2a-b)^2}+2b^{3/2}\right)^{1/2}\\ +\frac{11b}{27} < \mathrm{SB}(a,b) < \frac{a+2b}{3}.\end{aligned} \tag{29}$$

Let $p \in \mathbb{R}$ and let $a, b > 0$ with $a \neq b$. Then the arithmetic mean $A(a,b)$, logarithmic mean $L(a,b)$, geometric mean $G(a,b)$, and pth power mean $M_p(a,b)$ are defined by

$$\begin{aligned}A(a,b) &= \frac{a+b}{2}, \qquad L(a,b) = \frac{b-a}{\log b-\log a},\\ G(a,b) &= \sqrt{ab},\\ M_p(a,b) &= \left(\frac{a^p+b^p}{2}\right)^{1/p} \quad (p\neq 0),\\ M_0(a,b) &= \sqrt{ab} = G(a,b).\end{aligned} \tag{30}$$

It is well known that $M_p(a,b)$ is continuous and strictly increasing with respect to $p \in \mathbb{R}$ for fixed $a, b > 0$ with $a \neq b$; the main properties for the power mean are given in [22]. Recently, the arithmetic, logarithmic, geometric, and power means have been the subject of intensive research. In particular, many remarkable inequalities can be found in the literature [23–35].

Let $x = (1/2)\log(a/b)$; then (30) leads to

$$\frac{\sinh(x)}{x} = \frac{L(a,b)}{G(a,b)}, \qquad \cosh(px) = \left(\frac{M_p(a,b)}{G(a,b)}\right)^p. \tag{31}$$

From Theorem 1 and (31) we get the following.

Remark 6. Let $p, q \in (0,\infty)$; then the double inequality

$$\begin{aligned}\frac{1}{3p^2}M_p^p(a,b)G^{1-p}(a,b)+\left(1-\frac{1}{3p^2}\right)G(a,b)\\ < L(a,b) < \frac{1}{3q^2}M_q^q(a,b)G^{1-q}(a,b)\\ +\left(1-\frac{1}{3q^2}\right)G(a,b)\end{aligned} \tag{32}$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $p \leq \sqrt{15}/5$ and $q \geq 1$. In particular, the double inequality

$$\begin{aligned}\frac{5}{9}M_{\sqrt{15}/5}^{\sqrt{15}/5}(a,b)G^{1-\sqrt{15}/5}(a,b)+\frac{4}{9}G(a,b)\\ < L(a,b) < \frac{1}{3}A(a,b)+\frac{2}{3}G(a,b)\end{aligned} \tag{33}$$

holds for all $a, b > 0$ with $a \neq b$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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