Research Article

Operators on Spaces of Bounded Vector-Valued Continuous Functions with Strict Topologies

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Let X be a completely regular Hausdorff space, and let \((E, \|\cdot\|_E)\) and \((F, \|\cdot\|_F)\) be Banach spaces. Let \(C_0(X, E)\) be the space of all \(E\)-valued bounded, continuous functions defined on \(X\), equipped with the strict topologies \(\beta_z\), where \(z = \sigma, \infty, p, r, t\). General integral representation theorems of \(\beta_z\)-continuous linear operators \(T : C_0(X, E) \rightarrow F\) with respect to the corresponding operator-valued measures are established. Strongly bounded and \(\beta_z\)-continuous operators \(T : C_0(X, E) \rightarrow F\) are studied.

We extend to "the completely regular setting" some classical results concerning operators on the spaces \(C(X, E)\) and \(C_c(X, E)\), where \(X\) is a compact or a locally compact space.

1. Introduction and Terminology

Throughout the paper let \((E, \|\cdot\|_E)\) and \((F, \|\cdot\|_F)\) be real Banach spaces, and let \(E'\) and \(F'\) denote the Banach duals of \(E\) and \(F\), respectively. By \(B_{E'}\) and \(B_E\) we denote the closed unit ball in \(E'\) and \(E\), respectively. By \(\mathcal{L}(E, F)\) we denote the space of all bounded linear operators \(U : E \rightarrow F\). Given a locally convex space \((L, \mathcal{L})\) by \((L, \mathcal{L})'\) or \(L'_\sigma\) we will denote its topological dual. We denote by \(\sigma(L, K)\) the weak topology on \(L\) with respect to a dual pair \((L, K)\).

Assume that \(X\) is a completely regular Hausdorff space. Let \(C_0(X, E)\) stand for the Banach space of all bounded continuous, \(E\)-valued functions on \(X\) with the uniform norm \(\|\cdot\|\). We write \(C_0(X)\) instead of \(C_0(X, \mathbb{R})\). By \(C_0(X, E)'\) we denote the Banach dual of \(C_0(X, E)\). For \(f \in C_0(X, E)\) let \(\tilde{f}(t) = \|f(t)\|_E\) for \(t \in X\).

Let \(\mathcal{B}(\mathcal{S})\) (resp., \(\mathcal{B}(\mathcal{A})\)) be the algebra (resp., \(\sigma\)-algebra) of Baire sets in \(X\), which is the algebra (resp., \(\sigma\)-algebra) generated by the class \(\mathcal{X}\) of all zero sets of functions of \(C_0(X)\). By \(\mathcal{B}\) we denote the family of all cozero sets in \(X\). Let \(B(\mathcal{S}, E)\) stand for the Banach space of all totally \(\mathcal{S}\)-measurable functions \(f : X \rightarrow E\) (the uniform limits of sequences of \(E\)-valued \(\mathcal{S}\)-simple functions) provided with the uniform norm \(\|\cdot\|\) (see [1, 2]). We will write \(B(\mathcal{S})\) instead of \(B(\mathcal{S}, \mathbb{R})\).

Strict topologies \(\beta_z\) on \(C_0(X, E)\) and \(C_c(X, E)\) (for \(z = \sigma, \infty, p, r, t\)) play an important role in the topological measure theory (see [3–12] for definitions and more details). Recall that a subset \(H\) of \(C_0(X, E)\) is said to be solid if \(f_1 \in C_0(X, E)\) and \(f_2 \in H\) with \(\tilde{f}_1(t) \leq \tilde{f}_2(t)\) for \(t \in X\) imply that \(f_1 \in H\). Then \(\beta_z\) are locally convex-solid topologies on \(C_0(X, E)\); that is, they have a local base at 0 consisting of convex and solid sets (see [6, Theorem 8.1], [10, Theorem 5]). We have \(\beta_1 \subset \beta_\sigma \subset \beta_\infty \subset \beta_p \subset \beta_r\), which is the algebra (resp., \(\sigma\)-algebra) generated by the class \(\mathcal{X}\) of all zero sets of functions of \(C_0(X)\). By \(\mathcal{B}\) we denote the family of all cozero sets in \(X\). Let \(B(\mathcal{S}, E)\) stand for the Banach space of all totally \(\mathcal{S}\)-measurable functions \(f : X \rightarrow E\) (the uniform limits of sequences of \(E\)-valued \(\mathcal{S}\)-simple functions) provided with the uniform norm \(\|\cdot\|\) (see [1, 2]). We will write \(B(\mathcal{S})\) instead of \(B(\mathcal{S}, \mathbb{R})\).

Let \(C_0(X) \otimes E\) stand for the algebraic tensor product of \(C_0(X, E)\) and \(E\); that is, \(C_0(X) \otimes E\) is the space of all functions \(\sum_{i=1}^n (u_i \otimes x_i)\), where \(u_i \in C_0(X)\) and \(x_i \in E\) for \(i = 1, \ldots, n\), and \((u_i \otimes x_i)(t) = u_i(t)x_i\) for \(t \in X\). Then \(C_0(X) \otimes E\) is dense in \((C_0(X, E), \beta_\pi)\) for \(z = \infty, r, t\) (see [6, 8]). Moreover, \(C_0(X) \otimes E\) is dense in \((C_c(X, E), \beta_\pi)\) if \(X\) or \(E\) is a D-space (see [6, Theorem 5.2], [13]) and in \((C_c(X, E), \beta_\pi)\) if \(X\) is real-compact (see [10, Theorem 7]).

Let \(C_0(X, E)\) denote the Banach space of all continuous functions \(h : X \rightarrow E\) such that \(h(X)\) is a relatively compact set in \(E\), provided with the uniform norm \(\|\cdot\|\). Then \(C_0(X) \otimes E \subset C_c(X, E) \subset B(\mathcal{S}, E)\).
Linear operators from the spaces $C_{r^c}(X, E)$ and $C_b(X, E)$, equipped with the strict topologies $\beta_z(z = \sigma, \sigma_0, \tau, t)$ to a locally convex space $(F, \xi)$, were studied by Katsaras and Liu [14], Aguayo-Garrido, Nova-Yanéz and Sanchez [15, 16], and Khurana [17]. In particular, Katsaras and Liu found an integral representation of weakly compact operators $S : C_{r^c}(X, E) \to F$ and characterizations of $(\beta_{r^c}, \xi)$-continuous and weakly compact operators $S : C_{r^c}(X, E) \to F$ for $z = \sigma, \tau$ (see [14, Theorems 3, 4, 5]). Aguayo-Arrido and Nova-Yanéz derived a Riesz representation theorem for $(\beta_{r^c}, \xi)$-continuous and weakly compact operators $T : C_b(X, E) \to F$ for $z = \sigma, \tau$ in terms of their representing operator measures (see [15, Theorems 5 and 6]). If $X$ is a locally compact space, continuous operators on $C_b(X, E)$ were studied by Dobrakov (see [18]) and Mitter and Young (see [19]).

In this paper we develop the theory of continuous linear operators from $C_b(X, E)$, equipped with the strict topologies $\beta_z(z = \sigma, \sigma_0, \tau, t)$ to a Banach space $(F, \| \cdot \|_F)$. In particular, we extend to "the completely regular setting" some classical results of Brooks and Lewis (see [20, Theorems 5 and 6]). If $X$ is a locally compact space, continuous and strongly bounded operators $S : C_b(X, E) \to F$ for $z = \sigma, \tau$ in terms of their representing operator measures (see [18]) and Mitter and Young (see [19]).

In Section 2 we derive general Riesz representation theorems for $(\beta_{r^c}, \| \cdot \|_F)$-continuous linear operators $T : C_b(X, E) \to F$ for $z = \sigma, \sigma_0, \tau, t$ with respect to the corresponding measures $m : B \to L(E, F''')$ (see Theorems 9 and 14 below). Section 4 is devoted to the study of $(\beta_{r^c}, \| \cdot \|_F)$-continuous and strongly bounded operators $T : C_b(X, E) \to F$.

2. Integral Representation of Bounded Linear Operators on $C_{r^c}(X, E)$

Let $M(X)$ stand for the Banach lattice of all Baire measures on $\mathcal{B}$, provided with the norm $\| \nu \| = |\nu|(X)$ (the total variation of $\nu$). Due to the Alexandrov representation theorem $C_b(X)^\prime$ can be identified with $M(X)$ through the lattice isomorphism $M(X) \ni \nu \mapsto \varphi_\nu \in C_b(X)^\prime$, where $\varphi_\nu(u) = \int_X u \, dv$ for $u \in C_b(X)$ and $\| \varphi_\nu \| = \| \nu \|$ (see [4, Theorem 5.1]).

