Research Article

Solitary Waves of the Schrödinger Lattice System with Nonlinear Hopping

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Received 2 March 2015; Accepted 15 April 2015

Academic Editor: Dumitru Motreanu

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This paper is concerned with the nonlinear Schrödinger lattice with nonlinear hopping. Via variation approach and the Nehari manifold argument, we obtain two types of solution: periodic ground state and localized ground state. Moreover, we consider the convergence of periodic solutions to the solitary waves.

1. Introduction

In the last decades, a great deal of attention has been paid to study the existence of solitary waves for the lattice systems. They play a role in lots of physical models, such as nonlinear waves in crystals and arrays of coupled optical waveguides. The discrete nonlinear Schrödinger lattice is one of the most famous models in mathematics and physics. The existence and properties of discrete breathers (periodic in time and spatially localized) in discrete nonlinear Schrödinger lattice have been considered in a number of studies. One can see [1–9] and references therein.

In the present paper, we consider a variant of the discrete nonlinear Schrödinger lattice as follows:

$$i\dot{\psi}_l + (\Delta_d \psi)_l + \alpha \psi_l \sum_{j=1}^d (\mathcal{T}_j \psi)_l + \beta |\psi_l|^{2\sigma} \psi_l = 0, \quad (1)$$

$$l \in \mathbb{Z}^d,$$

where $\alpha, \beta \in \mathbb{R}$, $(\Delta_d \psi)_l = \sum_{m \in N_d} \psi_m - \psi_l$, and the nonlinear operator $\mathcal{T}$ is defined by

$$(\mathcal{T}_j \psi)_l = \left| \psi_{l_1, \dots, l_{j-1}, l_j+1, l_{j+1}, \dots, l_d} \right|^2 + \left| \psi_{l_1, \dots, l_{j-1}, l_j-1, l_{j+1}, \dots, l_d} \right|^2. \quad (2)$$

Here, N_d denotes the set of the nearest neighbors of the point $l \in \mathbb{Z}^d$.

Note that, for $\alpha = 0, \beta \neq 0$, (1) recovers the classical nonlinear Schrödinger lattice. For $\alpha \neq 0, \beta \neq 0$, (1) denotes the Schrödinger lattice with nonlinear hopping.

There has been a lot of interest in this equation as the modeling of waveguide arrays. Also, nonlinear hopping terms appear from Klein-Gordon and Fermi-Pasta-Ulam chains of anharmonic oscillators coupled with anharmonic intersite potentials or mixed FPU/KG chains. The generalized DNLS system with the nonlinear hopping terms has been derived as a perturbation of the integrable Ablowitz-Ladik system, by the rotating wave approximation on the FPU chain.

Karachalios et al. discuss the energy thresholds in the setting of DNLS lattice with nonlinear hopping terms by using fixed point method. The numerical results have also been obtained in their paper [10]. However, the Dirichlet boundary condition in Section 2 of their paper is not suitable. Since $\psi_{k+1} = \psi_{k+2} = 0$, we have $\psi_k = 0$ from the equation. With similar argument, the solution of lattice system under the Dirichlet boundary conditions is trivial. Our aim is to investigate the existence of nontrivial solitary waves for the infinite dimensional lattice (1). We firstly consider the k -periodic problem. In the paper of Karachalios et al., they “expect that the variational approach can be applied in the case of periodic boundary conditions, but the details have to be checked.” We obtain the nontrivial periodic solution by Nehari manifolds argument [11]. In Section 3, we obtain the solitary waves for the infinite dimensional lattice (1) by the concentration compactness method [12].

Here, we only consider the case of one dimension. That is, $d = 1$. The case of $d > 1$ is similar. It notes that, for classical nonlinear Schrödinger lattice, Weinstein discusses a connection among the dimensionality, the degree of the nonlinearity, and the existence of the excitation threshold [6]. Weinstein prove that if the degree of the nonlinearity σ satisfies $\sigma \geq 2/d$ where d is the dimension, then there exists a ground state for the total power which is greater than the excitation threshold and there is no ground state for the total power which is less than the excitation threshold. It is interesting that the power of solitary waves for the infinite dimensional lattice (1) always has a lower bound $\sigma \geq 1$ from our arguments.

The paper is organized as follows. In Section 2, we firstly consider the k -periodic problem. Note that the dimension in space variable is finite. We obtain the nontrivial periodic solution by Nehari manifolds argument. The existence of solitary waves is more complex. In Section 3, we follow the idea of [13–16] to obtain the solitary waves. The key point is to show the norms of periodic ground state are bounded. It is based on the concentration compactness. In Section 4, we concern the convergence of periodic ground states to solitary waves.

