Research Article

Some Generalized Difference Sequence Spaces Defined by a Sequence of Moduli in n -Normed Spaces

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We introduce some new generalized difference sequence spaces by means of ideal convergence, infinite matrix, and a sequence of modulus functions over n -normed spaces. We also make an effort to study several properties relevant to topological, algebraic, and inclusion relations between these spaces.

1. Introduction and Preliminaries

The concept of 2-normed spaces was initially developed by Gähler [1] in the middle of 1960s, while that of n -normed spaces one can see in Misiak [2]. Since then, many others have studied this concept and obtained various results; see Gunawan [3, 4] and Gunawan and Mashadi [5]. Let $n \in \mathbb{N}$ and X be a linear space over the field $\mathbb{R}$ of reals of dimension d , where $d \geq n \geq 2$. A real valued function $\|\cdot, \dots, \cdot\|$ on X^n satisfying the following four conditions:

- (1) $\|x_1, x_2, \dots, x_n\| = 0$ if and only if $x_1, x_2, \dots, x_n$ are linearly dependent in X ,
- (2) $\|x_1, x_2, \dots, x_n\|$ is invariant under permutation,
- (3) $\|\alpha x_1, x_2, \dots, x_n\| = |\alpha| \|x_1, x_2, \dots, x_n\|$ for any $\alpha \in \mathbb{R}$,
- (4) $\|x + x', x_2, \dots, x_n\| \leq \|x, x_2, \dots, x_n\| + \|x', x_2, \dots, x_n\|$,

is called a n -norm on X and the pair $(X, \|\cdot, \dots, \cdot\|)$ is called a n -normed space over the field $\mathbb{R}$. For more details about n -normed spaces, see [6, 7] and references therein.

The notion of difference sequence spaces was introduced by Kizmaz [8], who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$, and $c_0(\Delta)$. The notion was further generalized by Et and Çolak [9] by introducing the spaces $l_\infty(\Delta^m)$, $c(\Delta^m)$, and $c_0(\Delta^m)$. Later the concept has been studied by Bektaş et al. [10] and Et and Esi [11]. Another type of generalization of

the difference sequence spaces is due to Tripathy and Esi [12] who studied the spaces $l_\infty(\Delta_\nu)$, $c(\Delta_\nu)$, and $c_0(\Delta_\nu)$. Recently, Esi et al. [13] and Tripathy et al. [14] have introduced a new type of generalized difference operators and unified those as follows.

Let m and ν be nonnegative integers; then for Z , a given sequence space, we have

$$Z(\Delta_\nu^m) = \{x = (x_k) \in w : (\Delta_\nu^m x_k) \in Z\}, \quad (1)$$

for $Z = c, c_0$, and l_∞ where $\Delta_\nu^m x = (\Delta_\nu^m x_k) = (\Delta_\nu^{m-1} x_k - \Delta_\nu^{m-1} x_{k+\nu})$ and $\Delta_\nu^0 x_k = x_k$, for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation:

$$\Delta_\nu^m x_k = \sum_{v=0}^m (-1)^v \binom{m}{v} x_{k+\nu v}. \quad (2)$$

Taking $\nu = 1$, we get the spaces $l_\infty(\Delta^m)$, $c(\Delta^m)$, and $c_0(\Delta^m)$ studied by Et and Çolak [9]. Taking $\nu = m = 1$, we get the spaces $l_\infty(\Delta)$, $c(\Delta)$, and $c_0(\Delta)$ introduced and studied by Kizmaz [8]. For more details about difference sequence spaces, see [15–19].

A modulus function is a function $f : [0, \infty) \rightarrow [0, \infty)$ such that

- (1) $f(x) = 0$ if and only if $x = 0$,
- (2) $f(x + y) \leq f(x) + f(y)$, for all $x, y \geq 0$,

- (3) f is increasing,
 (4) f is continuous from the right at 0.

It follows that f must be continuous everywhere on $[0, \infty)$. The modulus function may be bounded or unbounded. For example, if we take $f(x) = x/(x+1)$, then $f(x)$ is bounded. If $f(x) = x^p$, $0 < p < 1$, then the modulus function $f(x)$ is unbounded. Subsequently, modulus function has been discussed in [20] and references therein.

In [21], Mursaleen introduced the idea of λ -statistical convergence by extending the concept of $[V, \lambda]$ summability. Further, Savaş and Das [22] unified the two approaches and gave new concepts of I -statistical convergence, I - S_λ -convergence, and I - $[V, \lambda]$ convergence. Later some pioneer works have been extended in this direction by numerous authors such as Belen and Mohiuddine [23], Das et al. [24], Gürdal and Sari [25], and references therein. Quite recently, many authors including Et et al. [26] and Maddox [27] have constructed some sequence spaces by using modulus function and difference sequences and investigate their properties.