By $M(X, E')$ we denote the set of all finitely additive measures $\mu : \mathcal{B} \to E'$ with the following properties:

(i) for each $x \in E$, the function $\mu_x : \mathcal{B} \to \mathbb{R}$ defined by $\mu_x(\mathcal{B}) = \mu(A)(x) \in M(X)$,

(ii) $|\mu|(X) < \infty$, where $|\mu|(A)$ stands for the variation of $\mu$ on $A \in \mathcal{B}$.

In view of [23, Theorem 2.5] $C_{r^c}(X, E)^\prime$ can be identified with $M(X, E')$ through the linear mapping $M(X, E') \ni \Phi \mapsto \Phi \in C_{r^c}(X, E)^\prime$, where $\Phi_h(h) = \int_X h \, d\mu$ for $h \in C_{r^c}(X, E)$ and $\| \Phi \| = |\mu|(X)$. Then one can embed $B(\mathcal{B}, E)$ into $C_{r^c}(X, E)^\prime$ by the mapping $\pi : B(\mathcal{B}, E) \to C_{r^c}(X, E)^\prime$, where for $g \in B(\mathcal{B}, E)$,

$$\pi(g)(F) := \int_X g \, d\mu \text{ for } \mu \in M(X, E').$$

(1)

Let $i_F : F \to F''$ denote the canonical embedding: that is, $i_F(y)(y') = y'(y)$ for $y \in F, y' \in F'$. Moreover, let $j_F : i_F(F) \to F$ stand for the left inverse of $i_F$; that is, $j_F \circ i_F = \text{id}_F$.

Assume that $S : C_{r^c}(X, E) \to F$ is a bounded linear operator. Let

$$\tilde{S} : S'' \circ \pi : B(\mathcal{B}, E) \to F'',$

where $S' : F' \to C_{r^c}(X, E)'$ and $S'' : C_{r^c}(X, E)'' \to F''$ denote the conjugate and biconjugate operators of $S$, respectively. Then we can define a measure $m : B \to L(E, F'')$ (called a representing measure of $S$) by

$$m(A)(x) = \tilde{S}(1_A \otimes x) = (S'' \circ \pi)(1_A \otimes x)$$

for $A \in \mathcal{B}, x \in E$.

(2)

Then $m(A) < \infty$, where the semivariation $\tilde{m}(A)$ of $m$ on $A \in \mathcal{B}$ is defined by $\tilde{m}(A) := \sup \| \sum m(A_j)(x_j) \|_{F''}$, where the supremum is taken over all finite $\mathcal{B}$-partitions $(A_j)$ of $A$ and $x_j \in B_{F'}$ for each $j$. For $y' \in F'$ let us put

$$m_{y'}(A)(x) := (m(A)(x))(y') \text{ for } A \in \mathcal{B}, x \in E.$$  

(4)

Let $|m_{y'}|(A)$ stand for the variation of $m_{y'}$ on $A$. Then (see [1, Section 4, Proposition 5])

$$\tilde{m}(A) = \sup \| m_{y'}(A)(y') \|_{F''}.$$  

(5)

The following general properties of the operator $\tilde{S} : B(\mathcal{B}, E) \to F''$ are well known (see [1, Section 6], [2, Section 1], [13, 24]):

$$\tilde{S}(g) = \int_X g \, d\mu \text{ for } g \in B(\mathcal{B}, E), \| \tilde{S} \| = \tilde{m}(X),$$

(6)

and for each $y' \in F'$,

$$\tilde{S}(g)(y') = \int_X g \, d\mu_{y'} \text{ for } g \in B(\mathcal{B}, E).$$

(7)

For $A \in \mathcal{B}$ let

$$\int_A g \, d\mu := \int_A 1_A \, d\mu \text{ for } g \in B(\mathcal{B}, E).$$

(8)

From the general properties of $\tilde{S}$ it follows that

$$\tilde{S}(C_{r^c}(X, E)) \subset i_F(F),$$

(9)

$$S(h) = j_F \left( \int_X h \, d\mu \right) \text{ for } h \in C_{r^c}(X, E).$$

(10)

Hence for each $y' \in F'$ we get

$$y'(S(h)) = \int_X h \, d\mu_{y'} \text{ for } h \in C_{r^c}(X, E).$$
and hence \( m_{y'} \in M(X, E') \). Moreover, we have
\[
\|S\| = \|S'\| = \sup \{ \|S'(y')\| : y' \in B_{F'} \} = \sup \{ \|y' \circ S\| : y' \in B_{F'} \} = \sup \{ \Phi_{m_{y'}}(y') : y' \in B_{F'} \} = \sup \{ [m_{y'}](X) : y' \in B_{F'} \},
\]
and using (5) we get
\[
\|S\| = \tilde{m}(X). \tag{12}
\]

By \( M(X, \mathcal{L}(E, F'')) \) we will denote the space of all measures \( m : \mathcal{B} \to \mathcal{L}(E, F'') \) such that \( \tilde{m}(X) < \infty \) and \( m_{y'} \in M(X, E') \) for each \( y' \in F' \). Thus the representing measure \( m \) of \( S \) belongs to \( M(X, \mathcal{L}(E, F'')) \).

For any \( x \in E \) define
\[
S_x(u) := S(u \otimes x) \text{ for } u \in C_b(X), \tag{13}
\]
\[
m_x(A) := m(A)(x) \text{ for } A \in \mathcal{B}. \tag{14}
\]

Then \( S_x : C_b(X) \to F \) is a bounded linear operator. Let \( \chi : B(\mathcal{B}) \to C_b(X)'' \) stand for the canonical embedding; that is, \( u \in B(\mathcal{B}) \),
\[
\chi(u)(\varphi) = \int_X ud\nu \text{ for } \nu \in M(X). \tag{15}
\]

Let \( \hat{S}_x := (S_x)'' \circ \chi : B(\mathcal{B}) \to F'' \).

Then
\[
\hat{S}_x(C_b(X)) \subset i_F(F), \tag{16}
\]
\[
S_x(u) = f_F(\hat{S}_x(u)) \text{ for } u \in C_b(X). \tag{17}
\]

The following lemma will be useful.

**Lemma 1.** Let \( S : C_b(X, E) \to F \) be a bounded linear operator. Then \( S''(\pi(1_A \otimes x)) = (S_x)''(\chi(1_A)) \) for each \( x \in E \) and every \( A \in \mathcal{B} \).

**Proof.** Let \( y' \in F' \). Then for each \( u \in C_b(X) \),
\[
(y' \circ S_x)(u) = y'(S(u \otimes x)) = \int_X (u \otimes x) dm_{y';} = \int_X ud_{m_{y'}} = \varphi_{m_{y'}}(u). \tag{18}
\]

Hence we have
\[
(S_x)''(\chi(1_A))(y') = \chi(1_A)(S'_x(y')) = \chi(1_A)(y' \circ S_x) = \chi(1_A)(\varphi_{m_{y'}}) = \int_X 1_A dm_{y'} = m_x(1_A)(y'). \tag{19}
\]

On the other hand, for each \( h \in C_{rc}(X, E) \), \( (y' \circ S)(h) = \int_X h dm_{y'} = \Phi_{m_{y'}}(h) \), and hence
\[
S''(\pi(1_A \otimes x)) \quad (\pi(1_A \otimes x))(S_x)''(\chi(1_A)) = (S_x)''(\chi(1_A)) = (S_x)''(\chi(1_A)). \tag{20}
\]

It follows that \( S''''(\pi(1_A \otimes x)) = (S_x)''''(\chi(1_A)) \), as desired. \( \square \)

From Lemma 1 for \( A \in \mathcal{B} \) and \( x \in E \) we get
\[
m_x(A) := \hat{S}(1_A \otimes x) = S''(\pi(1_A \otimes x)) = (S_x)''(\chi(1_A)) = m_x(1_A)(y'). \tag{21}
\]

Now we are ready to prove the following Bartle-Dunford-Schwartz type theorem (see [25, Theorem 5, pages 153-154]).

**Theorem 2.** Let \( S : C_{rc}(X, E) \to F \) be a bounded linear operator and let \( M(X, \mathcal{L}(E, F'')) \) be its representing measure. Then for each \( x \in E \) the following statements are equivalent.

(i) \( S_x : C_b(X) \to F \) is weakly compact.

(ii) \( m(A)(x) \in i_F(F) \) for each \( A \in \mathcal{B} \) and \( x \in E \).

(iii) \( m_x : \mathcal{B} \to F'' \) is strongly bounded.