2. Periodic Ground State

In this paper, we consider the following equation:

$$i\dot{\psi}_l + (\Delta_d \psi)_l + \alpha \psi_l (|\psi_{l+1}|^2 + |\psi_{l-1}|^2) + \beta |\psi_l|^{2\sigma} \psi_l = 0, \quad (3)$$

$$l \in \mathbb{Z},$$

where $\sigma \geq 1$.

To obtain breather, we seek the solution

$$\psi_l = e^{-i\omega t} u_l. \quad (4)$$

The equation of u_l is

$$\omega u_l + (\Delta_d u)_l + \alpha u_l (|u_{l+1}|^2 + |u_{l-1}|^2) + \beta |u_l|^{2\sigma} u_l = 0, \quad (5)$$

$$l \in \mathbb{Z}.$$

Actually, we give the proofs only in the focusing case with $\alpha, \beta > 0$ and $\omega < 0$. For the defocusing case with $\alpha, \beta < 0$ and $\omega > 4$, the argument is similar. Here, we omit the details.

In this section, we prove the existence of k -periodic solution which satisfies

$$u_{l+k} = u_l, \quad \text{for } l \in \mathbb{Z}, \quad (6)$$

where $k > 2$ is an integer.

Let

$$P_k = \left\{ l \in \mathbb{Z} \mid -\left\lfloor \frac{k}{2} \right\rfloor \leq l \leq k - \left\lfloor \frac{k}{2} \right\rfloor - 1 \right\}. \quad (7)$$

Consider the Banach space l_k^p with norm:

$$\|u\|_{l_k^p}^p = \sum_{l \in P_k} |u_l|^p. \quad (8)$$

We mention that

$$\|u\|_{l_k^q} \leq \|u\|_{l_k^p}, \quad 1 \leq p \leq q \leq \infty. \quad (9)$$

Denote that $\langle \cdot, \cdot \rangle_k$ is natural inner product in l_k^2 .

Define the functional

$$J_k(u) = \langle -\Delta_d u, u \rangle_k - \omega \langle u, u \rangle_k - \alpha \sum_{l \in P_k} |u_l|^2 |u_{l+1}|^2 - \frac{\beta}{\sigma + 1} \sum_{l \in P_k} |u_l|^{2\sigma+2} \quad (10)$$

and Nehari manifold

$$\mathcal{N}_k = \left\{ u \in l_k^2 \mid I_k(u) = \langle -\Delta_d u, u \rangle_k - \omega \langle u, u \rangle_k - \alpha \sum_{l \in P_k} |u_l|^2 (|u_{l+1}|^2 + |u_{l-1}|^2) - \beta \sum_{l \in P_k} |u_l|^{2\sigma+2} = 0, u \neq 0 \right\}. \quad (11)$$

Then, the minimizer of the constrained variational problem

$$m_k = \inf_{u \in \mathcal{N}_k} \{J_k(u)\} \quad (12)$$

is the nontrivial periodic solution of (5). We mention that the minimizer is called a periodic ground state.

Note that

$$0 \leq \langle \Delta_d u, u \rangle_k \leq 4 \|u\|_{l_k^2}^2, \quad \text{for } u \in l_k^2. \quad (13)$$

We want to obtain the periodic solution with prescribed frequency $\omega < 0$. With the Nehari manifold approach, we have one of the main results.

Theorem 1. Assume that the frequency $\omega < 0$ and $\alpha, \beta > 0$. There exists a positive k -periodic ground state u^k for (5).

Lemma 2. Under the assumptions of Theorem 1, the Nehari manifold $\mathcal{N}_k$ is nonempty.

Proof. For $t \geq 0$ and $u \neq 0$, define

$$\begin{aligned} \rho(t) &= I_k(\sqrt{t}u) \\ &= t \left(\langle -\Delta_d u, u \rangle_k - \omega \langle u, u \rangle_k \right) \\ &\quad - 2t^2 \alpha \sum_{l \in P_k} |u_l|^2 |u_{l+1}|^2 - t^{\sigma+1} \beta \sum_{l \in P_k} |u_l|^{2\sigma+2}. \end{aligned} \quad (14)$$

Then,

$$\begin{aligned} \rho'(t) &= \langle -\Delta_d u, u \rangle_k - \omega \langle u, u \rangle_k \\ &\quad - 4t\alpha \sum_{l \in P_k} |u_l|^2 |u_{l+1}|^2 - (\sigma + 1)t^\sigma \beta \sum_{l \in P_k} |u_l|^{2\sigma+2}. \end{aligned} \quad (15)$$

It holds that $\rho'(t) > 0$ for $t > 0$ small enough.