Let X and Y be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk} , where $n, k \in \mathbb{N}$. Then we say that A defines a matrix mapping from X into Y if, for every sequence $x = (x_k)_{k=0}^\infty \in X$, the sequence $Ax = \{A_n(x)\}_{n=0}^\infty$, the A -transform of x , is in Y , where

$$A_n(x) = \sum_{k=0}^\infty a_{nk} x_k \quad (n \in \mathbb{N}). \quad (3)$$

By (X, Y) , we denote the class of all matrices A such that $A : X \rightarrow Y$. Thus, $A \in (X, Y)$ if and only if the series on the right-hand side of (3) converges for each $n \in \mathbb{N}$ and every $x \in X$.

The matrix domain X_A of an infinite matrix A in a sequence space X is defined by

$$X_A = \{x = (x_k) : Ax \in X\}. \quad (4)$$

The approach constructing a new sequence space by means of the matrix domain of a particular limitation method has recently been employed by several authors (see [28]).

Let $\lambda = (\lambda_n)$ be a nondecreasing sequence of positive numbers such that $\lambda_{n+1} \leq \lambda_n + 1$, $\lambda_1 = 1$, $\lambda_n \rightarrow \infty$ as $n \rightarrow \infty$, and $I_n = [n - \lambda_n + 1, n]$.

Let $\mathbb{N}$ be a nonempty set. Then a family of sets $I \subseteq 2^{\mathbb{N}}$ (power set of $\mathbb{N}$) is said to be an ideal if I is additive; that is, $A, B \in I \Rightarrow A \cup B \in I$ and $A \in I, B \subseteq A \Rightarrow B \in I$. A nonempty family of sets $\mathcal{E}(I) \subseteq 2^{\mathbb{N}}$ is said to be filtered on $\mathbb{N}$ if and only if $\Phi \notin \mathcal{E}(I)$; for $A, B \in \mathcal{E}(I)$ we have $A \cap B \in \mathcal{E}(I)$ and for each $A \in \mathcal{E}(I)$ and $A \subseteq B$ implying $B \in \mathcal{E}(I)$.

An ideal $I \subseteq 2^{\mathbb{N}}$ is called nontrivial if $I \neq 2^{\mathbb{N}}$. A nontrivial ideal $I \subseteq 2^{\mathbb{N}}$ is called admissible if $\{\{x\} : x \in \mathbb{N}\} \subseteq I$. A nontrivial ideal is maximal if there cannot exist any nontrivial ideal $J \neq I$ containing I as a subset. For each ideal I , there exist a filter $\mathcal{E}(I)$ corresponding to I ; that is, $\mathcal{E}(I) = \{K \subseteq \mathbb{N} : K^c \in I\}$, where $K^c = \mathbb{N} \setminus K$.

Definition 1. A sequence $x = (x_k)$ in a n -normed space $(X, \|\cdot, \dots, \cdot\|)$ is said to be statistically convergent to some $L \in X$ if, for each $\varepsilon > 0$, the set $A(\varepsilon) = \{k \in \mathbb{N} : \|x_k - L, z_1, \dots, z_{n-1}\| \geq \varepsilon\}$ has its natural density zero.

Definition 2 (see [26]). A sequence $x = (x_k)$ is said to be λ_X^m -statistically convergent to the number L , if, for every $\varepsilon > 0$, $\lim_{n \rightarrow \infty} (1/\lambda_n) |\{k \in I_n : \|\Delta^m x_k - L\| \geq \varepsilon\}| = 0$. In this case one writes $S_\lambda(\Delta^m, X) - \lim_{k \rightarrow \infty} x_k = L$.

Definition 3 (see [22]). A sequence $x = (x_k)$ is said to be I - $[V, \lambda]$ summable to L , if, for any $\delta > 0$, $\{n \in \mathbb{N} : (1/\lambda_n) \sum_{k \in I_n} \|x_k - L\| \geq \delta\} \in I$, where $I_n = [n - \lambda_n + 1, n]$.

Definition 4 (see [22]). A sequence $x = (x_k)$ is said to be I - λ -statistically convergent or I - S_λ convergent to L if, for every $\varepsilon > 0$ and $\delta > 0$, $\{n \in \mathbb{N} : (1/\lambda_n) |\{k \in I_n : \|x_k - L\| \geq \delta\}| \geq \varepsilon\} \in I$. In this case one writes $x_k \rightarrow L(I-S_\lambda)$ or $I-S_\lambda - \lim_{k \rightarrow \infty} x_k = L$.

The following well-known lemma is required for establishing some important results in this paper.

Lemma 5. Let $\mathcal{F} = (f_k)$ be a sequence of modulus functions and $0 < \delta < 1$. Then for each $x > \delta$ one has $f_k(x) \leq 2f_k(1)x/\delta$.

Throughout the paper, $\mathbb{N}$ will denote the set of all positive integers. By S_X^n we denote the space of all sequences defined over n -normed space $(X, \|\cdot, \dots, \cdot\|)$.