**Proof.** (i)⇒(ii) Assume that \( S_x \) is weakly compact. Then by the Gantmacher theorem \( (S_x)^{''''}(C_b(X)''') \subset i_F(F) \) and \( (S_x)^{''''} : C_b(X)'''' \to F'''' \) is weakly compact (see [26, Theorem 17.2]). Hence \( \hat{S}_x(B(\mathcal{B})) \subset i_F(F) \) and \( \hat{S}_x : B(\mathcal{B}) \to F'''' \) is weakly compact. In view of (21) for each \( x \in E \), \( m_x(A) \in i_F(F) \) for \( A \in \mathcal{B} \) and \( m_x : \mathcal{B} \to F'''' \) is weakly compact (see [25, Theorem 1, page 148]). It follows that \( \{ f_F(m(A)(x)) : A \in \mathcal{B} \} \) is a relatively weakly compact set in \( F \) (see [24, Theorem 7]).

(ii)⇒(iii) It follows from [24, Theorem 7].

(iii)⇒(i) Assume that \( m_x \) is strongly bounded. Then by (21) \( \hat{S}_x : B(\mathcal{B}) \to F'''' \) is weakly compact and in view of (16) we derive that \( S_x \) is weakly compact. \( \square \)
3. Integral Representation of Continuous Linear Operators on $C_b(X,E)$

The spaces of all $\sigma$-additive, $u$-additive, perfect, $\tau$-additive, and tight members of $M(X)$ will be denoted by $M_\sigma(X)$, $M_\omega(X)$, $M_u(X)$, $M_\mu(X)$, and $M_\tau(X)$, respectively (see [3, 4]).

Then $(C_b(X), \beta_z^r) = (\{\eta : \eta \in M_\tau(X)\})$ for $z = \sigma, \omega, u, \mu, \tau$.

For the integration theory of functions $f \in C_b(X,E)$ with respect to $\mu \in M_\tau(X,E')$ we refer the reader to [6, page 197], [5, Definition 3.10], [4, page 375]. For $z = \sigma, \omega, u, \mu, \tau$, let

$$ M_z \left( X, E' \right) := \{ \mu \in M \left(X, E' \right) : \mu_x \in M_z \left(X \right) \text{ for each } x \in E \}. $$

(22)

Then $|\mu|$ is a member of $M_\tau(X)$ if $\mu$ is a member of $M_\tau(X,E')$ (see [5, Proposition 3.9], [6, Theorem 3.1], [10, Theorem 1]). For $\Phi \in C_b(X,E)^\tau$ let us put, for $u \in C_b(X)^\tau$,

$$ |\Phi| (u) := sup \left\{ |\Phi (f)| : f \in C_b(X,E), \int f \leq u \right\}. $$

(23)

It is known that $|\Phi| : C_b(X)^\tau \rightarrow \mathbb{R}^+$ is additive and positively homogeneous and can be extended to a linear functional on $C_b(X)$ (denoted by $|\Phi|$ again) by $|\Phi|(u) = |\Phi|(u^*) - |\Phi|(u^*)$ for $u \in C_b(X)$.

**Theorem 3.** Assume that $z = \sigma$ and $C_b(X) \otimes E$ is dense in $(C_b(X,E), \beta_z^p)$ (resp., $z = \omega$; $z = \mu$ and $C_b(X) \otimes E$ is dense in $(C_b(X,E), \beta_p \mu)$; $z = \tau$; $z = t$). Then the following statements hold.

(i) For a linear functional $\Phi$ on $C_b(X,E)$ the following conditions are equivalent.

(a) $\Phi$ is $\beta_z$-continuous.

(b) There exists a unique $\mu \in M_z(X,E')$ such that

$$ \Phi (f) = \Phi_{\mu} \left( f \right) = \int_X f d\mu \quad \text{for } f \in C_b(X,E). $$

(24)

(ii) For $\mu \in M_z(X,E')$, $|\Phi_{\mu}|(u) = \int_X u d|\mu| = \phi_{\mu}(u)$ for $u \in C_b(X)$.

**Proof.** (i) See [6, Theorems 5.3 and 4.2, Corollary 3.9], [5, Theorem 3.13], and [10, Theorem 8].

(ii) See [6, Theorem 2.1].

Assume that $\mathcal{M}$ is a subset of $M_z(X,E')$ and $sup_{\mu \in \mathcal{M}} |\mu|(X) < \infty$, where $z = \sigma, \omega, \mu, \tau, t$. Then we say that $\mathcal{M}$ satisfies the condition (C) if we have the following:

1. For $z = \sigma$: $sup_{\mu \in \mathcal{M}} |\mu|(Z_a) : \mu \in \mathcal{M} \rightarrow 0$ whenever $Z_a \downarrow 0, (Z_a) \in \mathcal{M}$.

2. For $z = \omega$: for every partition of unity $(u_\alpha)_{\alpha \in A}$ for $X$ and every $\varepsilon > 0$ there exists a finite set $A'$ such that

$$ \sum_{\alpha \in A} (1 - \sum_{\beta \in A'} u_\alpha) |\mu| < \varepsilon; $$

3. For $z = p$: for every continuous function $f$ from $X$ onto a separable metric space $Y$ and every $\varepsilon > 0$, there is a compact subset $K$ of $Y$ such that $sup_{\mu \in \mathcal{M}} |\mu|(X \setminus \overline{f}^{-1}(K)) \leq \varepsilon$;

4. For $z = t$: $sup_{\mu \in \mathcal{M}} |\mu|(Z_a) : \mu \in \mathcal{M} \rightarrow 0$ whenever $Z_a \downarrow 0, (Z_a) \in \mathcal{M}$.

5. For $z = t$: for every $\varepsilon > 0$ there exists a compact subset $K$ of $X$ such that $sup_{\mu \in \mathcal{M}} |\mu|(Z) : Z \in \mathcal{Z}, Z \subset X \setminus K \leq \varepsilon$ for each $\mu \in \mathcal{M}$.

The following lemmas will be useful.

**Lemma 4.** Assume that $\mathcal{M}$ is a subset of $M_z(X,E')$ and $sup_{\mu \in \mathcal{M}} |\mu|(X) < \infty$, where $z = \sigma$ and $C_b(X) \otimes E$ is $\beta_z$-dense in $(C_b(X,E), \beta_z^p)$; $z = \omega$; $z = \mu$ and $C_b(X) \otimes E$ is $\beta_p \mu$-dense in $(C_b(X,E), \beta_p \mu)$; $z = \tau$; $z = t$). Then the following statements are equivalent.

(i) $\{\Phi_{\mu} : \mu \in \mathcal{M}\}$ is $\beta_z$-equicontinuous.

(ii) $\{\Phi_{\mu} : \mu \in \mathcal{M}\}$ is $\beta_z$-equicontinuous.

(iii) $\{\phi_{\mu} : \mu \in \mathcal{M}\}$ is $\beta_z$-equicontinuous.

(iv) The condition (C) holds.

**Proof.** (i)⇒(ii) See [9, Lemma 2].

(ii)⇒(iii) It follows from Theorem 3.

(iii)⇒(iv) See [4, Theorem 11.14] for $z = \sigma$; [28, Proposition 3.6] for $z = \omega$; [28, Proposition 2.6] for $z = p$; [4, Theorem 11.24] for $z = \tau$; and [28, Proposition 1.1] for $z = t$.

**Lemma 5.** Assume that $z = \sigma$ and $C_b(X) \otimes E$ is $\beta_p \mu$-dense in $(C_b(X,E), \beta_p \mu)$; $z = \tau$; $z = t$). Then for $A \in \mathcal{B}$ the following statements hold.

(i) A functional $\Phi_A : C_r(X,E) \rightarrow \mathbb{R}$ defined by $\Phi_A (h) = \int_A h d\mu$ is $\beta_z$-continuous and can be uniquely extended to a $\beta_z$-continuous linear functional $\overline{\Phi}_A : C_b(X,E) \rightarrow \mathbb{R}$, and one will write the following:

$$ \int_A f d\mu := \overline{\Phi}_A (f) \quad \text{for } f \in C_b(X,E). $$

(25)

(ii) $\int_A f d\mu \leq \int_A \overline{\Phi}_A (f)$ for $f \in C_b(X,E)$.