Observe that

$$\begin{aligned} \rho''(t) &= -4\alpha \sum_{l \in P_k} |u_l|^2 |u_{l+1}|^2 - (\sigma + 1) \sigma t^{\sigma-1} \beta \sum_{l \in P_k} |u_l|^{2\sigma+2} \\ &< 0. \end{aligned} \quad (16)$$

Therefore, $\rho(t)$ admits a unique zero point $t^* \in (0, +\infty)$. This implies $\sqrt{t^*}u \in \mathcal{N}_k$. It completes the proof. $\square$

Lemma 3. *Under the assumptions of Theorem 1, for $u \in \mathcal{N}_k$, the function $J_k(\sqrt{t}u)$ has a unique critical point at $t = 1$, which is a global maximum.*

Proof. For $t > 0$ and $u \in \mathcal{N}_k$, we get

$$\begin{aligned} \theta(t) &= J_k(\sqrt{t}u) \\ &= t(\langle -\Delta_d u, u \rangle_k - \omega \langle u, u \rangle_k) \\ &\quad - t^2 \alpha \sum_{l \in P_k} |u_l|^2 |u_{l+1}|^2 - \frac{t^{\sigma+1} \beta}{\sigma + 1} \sum_{l \in P_k} |u_l|^{2\sigma+2}. \end{aligned} \quad (17)$$

Then,

$$\begin{aligned} \theta'(t) &= \langle -\Delta_d u, u \rangle_k - \omega \langle u, u \rangle_k \\ &\quad - 2t\alpha \sum_{l \in P_k} |u_l|^2 |u_{l+1}|^2 - t^\sigma \beta \sum_{l \in P_k} |u_l|^{2\sigma+2}. \end{aligned} \quad (18)$$

It holds that $\theta'(t) > 0$ for $t > 0$ small enough.

Note that

$$\theta''(t) = -2\alpha \sum_{l \in P_k} |u_l|^2 |u_{l+1}|^2 - \sigma t^{\sigma-1} \beta \sum_{l \in P_k} |u_l|^{2\sigma+2} < 0. \quad (19)$$

We can see that $t = 1$ is the unique maximum point of $\theta(t)$. This implies the proof. $\square$

Assume that u^k is the k -periodic solution of (5); we have

$$\begin{aligned} |\omega| \|u^k\|_{l_k^2}^2 &\leq \langle -\Delta_d u^k, u^k \rangle_k - \omega \langle u^k, u^k \rangle_k \\ &= 2\alpha \sum_{l \in P_k} |u_l^k|^2 |u_{l+1}^k|^2 + \beta \sum_{l \in P_k} |u_l^k|^{2\sigma+2} \\ &\leq \|u^k\|_{l_k^2}^2 \left(\beta \|u^k\|_{l_k^2}^{2\sigma} + 2\alpha \|u^k\|_{l_k^2}^2 \right). \end{aligned} \quad (20)$$

Therefore

$$\|u^k\|_{l_k^2} \geq C_1 > 0, \quad (21)$$

where C_1 is the unique positive solution of equation

$$\beta x^{2\sigma} + 2\alpha x^2 + \omega = 0. \quad (22)$$

Observe that C_1 is independent of k .

Thus, we get a lower bound of the power of the periodic solutions.

Theorem 4. *The power of the periodic solution must be greater than C_1 .*

Lemma 5. *Under the assumptions of Theorem 1, $J_k(u)$ is bounded below for all $u \in \mathcal{N}_k$.*

Proof. Let $u \in \mathcal{N}_k$. From the argument in (20) and (21), there exist $l_0 \in P_k$ and a positive constant C_2 such that

$$|u_{l_0}| > C_2 > 0. \quad (23)$$

Therefore,

$$\begin{aligned} J_k(u) &= \alpha \sum_{l \in P_k} |u_l|^2 |u_{l+1}|^2 + \frac{\sigma\beta}{\sigma+1} \sum_{l \in P_k} |u_l|^{2\sigma+2} \\ &> \frac{\sigma\beta}{\sigma+1} C_2^{2\sigma+2}. \end{aligned} \quad (24)$$

It completes the proof. $\square$

Lemma 6. *Under the assumptions of Theorem 1, the minimizer of the constrained variational problem $m_k = \inf_{u \in \mathcal{N}_k} \{J_k(u)\}$ could be attained.*

Proof. Assume that $\{u^n\}$ is a minimizing sequence. We can see that there exists a constant $M > 0$ such that

$$\max J_k(u^n) \leq M. \quad (25)$$

Thus,

$$|\omega| \|u^n\|_{l_k^2}^2 \leq \langle -\Delta_d u^n, u^n \rangle_k - \omega \langle u^n, u^n \rangle_k \leq M. \quad (26)$$

It holds that $\|u^n\|_{l_k^\infty}$ is bounded.

