Research Article

Jordan Type Inequalities for Hyperbolic Functions and Their Applications

Zhen-Hang Yang and Yu-Ming Chu

School of Mathematics and Computation Sciences, Hunan City University, Yiyang 413000, China

Correspondence should be addressed to Yu-Ming Chu; chuyuming2005@126.com

Received 3 August 2014; Accepted 2 September 2014

Academic Editor: Kehe Zhu

Copyright © 2015 Z.-H. Yang and Y.-M. Chu. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We present the best possible parameters \( p, q \in (0, \infty) \) such that the double inequality
\[
\left( \frac{1}{3} p^2 \right) \cosh(px) + 1 - \frac{1}{3} p^2 < \sinh(x)/x < \left( \frac{1}{3} q^2 \right) \cosh(qx) + 1 - \frac{1}{3} q^2
\]
holds for all \( x \in (0, \infty) \). As applications, some new inequalities for certain special function and bivariate means are found.

1. Introduction

The well known Jordan inequality [1] is given by
\[
\frac{2}{\pi} x < \sin(x) < x, \quad x \in \left( 0, \frac{\pi}{2} \right).
\]
(1)

During the past few years, the improvements, refinements, and generalizations for inequality (1) have attracted the attention of many researchers [2–13]. Recently, the hyperbolic counterpart and its generalizations have been the subject of intensive research.

Zhu [14] proved that the inequality
\[
\left( \frac{\sinh(x)}{x} \right)^q > p + (1 - p) \cosh(x)
\]
holds for all \( x > 0 \) if and only if \( q \geq 3(1 - p) \) if \( p \in (-\infty, 8/15] \cup (1, \infty) \).

In [3,15], Neuman and Sándor proved that
\[
\cosh^{4/3} \left( \frac{x}{2} \right) < \frac{\sinh(x)}{x} < \cosh^{3}(x)
\]
for all \( x > 0 \).

Klén et al. [5] proved that the double inequality
\[
\cosh^{1/4} (x) < \frac{\sinh(x)}{x} < \cosh^{1/2} (x)
\]
holds for all \( x \in (0, 1) \).

In [4], the authors proved that the double inequality
\[
\cosh^p (x) < \frac{\sinh(x)}{x} < \cosh^q (x)
\]
holds for all \( x \in (0, 1) \) if and only if \( p \leq 1/3 \) and \( q \geq \frac{\log(\sinh(1))}{\log(\cosh(1))} = 0.3721 \cdots \).

Zhu [16,17] proved that the inequalities
\[
(1 - \lambda) + \lambda \cosh^p (x) < \left( \frac{\sinh(x)}{x} \right)^p < (1 - \mu) + \mu \cosh^p (x),
\]
(2)

\[
\left( \frac{\sinh(x)}{x} \right)^q < (1 - \eta) + \eta \cosh^q (x),
\]
(3)

\[
\alpha + (1 - \alpha) e^{r(\text{tanh}(x) - 1)} < \left( \frac{\sinh(x)}{x} \right)^q < \beta + (1 - \beta) e^{r(\text{tanh}(x) - 1)}
\]
(4)

hold for all \( x > 0 \) if and only if \( \lambda \leq 0, \mu \geq 1/3, \eta \leq 1/3, \alpha \geq 1, \) and \( \beta \leq 1/2 \) if \( p \geq 4/5, q < 0, \) and \( r \geq 286/693 \).

Very recently, Yang [18] proved that the double inequality
\[
\left[ \cosh (px) \right]^{1/3 p^2} < \frac{\sinh(x)}{x} < \left[ \cosh (qx) \right]^{1/3 q^2}
\]
holds for all \( x > 0 \) if and only if \( p \geq \sqrt{5}/5 \) and \( q \leq 1/3 \).
The main purpose of this paper is to find the best possible parameters \( p, q \in (0, \infty) \) such that the double inequality \( \left( \frac{1}{3p^2} \right) \cosh(px) + 1 \geq \frac{\sinh(x)}{x} > \left( \frac{1}{3q^2} \right) \cosh(qx) + 1 \) holds for all \( x > 0 \) and present several new inequalities for certain special function and bivariate means.

2. Main Result

**Theorem 1.** Let \( p, q \in (0, \infty) \). Then the double inequality
\[
\frac{1}{3p^2} \cosh(px) + 1 - \frac{1}{3p^2} < \frac{\sinh(x)}{x} \left( \frac{1}{3q^2} \right) \cosh(qx) + 1 - \frac{1}{3q^2}
\]
holds for all \( x > 0 \) if and only if \( p \leq \sqrt{\frac{15}{5}} \) and \( q \geq 1 \).