**Proof.** (i) Assume that $(h_\alpha)$ is a net in $C_r(X,E)$ such that $h_\alpha \rightarrow 0$ for $\beta_z$. Then

$$ |\Phi_A (h_\alpha)| = \left| \int_A h_\alpha d\mu \right| \leq \int_A \overline{\Phi}_A (f) \leq \int_X \overline{\Phi}_A (f) \leq \varepsilon. $$

(26)

Since $\overline{\Phi}_A \rightarrow 0$ for $\beta_z$ in $C_b(X)$ and $|\mu| \in M_z(E)$, we obtain that $\Phi_A (h_\alpha) \rightarrow 0$; that is, $\Phi_A$ is $\beta_z$-continuous. Since $C_r(X,E)$ is dense in $(C_b(X,E), \beta_p \mu)$, $\Phi_A$ can be uniquely extended to a $\beta_z$-continuous linear functional $\overline{\Phi}_A : C_b(X,E) \rightarrow \mathbb{R}$ (see [29, Theorem 2.6]).
(ii) Assume that \( f \in C_b(X, E) \). Choose a net \((h_\alpha)\) in \( C_r^c(X, E) \) such that \( h_\alpha \to f \) for \( \beta_z \). Then \( h_\alpha \to \tilde{f} \) for \( \beta_\sigma \) in \( C_b(X) \). Then
\[
\left| \int_X \tilde{h}_\alpha \, d\mu - \int_X \tilde{f} \, d\mu \right| \leq \int_X |\tilde{h}_\alpha - \tilde{f}| \, d\mu,
\]
and hence \( \int_A \tilde{f} \, d\mu = \lim_\alpha \int_A \tilde{h}_\alpha \, d\mu \). Since \( \int_A f \, d\mu = \Phi(f) = \lim_\alpha \int_A h_\alpha \, d\mu \), we get
\[
\left| \int_A f \, d\mu \right| = \lim_\alpha \left| \int_A h_\alpha \, d\mu \right| \leq \lim_\alpha \int_A \tilde{h}_\alpha \, d\mu = \int_A \tilde{f} \, d\mu.
\]

For \( z = \sigma, \infty, p, r, t \) let us put
\[
M_z \left( X, \mathscr{D}(E, F''') \right) := \left\{ m : M(X, \mathscr{D}(E, F''')) : m_{y'} \in M_z \left( X, E' \right) \right\}
\]
(29) for each \( y' \in F' \).

**Lemma 6.** Assume that \( z = \sigma \) and \( C_b(X) \otimes E \) is \( \beta_\sigma \)-dense in \( C_r(X, E) \) (resp., \( z = \infty \); \( z = p \), and \( C_b(X) \otimes E \) is \( \beta_\sigma \)-dense in \( C_b(X, E) ; z = \tau ; \beta_\tau \)). Assume that \( m \in M_z(X, \mathscr{D}(E, F''')) \) and the set \( \{ m_{y'} : y' \in F' \} \) satisfies the condition \( (C_z) \). Then for \( A \in \mathcal{B} \) the following statements hold.

(i) An operator \( S_A : C_b(X, E) \to F''' \) defined by \( S_A(h) = \int_X \tilde{h} \, dm \) is \( \beta_\beta \)-continuous and can be uniquely extended to a \( \beta_\beta \)-continuous linear operator \( \tilde{S}_A : C_b(X, E) \to F''' \), and one will write the following.
\[
\int_A \tilde{f} \, dm = \tilde{S}_A(f) \quad \text{for } f \in C_b(X, E). \quad (30)
\]

(ii) For each \( y' \in F' \), \( \left( \int_A f \, dm \right)(y') = \int_A f \, dm_{y'}, \quad \text{for } f \in C_b(X, E). \)

**Proof.** (i) In view of Lemma 5 the set \( \{ \varphi_{m_{y'}} : y' \in B_{F'} \} \) is \( \beta_\beta \)-equicontinuous in \( C_b(X) \). Assume that \((h_\alpha)\) is a net in \( C_r(X, E) \) such that \( h_\alpha \to 0 \) for \( \beta_\sigma \). Let \( \varepsilon > 0 \) be given. Then there exists a neighborhood \( V_\varepsilon \) of 0 for \( \beta_\sigma \) in \( C_b(X) \) such that \( \sup_{y' \in B_{F'}} \int_X |m_{y'}(u)| \leq \varepsilon \) for \( u \in V_\varepsilon \). Since \( h_\alpha \to 0 \) for \( \beta_\sigma \) in \( C_b(X) \), choose \( \alpha_\varepsilon \) such that \( h_\alpha \in V_\varepsilon \) for \( \alpha \geq \alpha_\varepsilon \). Hence \( \sup_{y' \in B_{F'}} \int_X |h_\alpha \, d|m_{y'}| \leq \varepsilon \) for \( \alpha \geq \alpha_\varepsilon \). It follows that, for \( \alpha \geq \alpha_\varepsilon \) and each \( y' \in B_{F'} \),
\[
\left| \int_A h_\alpha \, dm_{y'} \right| \leq \int_X |h_\alpha \, d|m_{y'}| \leq \varepsilon,
\]
and hence,
\[
\left\| S_A(h_\alpha) \right\|_{F'''} = \sup \left\{ \left| S_A(h_\alpha) \right|(y') : y' \in B_{F'} \right\} \leq \varepsilon. \quad (32)
\]

This means that \( S_A : C_r(X, E) \to F''' \) is \( \beta_\beta \)-continuous. Since \( C_r(X, E) \) is \( \beta_\sigma \)-dense in \( C_b(X, E) \), \( \beta_\beta \), \( S_A \) possesses a unique \( \beta_\beta \)-continuous extension \( \tilde{S}_A : C_b(X, E) \to F''' \) (see [29, Theorem 2.6]). Let
\[
\int_A f \, dm = \tilde{S}_A(f) \quad \text{for } f \in C_b(X, E). \quad (33)
\]

(ii) Let \( f \in C_b(X, E) \). Choose a net \((h_\alpha)\) in \( C_r(X, E) \) such that \( h_\alpha \to f \) for \( \beta_\sigma \). By Lemma 5 and (7) for \( y' \in F' \) we have
\[
\left( \int_A f \, dm \right)(y') = \left( \lim_\alpha \left( \int_A h_\alpha \, dm \right) \right)(y') = \lim_\alpha \int_A h_\alpha \, dm_{y'} = \int_A f \, dm_{y'}. \quad (34)
\]

**Corollary 7.** Assume that \( z = \sigma \) and \( C_b(X) \otimes E \) is \( \beta_\sigma \)-dense in \( C_r(X, E) \) (resp., \( z = \infty \); \( z = p \) and \( C_b(X) \otimes E \) is \( \beta_\sigma \)-dense in \( C_b(X, E) ; z = \tau ; \beta_\tau \)). Assume that \( m \in M_z(X, \mathscr{D}(E, F''')) \) and the set \( \{ m_{y'} : y' \in B_{F'} \} \) satisfies the condition \( (C_z) \). Then for \( A \in \mathcal{B} \) the following statements hold:

**Proof.** (a) \( \left| m_{y'} \right| \quad (A) \)
\[
= \sup \left\{ \left| \int_A h \, dm \right| : h \in C_b(X) \otimes E, \left| h \right| \leq 1 \right\}
\]
\[
= \sup \left\{ \left| \int_A f \, dm \right| : f \in C_b(X, E), \left| f \right| \leq 1 \right\}. \quad (35)
\]

(b) \( \overline{m} \quad (A) \)
\[
= \sup \left\{ \left| \int_A h \, dm \right| : h \in C_b(X) \otimes E, \left| h \right| \leq 1 \right\}
\]
\[
= \sup \left\{ \left| \int_A f \, dm \right| : f \in C_b(X, E), \left| f \right| \leq 1 \right\}. \quad (35)
\]

In particular, if \( U \in \mathcal{D} \), then
\[
\left| m_{y'} \right| \quad (U) = \sup \left\{ \left| \int_U h \, dm \right| : h \in C_b(X) \otimes E, \left| h \right| \leq 1, \text{ supp } h \subset U \right\} \quad (36)
\]
\[
= \sup \left\{ \sum_{i=1}^{n} \left| \int_U u_i \, dm_{y',y''} \right| : y', y'' \right\}.
\]
where the supremum is taken over all finite disjoint supported collections \( \{u_1, \ldots, u_n\} \subset C_b(X) \) with \( \|u_i\| \leq 1 \) and \( \text{supp} u_i \subset U \) and \( \{x_1, \ldots, x_n\} \subset B_E \). One has

\[
(\text{d}) \quad \bar{m}(U) = \sup \left\{ \left\| \int_U h d\mu \right\|_{L^p} : h \in C_b(X) \otimes E, \right.
\]

\[
\|h\| \leq 1, \text{supp} h \subset U \bigg\}
\]

\[
= \sup \left\{ \left\| \int_U f d\mu \right\|_{L^p} : f \in C_b(X, E), \right. \n\]

\[
\|f\| \leq 1, \text{supp} f \subset U \bigg\}.
\]

Proof. Let \( A \in \mathcal{B} \) and \( y' \in F^* \). Then by Lemma 5 for \( f \in C_b(X, E) \) with \( \|f\| \leq 1 \) we have

\[
\left| \int_A f dm_{y'} \right| \leq \int_A f dm \left| m_{y'} \right|(A).
\]