Note that P_k is finite dimensional space. Passing to a subsequence, there exists u^k such that $u^{n_j} \rightarrow u^k$ in l_k^2 . Since the set l_k^2 is closed and the functional J_k is continuous, we obtain that $u^k \in \mathcal{N}_k$ and $J_k(u^k) = m_k$. $\square$

By Lagrange multiplier method, there exists some constant λ such that

$$\begin{aligned} &\lambda \left(2 \langle -\Delta_d u^k, v \rangle_k - 2\omega \langle u^k, v \rangle_k \right. \\ &\quad - 4\alpha \sum_{l \in P_k} u_l^k v_l (|u_{l+1}^k|^2 + |u_{l-1}^k|^2) - (2\sigma + 2) \beta \\ &\quad \cdot \sum_{l \in P_k} |u_l^k|^{2\sigma} u_l^k v_l \left. \right) + 2 \langle -\Delta_d u^k, v \rangle_k - 2\omega \langle u^k, v \rangle_k \\ &\quad - 2\alpha \sum_{l \in P_k} u_l^k v_l (|u_{l+1}^k|^2 + |u_{l-1}^k|^2) - 2\beta \sum_{l \in P_k} |u_l^k|^{2\sigma} u_l^k v_l \\ &= 0. \end{aligned} \quad (27)$$

Choose $v = u^k$. Note that $u^k \in \mathcal{N}_k$; it holds that

$$\begin{aligned} &\lambda \left(-2\alpha \sum_{l \in P_k} |u_l^k|^2 (|u_{l+1}^k|^2 + |u_{l-1}^k|^2) - \sigma\beta \sum_{l \in P_k} |u_l^k|^{2\sigma+2} \right) \\ &= 0. \end{aligned} \quad (28)$$

We have $\lambda = 0$. It implies that u^k is a nontrivial solution of (3).

Now, we prove that u^k is positive. Observe that

$$\langle -\Delta_d |u|, |u| \rangle - \omega \langle |u|, |u| \rangle \leq \langle -\Delta_d u, u \rangle - \omega \langle u, u \rangle. \quad (29)$$

Since that u^k is the nontrivial solution, then, there exists $t^{**} \in (0, 1]$ such that $\sqrt{t^{**}}|u^k| \in \mathcal{N}_k$. It is obvious that

$$J_k(\sqrt{t^{**}}|u^k|) \leq m_k. \quad (30)$$

Hence $J_k(\sqrt{t^{**}}|u^k|) = m_k$. We can assume that $u^k = \sqrt{t^{**}}|u^k|$.

Let $G(n, m)$ be the Green function of $-\Delta_d - \omega$. From [17], we have $G(n, m) > 0$ for $\omega < 0$. It obtains that

$$\begin{aligned} u_n^k &= \sum_{l \in \mathbb{Z}} G(n, l) \left(\alpha u_l^k \left(|u_{l+1}^k|^2 + |u_{l-1}^k|^2 \right) + \beta |u_l^k|^{2\sigma} u_l^k \right), \quad (31) \\ n &\in \mathbb{Z}. \end{aligned}$$

Since u^k is nonnegative, this holds $u_n^k > 0$ for all $n \in \mathbb{Z}$. It completes the proof of Theorem 1.

3. Localized Ground State

Here, we give some notations. Define the functional

$$\begin{aligned} J(u) &= \langle -\Delta_d u, u \rangle - \omega \langle u, u \rangle \\ &\quad - \alpha \sum_{l \in \mathbb{Z}} |u_l|^2 |u_{l+1}|^2 - \frac{\beta}{\sigma + 1} \sum_{l \in \mathbb{Z}} |u_l|^{2\sigma+2} \end{aligned} \quad (32)$$

and Nehari manifold

$$\begin{aligned} \mathcal{N} &= \left\{ u \in l^2 \mid I(u) = \langle -\Delta_d u, u \rangle - \omega \langle u, u \rangle \right. \\ &\quad \left. - \alpha \sum_{l \in \mathbb{Z}} |u_l|^2 (|u_{l+1}|^2 + |u_{l-1}|^2) - \beta \sum_{l \in \mathbb{Z}} |u_l|^{2\sigma+2} \right. \\ &\quad \left. = 0, u \neq 0 \right\}, \end{aligned} \quad (33)$$

where $\langle \cdot, \cdot \rangle$ is natural inner product in l^2 . Thus, we can see that the minimizer of the constrained variational problem

$$m = \inf_{u \in \mathcal{N}} \{J(u)\} \quad (34)$$

is the nontrivial solitary waves of (5). We call this minimizer a localized ground state. Similar to Lemmas 2 and 3, the results are obtained by replacing $J_k(u)$, $I_k(u)$ with $J(u)$, $I(u)$.