**Proof.** Let \( \lambda > 0 \) and let the function \( f_\lambda \) be defined on \((0, \infty)\) by
\[
f_\lambda(x) = \frac{\sinh(x)}{x} - \left( \frac{1}{3\lambda^2} \cosh(\lambda x) + 1 - \frac{1}{3\lambda^2} \right)
\]
Then making use of power series expansions and (9) we get
\[
f_\lambda(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n+1)!} - \left( \frac{1}{3\lambda^2} \sum_{n=0}^{\infty} \frac{(\lambda x)^{2n}}{(2n)!} + 1 - \frac{1}{3\lambda^2} \right)
\]
\[
= \sum_{n=2}^{\infty} \frac{3 - (2n + 1)\lambda^{2n-2}}{3(2n + 1)!} x^{2n}.
\]
Let
\[
a_n(\lambda) = 3 - (2n + 1)\lambda^{2n-2}.
\]
Then
\[
a_n(1) = 3 - (2n + 1) < 0
\]
for all \( n \geq 2 \). Consider
\[
a_2 \left( \frac{\sqrt{15}}{5} \right) = 0, \quad a_3 \left( \frac{\sqrt{15}}{5} \right) = \frac{12}{25} > 0,
\]
\[
a_{n+1} \left( \frac{\sqrt{15}}{5} \right) - a_n \left( \frac{\sqrt{15}}{5} \right) = \frac{4}{5^{n-1}} (n - 1) > 0
\]
for all \( n \geq 2 \).

It follows from (13) that
\[
a_2 \left( \frac{\sqrt{15}}{5} \right) = 0, \quad a_n \left( \frac{\sqrt{15}}{5} \right) > 0
\]
for all \( n \geq 3 \).

Therefore, inequality (8) holds for all \( x > 0 \) with \( p = \sqrt{15}/5 \) and \( q = 1 \) follows from (9)–(12) and (14).

Next, we prove that \( p \leq \sqrt{15}/5 \) and \( q \geq 1 \) if inequality (8) holds for all \( x > 0 \).

If the first inequality of (8) holds for all \( x > 0 \), then from (9) and (10) we have
\[
\lim_{x \to 0^+} f_p(x) = \frac{3 - 5p^2}{360} \geq 0
\]
and \( p \leq \sqrt{15}/5 \).

If the second inequality of (8) holds for all \( x > 0 \), then it follows from (9) that
\[
\lim_{x \to +\infty} f_q(x) e^{qx} = \frac{1 - e^{-2x}}{2x} e^{(1-q)x} - \frac{1 + e^{-2px}}{6q^2} - \left( 1 - \frac{1}{3q^2} \right) e^{-qx} \leq 0
\]
We clearly see that \( \lim_{x \to +\infty} (f_q(x)/e^{qx}) = +\infty \) if \( q < 1 \). Therefore, \( q \geq 1 \) follows from (16).

**Remark 2.** It is not difficult to verify that the bound
\[
g_p(x) = \frac{1}{3p^2} \cosh(px) + 1 - \frac{1}{3p^2}
\]
given in Theorem 1 is strictly increasing with respect to \( p \) on \((0, \infty)\) for fixed \( x \in (0, \infty) \).

**Remark 3.** Let \( p = \sqrt{15}/5 > 3/4 > \sqrt{2}/2 > 2/3 > \sqrt{3}/3 \) and \( q = 1 < 2\sqrt{3}/3 \). Then Theorem 1 and Remark 2 lead to
\[
\cosh \left( \frac{\sqrt{3}}{3} x \right) < \frac{3}{4} \cosh \left( \frac{2x}{3} \right) \left( \frac{1}{4} \right) \cosh \left( \frac{\sqrt{2}x}{2} \right) + \frac{1}{3}
\]
\[
< \frac{16}{27} \cosh \left( \frac{3x}{4} \right) + \frac{11}{27} < \frac{5}{9} \cosh \left( \frac{\sqrt{15}x}{5} \right)
\]
\[
+ \frac{4}{9} < \frac{\sinh(x)}{x}
\]
\[
< \frac{1}{3} \cosh(x) + \frac{2}{3} < \frac{1}{2} \cosh^2 \left( \frac{\sqrt{3}}{3} x \right) + \frac{1}{2}
\]
for all \( x > 0 \).

3. Applications

It is well known that
\[
\int_0^\infty \frac{x}{x \sinh(x)} dx = \frac{1}{2} \psi' \left( \frac{1}{2} \right),
\]
where \( \psi' \) is the trigamma function defined by
\[
\psi'(x) = \int_0^\infty \frac{e^{-xt}}{1 - e^{-t}} dt.
\]
Let
\[
Sh(x) = \int_0^x \frac{t}{x \sinh(t)} dt.
\]
Then Remark 3 leads to
\[
\int_0^x \frac{3}{\cosh(t) + 2} \, dt < \text{Sh}(x) < \int_0^x \frac{9}{5 \cosh(\sqrt{15} t/5) + 4} \, dt
\]
(22)
for all \( x > 0 \).

From (19) and (22) we get the following.