(37)

On the other hand, let \( \epsilon > 0 \) be given. Then there exist a finite \( \mathcal{B} \)-partition \( (A_i)_{i=1}^n \) of \( A \) and \( x_i \in B_E \), \( i = 1, \ldots, n \), such that

\[
\left| m_{y'} (A) - \frac{\epsilon}{3} \right| \leq \left| \sum_{i=1}^n \left( m_{y'} (A_i) (x_i) \right) (y') \right| = \left| \sum_{i=1}^n m_{x_i y'} (A_i) \right|.
\]

(39)

By the regularity of \( m_{x_i y'}, m_{y'} \in M_1(X) \) for \( i = 1, \ldots, n \), we can choose \( Z_i \subset Z \) such that \( m_{x_i y'} (A_i \setminus Z_i) \leq \epsilon/3n \) for \( i = 1, \ldots, n \). Choose pairwise disjoint \( V_i \subset Z \) for \( i = 1, \ldots, n \) such that \( m_{x_i y'} (V_i \setminus Z) \leq \epsilon/3n \). Then \( \{ V_i \}_{i=1}^n \) is a partition of \( X \). Choose \( x_i \in V_i \) for \( i = 1, \ldots, n \) such that \( \|m_{x_i y'} (V_i \setminus Z) \|_{L^p} \leq \epsilon/3n \). Hence we get

\[
\left| m_{y'} (A) - \frac{\epsilon}{3} \right| \leq \left| \sum_{i=1}^n m_{x_i y'} (A_i \setminus Z_i) \right|
\]

\[
+ \left| \sum_{i=1}^n \left( \int_{Z_i} v_i d m_{x_i y'} \right) - \sum_{i=1}^n \left( \int_{V_i \setminus A_i} v_i d m_{x_i y'} \right) \right|
\]

\[
+ \left| \int_A h d m_{y'} \right|
\]

\[
\leq \sum_{i=1}^n \left| m_{x_i y'} (A_i \setminus Z_i) \right| + \sum_{i=1}^n \left| m_{x_i y'} (V_i \setminus Z_i) \right|
\]

\[
+ \left| \int_A h d m_{y'} \right|
\]

\[
\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \left| \int_A h d m_{y'} \right|
\]

(40)

and hence \( |m_{y'}(A)| \leq \int_A h d m_{y'} + \epsilon \). Thus the proof of (a) is complete.

In view of (5), (a), and Lemma 6 we get

\[
\bar{m}(A) = \sup \left\{ \left| m_{y'} (A) : y' \in B_E \right| \right. \n\]

\[
= \sup \left\{ \left| \int_A h d\mu \right| (y') : h \in C_b(X) \otimes E, \right. \n\]

\[
\|h\| \leq 1, y' \in B_E \bigg\}
\]

\[
= \sup \left\{ \left| \int_A f d\mu \right| (y') : f \in C_b(X, E), \right. \n\]

\[\|f\| \leq 1, y' \in B_E \bigg\}
\]

\[
= \sup \left\{ \left| \int_A h d\mu \right|_{L^p} : h \in C_b(X) \otimes E, \|h\| \leq 1 \right\}
\]

\[
= \sup \left\{ \left| \int_A f d\mu \right|_{L^p} : f \in C_b(X, E), \|f\| \leq 1 \right\};
\]

(41)

that is, (b) holds.

Assume now that \( U \in \mathcal{D} \). Let \( U_i = V_i \cap U \in \mathcal{D} \) for \( i = 1, \ldots, n \). Then \( |m_{x_i y'} (U_i \setminus Z_i)| \leq |m_{x_i y'} (V_i \setminus Z_i)| \leq \epsilon/3n \) for \( i = 1, \ldots, n \). For \( i = 1, \ldots, n \), choose \( u_i \in C_b(X) \) with \( 0 \leq u_i \leq 1_X \), \( u_i|_{Z_i} \equiv 1 \), and \( u_i|_{X \setminus U_i} \equiv 0 \). Let \( h_i = \sum_{J=0}^n u_i \otimes x_i \). Then \( \|h_i\| \leq 1 \) and \( \text{supp} h \subset U \); and hence by (a), \( |m_{y'} (U)| \leq \left| \int_U h_i d m_{y'} \right| + \epsilon \). Note that \( \int_U h_i d m_{y'} = \sum_{i=1}^n \int_U u_i d m_{x_i y'} \), where \( \text{supp} u_i \) are pairwise disjoint and \( u_i \subset U \) for \( i = 1, \ldots, n \). Thus (c) holds.

Using (c) we easily show that (d) holds. Thus the proof is complete.

Definition 8. Let \( T : C_b(X, E) \rightarrow F \) be a bounded linear operator. Then the measure \( m \in M(X, \mathcal{L}(E, F^*)) \) defined by

\[
m(A) (x) := \left( \left( T|_{C_b(X, E)} \right)^* \circ x \right) (1_A \otimes x)
\]

(42)

for \( A \in \mathcal{B}, x \in E \)

will be called a representing measure of \( T \).

Now we state general Riesz representation theorems for continuous linear operators on \( C_b(X, E) \), provided with the strict topologies \( \beta_z \), where \( z = \sigma, \alpha, p, r, t \).

Theorem 9. Assume that \( z = \sigma \) and \( C_b(X, E) \) is \( \beta_z \)-dense in \( C_b(X, E) \) (resp., \( z = \alpha \); \( z = p \), and \( C_b(X, E) \) is \( \beta_z \)-dense in \( C_b(X, E) ; z = r, z = t \).

(i) Let \( T : C_b(X, E) \rightarrow F \) be a \( \beta_z \)-continuous linear operator and let \( m \in M(X, \mathcal{L}(E, F^*)) \) be its representing measure. Then the following statements hold.

(i) \( m \in M(X, \mathcal{L}(E, F^*)) \) and \( m_{y'} : y' \in B_E \) satisfies the condition (Cz).

(ii) For each \( y' \in F^* \), \( y'(T(f)) = \int_X f dm_{y'} \) for \( f \in C_b(X, E) \).
(iii) For each \( f \in C_b(X,E) \) and \( A \in B \) there exists a unique vector in \( F'' \), denoted by \( \int_A fdm \), such that \( (\int_A fdm)(y') = \int_A f dm_{y'} \) for each \( y' \in F' \).

(iv) For each \( A \in B \), the mapping \( C_b(X,E) \ni f \mapsto \int_A f dm \in F'' \) is a \((\beta_{\Sigma}, \| \cdot \|_{F''})\)-continuous linear operator.

(v) For \( f \in C_b(X,E) \), \( \int_X f dm \in i_F(F) \) and \( T(f) = j_F(\int_X f dm) \).

(vi) \( \| T \| = \tilde{m}(X) \).

Let \( m_\circ \in M(X, \mathcal{L}(E,F'')) \) stand for the representing measure of \( T \). Note that, for \( A \in B \), \( x \in E \), and \( y' \in F' \) we have

\[
(m_\circ(A)(x))(y') = \left( \left( \left( T|_{C_b(X,E)} \right)' \circ \pi \right)(1_A \otimes x) \right)(y')
\]

\[
= \pi(1_A \otimes x) \left( \left( T|_{C_b(X,E)} \right)'(y') \right)
\]

\[
= \pi(1_A \otimes x) \left( y' \circ \left( T|_{C_b(X,E)} \right) \right)
\]

\[
= \int_X (1_A \otimes x) dm_{y'},
\]

\[
= (m(A)(x))(y');
\]

that is, \( m_\circ = m \). By the first part of the proof (ii) and (vi) hold. Thus the proof is complete. \( \square \)

Following [14, 27] by \( M_a(\mathcal{B}a) \) we denote the space of all bounded countably additive, real-valued, regular (with respect to zero sets) measures on \( \mathcal{B}a \).

We define \( M_a(\mathcal{B}a, E') \) to be the set of all measures \( \mu : \mathcal{B}a \to E' \) such that the following two conditions are satisfied.

(i) For each \( x \in E \), the function \( \mu_x : \mathcal{B}a \to \mathbb{R} \), defined by \( \mu_x(A) = \mu(A)(x) \) for \( A \in \mathcal{B}a \), belongs to \( M_a(\mathcal{B}a) \).

(ii) \( |\mu| < \infty \), where for each \( A \in \mathcal{B}a \), we define \( |\mu| = \sup \sum \mu(A_i) \chi_i \), where the supremum is taken over all finite \( \mathcal{B}a \)-partitions \( (A_i) \) of \( A \) and all finite collections \( \chi_i \in B_E \).