In this section, to obtain the localized ground state u satisfying

$$\lim_{l \rightarrow \infty} |u_l| = 0, \quad (35)$$

we follow the idea of [13]. We want to pass to the limit as $k \rightarrow \infty$. The key point is the following result.

Lemma 7. *Under the assumptions of Theorem 1, let u^k be the k -periodic solution. Therefore, the sequences m_k and $\|u^k\|_{l_k^2}$ are bounded.*

Proof. First, we concern the sequences m_k which are bounded. From similar argument of Lemma 2, this holds that, for any given $u \in l^2$, there exists t' such $I(\sqrt{t'}u) < 0$. Since the sequences with finite support are dense in l^2 , therefore, there exists $\tilde{u}$ with finite support such that $I(\tilde{u}) < 0$. It obtains that there exists t'' such that $I(\sqrt{t''}\tilde{u}) = 0$. For k large enough, we have $\text{supp } \sqrt{t''}\tilde{u} \subset P_k$. We can get $\tilde{v}^k \in l_k^2$ such that $\tilde{v}_l^k = \sqrt{t''}\tilde{u}_l$ for $l \in P_k$. This holds that $I_k(\tilde{v}^k) = I(\sqrt{t''}\tilde{u}) = 0$. And $m_k \leq J_k(\tilde{v}^k) = J(\sqrt{t''}\tilde{u})$ is bounded.

Second, we prove that $\|u^k\|_{l_k^2}$ is uniformly bounded. Assume that $\|u^k\|_{l_k^2}$ is unbounded. Passing to a subsequence which is still denoted by itself, we have $\|u^k\|_{l_k^2} \rightarrow \infty$ for $k \rightarrow \infty$. Let $v^k = u^k / \|u^k\|_{l_k^2}$. One of the following should hold:

- (i) v^k is vanishing; that is, $\|v^k\|_{l^\infty} \rightarrow 0$;
- (ii) v^k is not vanishing; passing to a subsequence which is still denoted by itself, there exists $\delta > 0$ and $b^k \in \mathbb{Z}$ such that $|v_{b^k}^k| > \delta$ for all k .

Now, we rule out case (i). This holds that

$$\begin{aligned} 0 &= \frac{I_k(u^k)}{\|u^k\|_{l_k^2}^2} \\ &= \langle (-\Delta_d - \omega) v^k, v^k \rangle_k \\ &\quad - 2\alpha \sum_{l \in P_k} |v_l^k|^2 |u_{l-1}^k|^2 - \beta \sum_{l \in P_k} |v_l^k|^2 |u_l^k|^{2\sigma}. \end{aligned} \quad (36)$$

Hence,

$$\begin{aligned} |\omega| &= |\omega| \|v^k\|_{l_k^2} \leq \langle (-\Delta_d - \omega) v^k, v^k \rangle_k \\ &= 2\alpha \sum_{l \in P_k} |v_l^k|^2 |u_{l-1}^k|^2 + \beta \sum_{l \in P_k} |v_l^k|^2 |u_l^k|^{2\sigma}. \end{aligned} \quad (37)$$

Assume that

$$\begin{aligned} A_k &= \{l \in P_k \mid |u_l^k| < M_0\}, \\ B_k &= P_k \setminus A_k, \end{aligned} \quad (38)$$

where $M_0 > 0$ is a constant which is defined below.

Let M_0 be small enough such that

$$\begin{aligned} &2\alpha \sum_{l \in A_k} |v_l^k|^2 |u_{l-1}^k|^2 + \beta \sum_{l \in A_k} |v_l^k|^2 |u_l^k|^{2\sigma} \\ &< 2\alpha M_0^2 \sum_{l \in A_k} |v_{l+1}^k|^2 + \beta M_0^{2\sigma} \sum_{l \in A_k} |v_l^k|^2 \leq \frac{|\omega|}{2}. \end{aligned} \quad (39)$$

Combine with (37), we have

$$\frac{|\omega|}{2} \leq \liminf_{k \rightarrow \infty} 2\alpha \sum_{l \in B_k} |v_{l+1}^k|^2 |u_l^k|^2 + \beta \sum_{l \in B_k} |v_l^k|^2 |u_l^k|^{2\sigma}. \quad (40)$$

From the argument above, there exists a constant $M' > 0$ such that

$$m_k = \sum_{l \in P_k} \alpha |u_l^k|^2 |u_{l+1}^k|^2 + \frac{\sigma\beta}{\sigma+1} \sum_{l \in P_k} |u_l^k|^{2\sigma+2} < M'. \quad (41)$$

Hence, $\|u^k\|_{l_k^{2\sigma+2}}$ is uniformly bounded.