**Remark 4.** For all \( x > 0 \) one has
\[
\sqrt{3} \log\left(\frac{e^x + 2 - \sqrt{3} e^x + 2 + \sqrt{3}}{2 + \sqrt{3}}\right) < \text{Sh}(x) < 2 \sqrt{\frac{15}{2} \arctan\left(\frac{5}{3} e^{\sqrt{15} x/5} + \frac{4}{3}\right) - \arctan(3)}.
\]
(23)
In particular, we have
\[
4.5620 \cdots = 2 \sqrt{3} \log\left(2 + \sqrt{3}\right) < \psi'(\frac{1}{2}) < 2 \sqrt{\frac{15}{2} \arctan\left(\frac{5}{3} e^{\sqrt{15} x/5} + \frac{4}{3}\right) - \arctan(3)}.
\]
(24)

For \( a, b > 0 \), the Schwab-Borchardt mean \( SB(a, b) \) [19–21] is given by
\[
SB(a, b) = \frac{\sqrt{b^2 - a^2}}{\arccos(a/b)} \quad (a < b),
\]
(25)
\[
SB(a, b) = a \quad (a = b),
\]
(26)
\[
SB(a, b) = \frac{\sqrt{a^2 - b^2}}{\cosh^{-1}(a/b)} \quad (a > b).
\]
(27)

Let \( a > b \) and let \( x = \cosh^{-1}(a/b) \). Then \( \cosh(x) = a/b \) and \( \sqrt{a^2 - b^2}/\cosh^{-1}(a/b) = b \sinh(x)/x \). It follows from Remark 3 and (26) that
\[
\left[\frac{16}{27} \cosh\left(\frac{3x}{4}\right) + \frac{11}{27}\right] b < SB(a, b) < \left[\frac{1}{3} \cosh(x) + \frac{2}{3} \right] b.
\]
(28)

Note that
\[
\left[\frac{1}{3} \cosh(x) + \frac{2}{3}\right] b = \frac{a + 2b}{3}.
\]
\[
\left[\frac{16}{27} \cosh\left(\frac{3x}{4}\right) + \frac{11}{27}\right] b = \frac{8b^{1/4}}{27} \left(\sqrt{2(a + b)(2a - b)^2 + 2b^{3/2}}\right)^{1/2} + \frac{11b}{27}.
\]
(29)

From (27) and (28) we get the following.

**Remark 5.** Let \( a > b > 0 \); then the Schwab-Borchardt mean \( SB(a, b) \) satisfies the double inequality
\[
\frac{8b^{1/4}}{27} \left(\sqrt{2(a + b)(2a - b)^2 + 2b^{3/2}}\right)^{1/2} + \frac{11b}{27} < SB(a, b) < \frac{a + 2b}{3}.
\]
(30)

Let \( p \in \mathbb{R} \) and let \( a, b > 0 \) with \( a \neq b \). Then the arithmetic mean \( A(a, b) \), logarithmic mean \( L(a, b) \), geometric mean \( G(a, b) \), and \( p \)th power mean \( M_p(a, b) \) are defined by
\[
A(a, b) = \frac{a + b}{2}, \quad L(a, b) = \frac{b - a}{\log b - \log a},
\]
\[
G(a, b) = \sqrt{ab},
\]
(30)
\[
M_p(a, b) = \left(\frac{a^p + b^p}{2}\right)^{1/p} \quad (p \neq 0),
\]
\[
M_0(a, b) = \sqrt{ab} = G(a, b).
\]
(31)

It is well known that \( M_p(a, b) \) is continuous and strictly increasing with respect to \( p \in \mathbb{R} \) for fixed \( a, b > 0 \); the main properties for the power mean are given in [22]. Recently, the arithmetic, logarithmic, geometric, and power means have been the subject of intensive research. In particular, many remarkable inequalities can be found in the literature [23–35].

From Theorem 1 and (31) we get the following.

**Remark 6.** Let \( p, q \in (0, \infty) \); then the double inequality
\[
\frac{1}{3p^2} M_p^p(a, b) G^{1-p} \left(\frac{a^p + b^p}{2}\right) + \left(1 - \frac{1}{3p^2}\right) G(a, b) < L(a, b) < \frac{1}{3q^2} M_q^q(a, b) G^{1-q} \left(\frac{a^q + b^q}{2}\right) + \left(1 - \frac{1}{3q^2}\right) G(a, b)
\]
(32)
holds for all \( a, b > 0 \) with \( a \neq b \) if and only if \( p \leq \sqrt{15}/5 \) and \( q \geq 1 \). In particular, the double inequality
\[
\frac{5}{9} M_{\sqrt{15}/5}^{\sqrt{15}/5}(a, b) G^{1-\sqrt{15}/5} \left(\frac{a^{\sqrt{15}/5} + b^{\sqrt{15}/5}}{2}\right) + \frac{4}{9} G(a, b)
\]
(33)
holds for all \( a, b > 0 \) with \( a \neq b \).

**Conflict of Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.

**Acknowledgments**

This research was supported by the Natural Science Foundation of China under Grants 61374086 and 11171307 and the Natural Science Foundation of Zhejiang Province under Grant LY13A010004.
References

Submit your manuscripts at
http://www.hindawi.com