It is known that if \( \mu \in M_a(\mathcal{B}a, E') \), then \( |\mu| \in M_a(\mathcal{B}a) \) (see [27, Lemma 2.1]).

The following result will be of importance (see [27, Theorem 2.5]).

Theorem 10. Let \( \mu \in M_a(\mathcal{B}a, E') \). Then \( \mu \) possesses a unique extension \( \overline{\mu} \in M_a(\mathcal{B}a, E') \) and \( |\overline{\mu}|(X) = |\mu|(X) \).

Arguing as in the proof of Lemma 6 we can obtain the following lemma.

Lemma 11. Assume that \( C_b(X) \otimes E = \beta_{\Sigma} \)-dense in \( C_b(X,E) \) and \( \mu \in M_a(\mathcal{B}a, E') \). Then for \( A \in \mathcal{B}a \) the following statements hold.

(i) A functional \( \Phi_A : C_{rc}(X,E) \to \mathbb{R} \) defined by \( \Phi_A(h) = \int_A h dm = \beta_{\Sigma}dm_{\overline{\mu}} \) is a \( \beta_{\Sigma} \)-continuous linear functional on \( C_b(X,E) \) and it is uniquely extended to a \( \beta_{\Sigma} \)-continuous linear functional \( \overline{\Phi_A} : C_b(X,E) \to \mathbb{R} \), and one will write the following:

\[
\int_A f dm := \overline{\Phi_A}(f) \quad \text{for } f \in C_b(X,E).
\]

(ii) For \( f \in C_b(X,E) \), \( |\int_A f dm| \leq \int_A f|d|\overline{\mu} |.

By \( M_a(X, \mathcal{L}(E,F)) \) we will denote the space of all operator measures \( m : \mathcal{B} \to \mathcal{L}(E,F) \) such that \( \tilde{m}(X) < \infty \) and
$m_y \in M_\sigma(X, E')$ for each $y' \in F'$. By $M_\sigma(\mathcal{B}a, L(E, F))$ we will denote the space of all operator measures $m : \mathcal{B}a \to L(E, F)$ with $\bar{m}(X) < \infty$ such that $m_y \in M_\sigma(\mathcal{B}a, E')$ for each $y' \in F'$.

Remark 12. Note that in view of the Orlicz-Pettis theorem every $m \in M_\sigma(\mathcal{B}a, L(E, F))$ is countably additive in the strong operator topology; that is, for each $x \in E$, the measure $m_x : \mathcal{B}a \to F$ defined by $m_x(A) := m(A)(x)$ for $A \in \mathcal{B}a$ is countably additive. Moreover, in view of [30, Theorem 2] for each $x \in E$, $m_x$ is inner regular by zero sets and outer regular by cozero sets; that is, for each $A \in \mathcal{B}a$ and $\varepsilon > 0$ there exist $Z \in \mathcal{L}$ with $Z \subset A$ and $P \in \mathcal{P}$ with $A \subset P$ such that $\|m_x\|(A \setminus Z) \leq \varepsilon$ and $\|m_x\|(P \setminus A) \leq \varepsilon,$ (where $\|m_x\|(A)$ denotes the semivariation of $m_x$ on $A \in \mathcal{B}a$).

According to [14, Theorem 7] we have the following theorem.

**Theorem 13.** Assume that $m \in M_\sigma(X, L(E, F))$ and $\{m(A)(x) : A \in \mathcal{B}\}$ is a relatively weakly compact subset of $F$ for each $x \in E$. Then $m$ possesses a unique extension $\bar{m} \in M_\sigma(\mathcal{B}a, L(E, F))$ such that $\bar{m}(X) = \bar{m}(X)$.

For a linear operator $T : C_b(X, E) \to F$ and $x \in E$ let $T_x(u) := T(u \otimes x)$ for $u \in C_b(X)$. For $m \in M_\sigma(\mathcal{B}a, L(E, F')$) and $x \in E$ let $m_x(A) := m(A)(x)$ for $A \in \mathcal{B}a$.

**Theorem 14.** Assume that $C_b(X) \otimes E$ is $\beta_\sigma$-dense in $C_b(X, E)$.

(i) Let $T : C_b(X, E) \to F$ be a $(\beta_\sigma, \|\cdot\|_p)$-continuous linear operator such that $T_x : C_b(X) \to F$ is weakly compact for each $x \in E$, and let $m \in M_\sigma(X, L(E, F'))$ be the representing measure of $T$. Then the following statements hold.

(i) $m \in M_\sigma(X, L(E, F'))$ and $\bar{m}(Z_n) \to 0$ whenever $Z_n \uparrow 0, (Z_n) \subset \mathcal{L}$.

(ii) $m(A)(x) \in i_p(F)$, for each $A \in \mathcal{B}, x \in E$, and the measure $m_x : \mathcal{B} \to L(E, F)$, defined by $m_x(A) := j_p(m(A)(x))$ for $A \in \mathcal{B}, x \in E$, belongs to $M_\sigma(\mathcal{B}a, L(E, F))$ and possesses a unique extension $\bar{m} \in M_\sigma(\mathcal{B}a, L(E, F))$ with $\bar{m}(X) = \bar{m}(X)$ which is countably additive both in the strong operator topology and in the weak star operator topology. Moreover, $\bar{m}_x = \bar{m}_x$ for $y' \in F'$.

(iii) For every $x \in C_b(X, E)$ and $A \in \mathcal{B}a$ there exists a unique vector in $F$, denoted by $\int_A f \, d\bar{m}$, such that, for each $y' \in F'$, $y'\left(\int_A f \, d\bar{m}\right) = \lim_{\alpha} \int_A h_{\alpha} \, d\bar{m}$.

(iv) For each $A \in \mathcal{B}a$, the mapping $T_A : C_b(X) \to F$ defined by $T_A(f) = \int_A f \, d\bar{m}$ is a $(\beta_\sigma, \|\cdot\|_p)$-continuous linear operator.

(v) $T(f) = T_X(f) = \int_X f \, d\bar{m}$ for $f \in C_b(X, E)$.

(ii) Let $m \in M_\sigma(X, L(E, F'))$ be such that $\bar{m}(Z_n) \to 0$ whenever $Z_n \downarrow 0, (Z_n) \subset \mathcal{L}$ and for each $x \in E,$ let $m_x : \mathcal{B} \to F'$ be strongly bounded. Then the operator $T : C_b(X, E) \to F$ defined by $T(f) = j_p(\int_X f \, d\bar{m})$ is $(\beta_\sigma, \|\cdot\|_p)$-continuous and $T_x : C_b(X) \to F$ is weakly compact for each $x \in E$, and the statements (ii)–(v) hold.

Proof. (i) It follows from Theorem 9.

(ii) In view of Theorem 2 $m(A)(x) \in i_p(F)$ for $A \in \mathcal{B}, x \in E$, and $\{m_x(A)(x) : A \in \mathcal{B}\}$ is a relatively weakly compact in $F$ for each $x \in E$. Since $m_x \in M_\sigma(X, L(E, F))$, by Theorem 10 $m_x$ possesses a unique extension $\bar{m} \in M_\sigma(\mathcal{B}a, L(E, F))$ with $\bar{m}(X) = \bar{m}(X)$ by the Orlicz-Pettis theorem $\bar{m}$ is countably additive in the strong operator topology. Moreover, since, for each $y' \in F'$, $\|\bar{m}_x\|_p \in M_\sigma(\mathcal{B}a) = ca(\mathcal{B}a, E')$. This means that $m : \mathcal{B}a \to L(E, F)$ is countably additive in the weak star operator topology.

Let $y' \in F'$. Then for $A \in \mathcal{B}a$ and $x \in E$ we have $\|\bar{m}_x\|_p = m_x(A)(x)$, and by Theorem 10, $\|\bar{m}_x\|_p = m_x$.

(iii) For $A \in \mathcal{B}a$, let $S_A(h) := \int_A f \, d\bar{m}$ for $h \in C_b(X, E)$. Proceeding as in the proof of Lemma 6 we can show that $S_A : C_b(X, E) \to F$ is a $(\beta_\sigma, \|\cdot\|_p)$-continuous linear operator, and hence $S_A$ possesses a unique $(\beta_\sigma, \|\cdot\|_p)$-continuous linear extension $T_A : C_b(X, E) \to F$ (see [29, Theorem 2.6]). Let us write the following:

$$\int_A f \, d\bar{m} := T_A(f) \quad \text{for } f \in C_b(X, E).$$

Let $f \in C_b(X, E)$ and $x \in E$. Choose a net $(h_{\alpha})$ in $C_b(X, E)$ such that $h_{\alpha} \to f$ for $\beta_\sigma$. For each $y' \in F'$, $\bar{m}_y = \bar{m}_y$ (see (i)) and by Lemma 11 we have

\[
y'(\int_A f \, d\bar{m}) = y'(\lim_{\alpha} \int_A h_{\alpha} \, d\bar{m}) = \lim_{\alpha} y'(\int_A h_{\alpha} \, d\bar{m}) = \lim_{\alpha} \int_A h_{\alpha} \, d\bar{m}_y = \int_A f \, d\bar{m}_y.
\]

(iv) It follows from the proof of (ii).