By Hölder's inequality, we have

$$\begin{aligned} \sum_n |v_n|^2 |u_n|^2 &\leq \|u\|_{l^{2\sigma+2}}^2 \|v\|_{l^{(2\sigma+2)/\sigma}}^2, \\ \sum_n |v_n|^2 |u_n|^{2\sigma} &\leq \|u\|_{l^{2\sigma+2}}^{2\sigma} \|v\|_{l^{2\sigma+2}}^2, \end{aligned} \quad (42)$$

$$\|v\|_{l^p} \leq \|v\|_{l^\infty}^{(p-2)/p} \|v\|_{l^2}^{2/p}, \quad \text{for } p > 2.$$

Since v^k is vanishing, we can see that

$$\lim_{k \rightarrow \infty} \|v^k\|_{l_k^p} = 0, \quad \text{for } p > 2. \quad (43)$$

It concludes that

$$\liminf_{k \rightarrow \infty} 2\alpha \sum_{B_k} |v_{l+1}^k|^2 |u_l^k|^2 + \beta \sum_{B_k} |v_l^k|^2 |u_l^k|^{2\sigma} \rightarrow 0, \quad (44)$$

as $k \rightarrow \infty$.

It contradicts with (40).

Let us rule out the nonvanishing case. By the discrete translation invariance, we can assume that $b_k = 0$. Since $\|v^k\|_{l_k^2} = 1$, there exists $v = \{v_l\}$ such that $v_l^k \rightarrow v_l$ for all $l \in \mathbb{Z}$. It is obvious that $v \in l^2$, $\|v\|_{l^2} \leq 1$, and $|v_0| \geq \delta$.

Since $|v_0| \neq 0$, then $|u_0^k| \rightarrow \infty$, as $k \rightarrow \infty$. On the other hand, we have

$$\frac{\sigma\beta}{\sigma+1} |u_0^k|^{2\sigma+2} \leq m_k \leq M'. \quad (45)$$

It is a contradiction. $\square$

Theorem 8. Assume that the frequency $\omega < 0$ and $\alpha, \beta > 0$. There exists a positive localized ground state u for (5).

Proof. Let $u^k \in l_k^2$ be a periodic ground state. From Lemma 7, the sequence $\|u^k\|_{l_k^2}$ is bounded. Therefore, u^k is either vanishing or nonvanishing. In the case of vanishing, we have $\lim_{k \rightarrow \infty} \|u^k\|_{l_k^p} \rightarrow 0$, for $p > 2$. This holds that

$$\begin{aligned} |\omega| \|u^k\|_{l_k^2}^2 &\leq \langle (-\Delta_d - \omega) u^k, u^k \rangle_k \\ &= 2\alpha \sum_{l \in P_k} |u_l^k|^2 |u_{l-1}^k|^2 + \beta \sum_{l \in P_k} |u_l^k|^{2\sigma+2} \\ &\leq 2\alpha \sum_{l \in P_k} |u_l^k|^4 + \beta \sum_{l \in P_k} |u_l^k|^{2\sigma+2} \rightarrow 0. \end{aligned} \quad (46)$$

It is a contradiction.

Thus, the sequence u^k is nonvanishing. By the discrete translation invariance, we assume that $|u_0^k| \geq \delta > 0$. There exists $u = \{u_l\}$ such that $u_l^k \rightarrow u_l$ for all $l \in \mathbb{Z}$. It is obvious that $u \in l^2$ and $u \neq 0$. Also, we obtain that u is a nontrivial solution for (5) by pointwise limits. Now, we want to prove that u is a localized ground state.

Let L be a positive integer such that

$$\begin{aligned} \liminf_{k \rightarrow \infty} J_k(u^k) &\geq \liminf_{k \rightarrow \infty} \alpha \sum_{-L \leq l \leq L} |u_l^k|^2 |u_{l-1}^k|^2 + \frac{\sigma\beta}{\sigma+1} \sum_{-L \leq l \leq L} |u_l^k|^{2\sigma+2} \\ &\geq \alpha \sum_{-L \leq l \leq L} |u_l|^2 |u_{l-1}|^2 + \frac{\sigma\beta}{\sigma+1} \sum_{-L \leq l \leq L} |u_l|^{2\sigma+2}. \end{aligned} \quad (47)$$

Let $L \rightarrow \infty$; it obtains that

$$\liminf_{k \rightarrow \infty} J_k(u^k) \geq J(u) \geq m, \quad (48)$$

$$\liminf_{k \rightarrow \infty} m_k \geq m.$$

For any given $\epsilon > 0$, let $u' \in N$ such that

$$J(u') = \alpha \sum_{l \in \mathbb{Z}} |u_l'|^2 |u_{l-1}'|^2 + \frac{\sigma\beta}{\sigma+1} \sum_{l \in \mathbb{Z}} |u_l'|^{2\sigma+2} < m + \epsilon. \quad (49)$$