(v) Let $f \in C_b(X, E)$. In view of Theorem 9, for each $y' \in F'$, $y'(T(f)) = \int_X f \, d\bar{m}_y$. On the other hand by (ii) for $y' \in F'$ we have $y'(\int_X f \, d\bar{m}) = \int_X f \, d\bar{m}_y$. It follows that $T(f) = \int_X f \, d\bar{m}$.

(II) Since $\|\bar{m}_y\|_p \in B_{p'}$ satisfies the condition $(C_\sigma)$, by Theorem 9 for $f \in C_b(X, E)$, $\int_X f \, d\bar{m}$ is $i_p(F)$ and the mapping $T : C_b(X, E) \to F$ defined by $T(f) := j_p(\int_X f \, d\bar{m})$ is $(\beta_\sigma, \|\cdot\|_p)$-continuous linear operator, and $\bar{m}$ coincides with the representing measure $T$. Hence in view of Theorem 2 $T_x : C_b(X) \to F$ is a weakly compact operator. Thus by the first part of the proof the statements (ii)–(v) are satisfied.

\[\square\]

4. Strongly Bounded Operators on $C_b(X, E)$

Definition 15. A bounded linear operator $T : C_b(X, E) \to F$ is said to be strongly bounded if its representing measure...
\( m \in M(X, \mathcal{L}(E, F^\prime)) \) is strongly bounded; that is, \( \bar{m}(A_n) \to 0 \) whenever \( (A_n) \) is a pairwise disjoint sequence in \( \mathcal{B}a \).

Note that \( m \in M(X, \mathcal{L}(E, F^\prime)) \) is strongly bounded if and only if the family \( \{m_{\gamma} : \gamma \in B_P\} \) is uniformly strongly additive.

Now we are ready to state our main results that extend some classical results of Lewis ([20, Theorem 5], [31, Lemma 1]) and Brooks and Lewis (see [22, Theorem 2.1], [21, Theorem 5.2]) concerning operators on the spaces \( C_\infty(X, E) \) and \( C_0(X, E) \), where \( X \) is a compact or a locally compact space, respectively.

**Theorem 16.** Assume that \( C_0(X) \otimes E \) is \( \beta_a\)-dense in \( C_\infty(X, E) \). Let \( T : C_\infty(X, E) \to F \) be a \( (\beta_a, \|\cdot\|_E)\)-continuous linear operator and let \( m \in M(X, \mathcal{L}(E, F^\prime)) \) be its representing measure. Then \( m \in M_\sigma(X, \mathcal{L}(E, F^\prime)) \) and the following statements are equivalent.

(i) \( T \) is strongly bounded.

(ii) \( \sup \{ |m_{\gamma}|(A_n) : y^\prime \in B_P \} \to 0 \) whenever \( (A_n) \downarrow 0 \), \( (A_n) \subset \mathcal{B}a \) (here \( m_{\gamma} \in M_\sigma(\mathcal{B}a, E) \) denotes the unique extension of \( m_{\gamma} \in M_\sigma(X, E) \)).

(iii) If \( (A_n) \) is a sequence in \( \mathcal{B}a \) such that \( A_n \downarrow 0 \), then there exists a nested sequence \( (U_n) \) in \( \mathcal{P} \) such that \( A_n \subset U_n \) for \( n \in \mathbb{N} \) and \( \sup \|T(f)\|_F : f \in C_\infty(X, E), \|f\| \leq 1 \) and \( \sup f \subset U_n \).

**Proof.** In view of Theorem 9 \( m \in M_\sigma(X, \mathcal{L}(E, F^\prime)) \).

(i)\(\Rightarrow\)(ii) Assume that \( T \) is strongly bounded. Since the family \( \{m_{\gamma} : y^\prime \in B_P \} \) is uniformly strongly additive, according to [25, Lemma 1, page 26] the family \( \{m_{\gamma} \} : y^\prime \in B_P \) is uniformly countably additive (see Theorem 16).

(ii)\(\Rightarrow\)(i) It follows from [25, Lemma 1, page 26].

(ii)\(\Rightarrow\)(iii) Assume that (ii) holds and \( (A_n) \) is a sequence in \( \mathcal{B}a \) such that \( A_n \downarrow 0 \). Then there exists \( \lambda \in \mathcal{C}a(\mathcal{B}a) \) such that \( |(m_{\gamma})(A_n) : y^\prime \in B_P| \leq \epsilon/2 \) whenever \( \lambda(A) \leq \delta \) and \( A \in \mathcal{B}a \). Since \( \lambda \) is zero-set regular, there exists a nested sequence \( (U_n) \) in \( \mathcal{P} \) so that \( A_n \subset U_n \) and \( \lambda(U_n \setminus A_n) \leq \delta \) for \( n \in \mathbb{N} \). Hence \( \sup \{m_{\gamma}(U_n \setminus A_n) : y^\prime \in B_P\} \leq \epsilon/2 \) for \( n \in \mathbb{N} \). In view of (ii) there exists \( n_\epsilon \in \mathbb{N} \) such that \( \sup \{m_{\gamma}(U_n \setminus A_n) : y^\prime \in B_P\} \leq \epsilon/2 \) for \( n \geq n_\epsilon \). Hence \( \sup \{m_{\gamma}(U_n) : y^\prime \in B_P\} \leq \epsilon \) for \( n \geq n_\epsilon \); that is, \( \sup \{m_{\gamma}(U_n) : y^\prime \in B_P\} \to 0 \).

Let \( f \in C_\infty(X, E), \|f\| \leq 1 \), and \( sup f \subset U_n \). Then by Theorem 9 we have

\[
\|T(f)\|_F = \sup \left\{ \left\| \int_X f dm_{\gamma} y^\prime : y^\prime \in B_P \right\| \right. \\
\leq \sup \left\{ \left\| \int_X f dm_{\gamma} y^\prime : y^\prime \in B_P \right\| \right. \\
\leq \sup \left\{ \left| m_{\gamma}(U_n) : y^\prime \in B_P \right| \right. \\
\leq \sup \left\{ \left| m_{\gamma}(U_n) : y^\prime \in B_P \right| \right. \\
\leq \frac{\epsilon}{4} + \frac{\epsilon}{4} + \left\| \int_{U_n} h_\epsilon dm_{\gamma} y^\prime \right| \\
\leq \frac{\epsilon}{4} + \frac{\epsilon}{4} + \left\| \int_{U_n} h_\epsilon dm_{\gamma} y^\prime \right| \\
\leq \frac{\epsilon}{4} + \frac{\epsilon}{4} + \left\| \int_{U_n} h_\epsilon dm_{\gamma} y^\prime \right|. 
\]

(iii)\(\Rightarrow\)(i) Assume that (iii) holds and \( A_n \downarrow 0 \), \( (A_n) \subset \mathcal{B}a \). Then there exists a nested sequence \( (U_n) \) in \( \mathcal{P} \) such that \( A_n \subset U_n \) for \( n \in \mathbb{N} \) and

\[
\sup \{\|T(f)\|_F : f \in C_\infty(X, E), \|f\| \leq 1, sup f \subset U_n \} \to 0. 
\]
Hence
\[
\left| \int_{U_n} h_d d m_{y'} \right| \geq |\overline{m}_{y'}| \left( A_{n_\varepsilon} \right) - \frac{3}{4} \varepsilon \geq \frac{1}{4} \varepsilon.
\]
\[
\|T(h_\varepsilon)\|_F \geq \left| y'_{\omega} \left( T(h_\varepsilon) \right) \right| = \left| \int_X h_d d m_{y'} \right| \geq \frac{1}{4} \varepsilon.
\]
(54)

Thus we get a contradiction to \(\|T(h_\varepsilon)\|_F \leq (1/8)\varepsilon\).

Thus the proof is complete.

**Theorem 17.** Assume that \(C_{0}(X) \otimes E\) is \(\beta_{0}\)-dense in \(C_{0}(X,E)\). Let \(T : C_{0}(X,E) \rightarrow F\) be a \((\beta_{0}, \|\cdot\|_E)\)-continuous and strongly bounded operator and let \(m \in M(X, L(E,F))\) be its representing measure. Then the following statements hold.