Choose $t_1 > 1$ such that

$$\begin{aligned} J(t_1 u') &< m + \epsilon, \\ I(t_1 u') &< 0. \end{aligned} \quad (50)$$

From density argument, there exists a finite supported sequence $v = \{v_l\}$ sufficiently close to $t_1 u'$ in l^2 such that

$$I(v) < 0,$$

$$\alpha \sum_{l \in \mathbb{Z}} |v_l|^2 |v_{l-1}|^2 + \frac{\sigma\beta}{\sigma+1} \sum_{l \in \mathbb{Z}} |v_l|^{2\sigma+2} < m + \epsilon. \quad (51)$$

Thus, there exists $t_2 \in (0, 1)$ such that $t_2 v \in \mathcal{N}$ and $J(t_2 v) < m + \epsilon$.

Choose k large enough such that P_k contains the support of v . Let $v^k \in l_k^2$ such that $v_l^k = t_2 v_l$ for $l \in P_k$. It concludes that

$$\begin{aligned} I_k(v^k) &= I(t_2 v), \\ J_k(v^k) &= J(t_2 v) < m + \epsilon. \end{aligned} \quad (52)$$

It implies

$$\limsup_{k \rightarrow \infty} m_k < m + \epsilon. \quad (53)$$

Combining with (48), we have $\lim_{k \rightarrow \infty} m_k = m$. It completes the proof. $\square$

Remark 9. With similar argument in (20) and (21), the power of the localized ground state has a lower bound $C_1 > 0$.

4. Global Convergence

Theorem 10. Let $u^k \in l_k^2$ be the periodic ground state to (5). Then, there exists a ground state $u \in l^2$ such that u^k is strongly convergent to u in l_k^2 after some discrete translation.

Proof. Let $u^k \in l_k^2$ be the periodic ground state and $b_k \in \mathbb{Z}$. Now, we consider a translation

$$u_l^k = u_{l+b_k}^k. \quad (54)$$

From the argument above, we can assume that $u_l^k \rightarrow u_l$ for all $l \in \mathbb{Z}$ where u is a ground state. We want to prove that $\|u^k - u\|_{l_k^2}$ is convergent to 0 as $k \rightarrow \infty$. First, it concludes that

$$\begin{aligned} J_k(u^k - u) &\rightarrow 0, \\ I_k(u^k - u) &\rightarrow 0, \\ \text{as } k &\rightarrow \infty. \end{aligned} \quad (55)$$

Indeed,

$$\begin{aligned} J_k(u^k - u) &= \langle -\Delta_d(u^k - u), (u^k - u) \rangle \\ &\quad - \omega \langle (u^k - u), (u^k - u) \rangle \\ &\quad - \alpha \sum_{l \in P_k} |u_l^k - u_l|^2 |u_{l+1}^k - u_{l+1}|^2 \\ &\quad - \frac{\beta}{\sigma + 1} \sum_{l \in P_k} |u_l^k - u_l|^{2\sigma+2} \\ &= J_k(u^k) - J_k(u) \\ &\quad - 2 \langle -\Delta_d(u^k - u), u \rangle \\ &\quad - 2\omega \langle (u^k - u), u \rangle \\ &\quad - \alpha \sum_{l \in P_k} |u_l^k - u_l|^2 |u_{l+1}^k - u_{l+1}|^2 \\ &\quad - \frac{\beta}{\sigma + 1} \sum_{l \in P_k} |u_l^k - u_l|^{2\sigma+2} \\ &\quad + \alpha \sum_{l \in P_k} |u_l^k|^2 |u_{l+1}^k|^2 \\ &\quad + \frac{\beta}{\sigma + 1} \sum_{l \in P_k} |u_l^k|^{2\sigma+2} \\ &\quad - \alpha \sum_{l \in P_k} |u_l|^2 |u_{l+1}|^2 \\ &\quad - \frac{\beta}{\sigma + 1} \sum_{l \in P_k} |u_l|^{2\sigma+2}. \end{aligned} \quad (56)$$

Similar to the argument in [13], it obtains that $J_k(u^k) \rightarrow J(u) = m$ and $-2\langle -\Delta_d(u^k - u), u \rangle - 2\omega \langle (u^k - u), u \rangle \rightarrow 0$, as $k \rightarrow \infty$.

Since $\|u^k\|_{l_k^2}$ and $\|u\|_{l^2}$ are bounded. For any given $\epsilon > 0$, there exists $M > 0$ such that $\sum_{|l| \geq M} |u_l|^2 < \epsilon$. Therefore, we have

$$\begin{aligned} &\alpha \sum_{l \in P_k, |l| \geq M} |u_l^k|^2 |u_{l+1}^k|^2 \\ &\quad - \alpha \sum_{l \in P_k, |l| \geq M} |u_l^k - u_l|^2 |u_{l+1}^k - u_{l+1}|^2 \\ &\leq \alpha \sum_{l \in P_k, |l| \geq M} (|u_l^k|^2 - |u_l^k - u_l|^2) |u_{l+1}|^2 \\ &\quad + \alpha \sum_{l \in P_k, |l| \geq M} |u_l^k - u_l|^2 (|u_{l+1}^k|^2 - |u_{l+1}^k - u_{l+1}|^2) \\ &\leq \alpha \left(2 \|u^k\|_{l_k^2}^2 \|u\|_{l^2}^2 + \|u\|_{l^2}^2 \right) \sum_{l \in P_k, |l| \geq M} |u_{l+1}|^2 \\ &\quad + \alpha \left(\|u^k\|_{l_k^2}^2 + \|u\|_{l^2}^2 \right) \left(2 \|u^k\|_{l_k^2}^2 + \|u\|_{l^2}^2 \right) \\ &\quad \cdot \left(\sum_{l \in P_k, |l| \geq M} |u_{l+1}|^2 \right)^{1/2} \lesssim_{M', \|u\|_{l^2}^2} \epsilon \end{aligned} \quad (57)$$

for k large enough.

Also, we have

$$\begin{aligned} &-\frac{\beta}{\sigma + 1} \sum_{l \in P_k, |l| \geq M} |u_l^k - u_l|^{2\sigma+2} \\ &\quad + \frac{\beta}{\sigma + 1} \sum_{l \in P_k, |l| \geq M} |u_l^k|^{2\sigma+2} \lesssim_{M', \|u\|_{l^2}^2} \epsilon \end{aligned} \quad (58)$$

for k large enough.

On the other hand, from the point limits, we have that

$$\begin{aligned} &-\alpha \sum_{l \in P_k, |l| < M} |u_l^k - u_l|^2 |u_{l+1}^k - u_{l+1}|^2 \\ &\quad - \frac{\beta}{\sigma + 1} \sum_{l \in P_k, |l| < M} |u_l^k - u_l|^{2\sigma+2} \\ &\quad + \alpha \sum_{l \in P_k, |l| < M} |u_l^k|^2 |u_{l+1}^k|^2 \\ &\quad + \frac{\beta}{\sigma + 1} \sum_{l \in P_k, |l| < M} |u_l^k|^{2\sigma+2} \\ &\quad - \alpha \sum_{l \in P_k, |l| < M} |u_l|^2 |u_{l+1}|^2 - \frac{\beta}{\sigma + 1} \sum_{l \in P_k, |l| < M} |u_l|^{2\sigma+2} \end{aligned} \quad (59)$$

$< \epsilon$

for k large enough.

Combine with Hölder inequality, $J^k(u'^k - u) \rightarrow 0$ holds.

With similar argument, we obtain $I^k(u'^k - u) \rightarrow 0$. Therefore,

$$\begin{aligned} J^k(u'^k - u) - I^k(u'^k - u) \\ = \alpha \sum_{l \in P_k} |u_l'^k - u_l|^2 |u_{l-1}'^k - u_{l-1}|^2 + \frac{\sigma\beta}{\sigma+1} \\ \cdot \sum_{l \in P_k} |u_l'^k - u_l|^{2\sigma+2} \rightarrow 0. \end{aligned} \quad (60)$$

Since $\|\cdot\|_{l_k^\infty} \leq \|\cdot\|_{l_k^{2\sigma+2}}$, we have $\|u'^k - u\|_{l_k^\infty} \rightarrow 0$. From Lemma 7, it is known that $\|u'^k - u\|_{l_k^p} \rightarrow 0$ for $p > 2$. Hence,

$$\begin{aligned} |\omega| \|u'^k - u\|_{l_k^2}^2 &\leq \langle -\Delta_d(u'^k - u), (u'^k - u) \rangle \\ &\quad - \omega \langle (u'^k - u), (u'^k - u) \rangle \\ &= 2\alpha \sum_{l \in P_k} |u_l'^k - u_l|^2 |u_{l+1}'^k - u_{l+1}|^2 + \beta \\ &\quad \cdot \sum_{l \in P_k} |u_l'^k - u_l|^{2\sigma+2} \leq 2\alpha \|u'^k - u\|_{l_k^4}^4 \\ &\quad + \beta \|u'^k - u\|_{l_k^{2\sigma+2}}^{2\sigma+2} \rightarrow 0. \end{aligned} \quad (61)$$

It completes the proof. $\square$

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The author sincerely thanks Professor Yong Li for many useful suggestions. The author also thanks the referees for their comments that improved this paper. This work was supported by NSF of China (NSFC) Grant no. 11401250.

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