(i) \(m \in M_{\sigma}(X, L(E,F))\) and \(m(A)(x) = \omega_{F}(A)(x)\) for \(A \in \mathcal{B}, x \in E\), and the measure \(m_{F} : \mathcal{B} \rightarrow L(E,F)\), defined by \(m_{F}(A)(x) := \omega_{F}(m(A)(x))\) for \(A \in \mathcal{B}, x \in E\), belongs to \(M_{\sigma}(X, L(E,F))\) and possesses a unique extension \(\overline{m} \in M_{\sigma}(\mathcal{B}_{a}, L(E,F))\) with \(\overline{m}(X) = \overline{m}(X) = \overline{m}(X)\) which is variationally semiregular; that is, \(\overline{m}(A_{n}) \rightarrow 0\) whenever \(A_{n} \downarrow 0, (A_{n}) \subset \mathcal{B}_{a}\).

(ii) For every \(f \in C_{0}(X,E)\) and \(A \in \mathcal{B}_{a}\) there exists a unique vector in \(F\), denoted by \(\int f \overline{d m}\), such that, for each \(y' \in F'\), \(y'(\int f \overline{d m}) = \int f \overline{d m}_{y'}\).

(iii) For each \(A \subset \mathcal{B}_{a}\), \(\int f_{n} \overline{d m} \rightarrow 0\) whenever \((f_{n})_{n} \subset C_{0}(X,E)\) is a uniformly bounded sequence in \(C_{0}(X,E)\) such that \(f_{n}(t) \rightarrow 0\) for \(t \in X\).

(iv) \(T(f) = \int f \overline{d m}\) for \(f \in C_{0}(X,E)\).

(v) \(T(f_{n}) \rightarrow 0\) whenever \((f_{n})_{n} \subset C_{0}(X,E)\) is a uniformly bounded sequence in \(C_{0}(X,E)\) such that \(f_{n}(t) \rightarrow 0\) for \(t \in X\).

**Proof.** (i) Note that, for \(x \in E\), \(\|m_{x}(A)\|_{F'} \leq \overline{m}(A)\|x\|_{E}\) for \(A \in \mathcal{B}\). Hence \(m_{x} : \mathcal{B} \rightarrow F'\) is strongly bounded, and by Theorems 2 and 14 \(m(A)(x) = \omega_{F}(m(A)(x))\) for \(A \in \mathcal{B}, x \in E\), belongs to \(M_{\sigma}(X, L(E,F))\) and possesses a unique extension \(\overline{m} \in M_{\sigma}(\mathcal{B}_{a}, L(E,F))\) with \(\overline{m}(X) = \overline{m}(X) = \overline{m}(X)\). Since \(\overline{m}_{y'} = \overline{m}_{y'}\) for \(y' \in F'\), by Theorem 16 we have \(\overline{m}(A_{n}) = \sup \{\|\overline{m}_{y'}| (A_{n}): y' \in F'\} \rightarrow 0\) whenever \(A_{n} \downarrow 0, (A_{n}) \subset \mathcal{B}_{a}\).

(ii) It follows from Theorem 14 because for each \(x \in E\), \(T_{X} : C_{0}(X,E) \rightarrow F\) is weakly compact (see Theorem 2).

(iii) In view of (i) there exists \(\lambda \in ca(\mathcal{B}_{a})\) such that \(\{\|\overline{m}_{y'}|: y' \in F'\}\) is \(\lambda\)-continuous (see [25, Theorem 4, pages 11-12]). Let \((f_{n})_{n}\) be a sequence in \(C_{0}(X,E)\) such that \(\sup \|f_{n}\|_{F'} = M < \infty\) and \(f_{n}(t) \rightarrow 0\) for every \(t \in X\). Let \(\varepsilon > 0\) be given. Then there exists \(\delta > 0\) such that \(\sup \|\overline{m}_{y'}| (A_{n})(x): a \in A_{n}\} \leq \delta \) whenever \(\|A_{n}\|_{\sigma} < \delta\) \(E\in \mathcal{B}_{a}\). Since \(\overline{f}_{n} \in B(\mathcal{B}_{a})\) for \(n \in \mathbb{N}\), by the Egoroff theorem there exists \(A_{\delta} \subset \mathcal{B}_{a}\) with \(\lambda(X \setminus A_{\delta}) \leq \delta\) and \(\sup_{E \in A_{\delta}} \overline{f}_{n}(t) \rightarrow 0\) whenever \(\lambda(X \setminus A_{\delta}) \leq \delta\) and \(\sup_{E \in A_{\delta}} \overline{f}_{n}(t) \leq \varepsilon/2 \overline{m}(X)\) for \(n \geq n_{\varepsilon}\).

\[\int T_{X}(f_{n}) \overline{d m} = \int f_{n} \overline{d m}_{y'} \rightarrow 0\] whenever \((f_{n})_{n} \subset C_{0}(X,E)\) is a uniformly bounded sequence in \(C_{0}(X,E)\) such that \(f_{n}(t) \rightarrow 0\) for \(t \in X\).

\[
\|T(f_{n})\|_{F} \leq \frac{\varepsilon}{2} \overline{m}(X) + M \cdot \frac{\varepsilon}{2} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]

Hence \(\|T(f_{n})\|_{F} \rightarrow 0\) as \(n \rightarrow \infty\).

\[\int T_{X}(f_{n}) \overline{d m} = \int f_{n} \overline{d m}_{y'} \rightarrow 0\] whenever \((f_{n})_{n} \subset C_{0}(X,E)\) is a uniformly bounded sequence in \(C_{0}(X,E)\) such that \(f_{n}(t) \rightarrow 0\) for \(t \in X\).

\[\int T_{X}(f_{n}) \overline{d m} = \int f_{n} \overline{d m}_{y'} \rightarrow 0\] whenever \((f_{n})_{n} \subset C_{0}(X,E)\) is a uniformly bounded sequence in \(C_{0}(X,E)\) such that \(f_{n}(t) \rightarrow 0\) for \(t \in X\).

\[\int T_{X}(f_{n}) \overline{d m} = \int f_{n} \overline{d m}_{y'} \rightarrow 0\] whenever \((f_{n})_{n} \subset C_{0}(X,E)\) is a uniformly bounded sequence in \(C_{0}(X,E)\) such that \(f_{n}(t) \rightarrow 0\) for \(t \in X\).
variationally semiregular, in view of [33, Proposition 2.2] we have
\[
\lim_{n \to \infty} \sum_{i=1}^{n} T(f_i) = \lim_{n \to \infty} \int_X S_n \, d\mathbb{M} = \int_X g \, d\mathbb{M} \in E.
\] (56)

Hence \( \sum_{i=1}^{\infty} T(f_i) = \int_X g \, d\mathbb{M} \). Finally, if \((n_i)\) is any permutation of \(\mathbb{N}\), then \(\lim_{n \to \infty} \sum_{j=1}^{n_i} f_{n_i}(t) = g(t)\) for \(t \in X\). Then
\[
\sum_{i=1}^{\infty} T(f_{n_i}) = \int_X g \, d\mathbb{M},
\]
as desired. \(\square\)

Remark 19. A related result to Corollary 18 for strongly bounded operators on the space \(C_b(X,E)\) of \(E\)-valued continuous functions vanishing at infinity defined on a locally compact space \(X\) was obtained by Brooks and Lewis (see [21, Theorem 5.2]).

Recall that a Banach space \(E\) is said to be a Schur space if every weakly convergent sequence in \(E\) is norm convergent.

As a consequence of Theorem 17 we derive the following Dunford-Pettis type theorem for operators on \(C_b(X,E)\).

Theorem 20. Assume that \(C_b(X) \otimes E\) is \(\beta_{\sigma}\)-dense in \(C_b(X,E)\), where \(E\) is a Schur space. Let \(T : C_b(X,E) \to F\) be a \((\beta_{\sigma}, \|\cdot\|_E)\)-continuous and strongly bounded operator. Then \(T(f_n) \to 0\) in \(F\) whenever \((f_n)\) is a \(\sigma(C_b(X,E),M_E(X,E'))\) convergent to \(0\) sequence in \(C_b(X,E)\).

Proof. Assume that \(f_n \to 0\) for \(\sigma(C_b(X,E),M_E(X,E'))\). Then according to [11, Corollary 5], we obtain that \(\sup_n \|f_n\| < \infty\) and \(f_n(t) \to 0\) in \(\sigma(E,E')\) for each \(t \in X\). It follows that \(\|f_n(t)\|_E \to 0\) for \(t \in X\) because \(E\) is supposed to be a Schur space. Using Theorem 17 we derive that \(T(f_n) \to 0\) in \(F\), as desired. \(\square\)

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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