Let $I \subseteq 2^{\mathbb{N}}$ be an admissible ideal, $\mathcal{F} = (f_k)$ a sequence of modulus functions, $p = (p_k)$ a bounded sequence of positive (strictly) real numbers, $u = (u_k)$ a sequence of positive real numbers, $A = (a_{nk})$ an infinite matrix, and $(X, \|\cdot, \dots, \cdot\|)$ a n -normed space. Then, for every $\delta > 0$ and $z_1, \dots, z_{n-1} \in X$, we define the following sequence spaces:

$$\begin{aligned} & V_{\mathcal{F}}^I [\Delta_v^m, \lambda, A, u, p, \|\cdot, \dots, \cdot\|]_0 \\ &= \left\{ x = (x_k) \in S_X^n : \right. \\ & \quad \left\{ n \in \mathbb{N} : \right. \\ & \quad \left. \frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} [f_k(\|u_k \Delta_v^m x_k, z_1, \dots, z_{n-1}\|)]^{p_k} \geq \delta \right\} \in I \left. \right\}, \\ & V_{\mathcal{F}}^I [\Delta_v^m, \lambda, A, u, p, \|\cdot, \dots, \cdot\|] \\ &= \left\{ x = (x_k) \in S_X^n : \right. \\ & \quad \left\{ n \in \mathbb{N} : \right. \\ & \quad \left. \frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} [f_k(\|u_k \Delta_v^m x_k - L, z_1, \dots, z_{n-1}\|)]^{p_k} \geq \delta \right\} \\ & \quad \in I \left. \right\}, \end{aligned} \quad (5)$$

for some $L > 0$, and

$$V_{\mathcal{F}}^I[\Delta_v^m, \lambda, A, u, p, \|\cdot, \dots, \cdot\|]_{\infty} = \left\{ x = (x_k) \in S_X^n : \right. \\ \left. \begin{aligned} & n \in \mathbb{N} : \\ & \frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} [f_k(\|u_k \Delta_v^m x_k - L, z_1, \dots, z_{n-1}\|)]^{p_k} \geq K \end{aligned} \right\} \\ \in I \Bigg\}, \quad (6)$$

for some $K > 0$.

If we take $\mathcal{F}(x) = f_k(x) = x$, $u = (u_k) = 1$ for all $k \in \mathbb{N}$, $v = 1$, and $A = (a_{nk}) = I$ (identity matrix) and n -normed space is replaced by 2-normed space then we get the sequence spaces defined by Kumar et al. [29].

The following inequality will be used throughout the paper. Let $p = (p_k)$ be a sequence of positive real numbers with $0 < p_k \leq \sup_k p_k = H$, and let $D = \max\{1, 2^{H-1}\}$. Then, for the factorable sequences (a_k) and (b_k) in the complex plane, we have

$$|a_k + b_k|^{p_k} \leq D(|a_k|^{p_k} + |b_k|^{p_k}). \quad (7)$$

2. Main Results

The main purpose of this section is to study some topological properties and some inclusion relations between the sequence spaces which we have defined above.

Theorem 6. Suppose $\mathcal{F} = (f_k)$ is a sequence of modulus functions, $p = (p_k)$ is a bounded sequence of strictly positive real numbers, $u = (u_k)$ is a sequence of positive real numbers, and $A = (a_{nk})$ is an infinite matrix. Then the spaces $V_{\mathcal{F}}^I[\Delta_v^m, \lambda, A, u, p, \|\cdot, \dots, \cdot\|]_0$, $V_{\mathcal{F}}^I[\Delta_v^m, \lambda, A, u, p, \|\cdot, \dots, \cdot\|]$, and $V_{\mathcal{F}}^I[\Delta_v^m, \lambda, A, u, p, \|\cdot, \dots, \cdot\|]_{\infty}$ are linear spaces over the real field $\mathbb{R}$.

Proof. The proof of the theorem is easy so we omit it. $\square$

For the next result we shall define the following sequence space:

$$V_{\mathcal{F}}[\Delta_v^m, \lambda, A, u, p, \|\cdot, \dots, \cdot\|]_{\infty} = \left\{ x = (x_k) \in S_X^n : \right.$$

$$\left. \begin{aligned} & \sup_n \frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} [f_k(\|u_k \Delta_v^m x_k - L, z_1, \dots, z_{n-1}\|)]^{p_k} \\ & < \infty \end{aligned} \right\}. \quad (8)$$

Theorem 7. Let $\mathcal{F} = (f_k)$ be a sequence of modulus functions and $p = (p_k)$ a bounded sequence of strictly positive real numbers. Then the space $V_{\mathcal{F}}[\Delta_v^m, \lambda, A, u, p, \|\cdot, \dots, \cdot\|]_0$ is a paranormed space with the paranorm:

$$g(x) = \sup_n \left(\frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} [f_k(\|u_k \Delta_v^m x_k, z_1, \dots, z_{n-1}\|)]^{p_k} \right)^{1/G}, \quad (9)$$

where $G = \max(1, H)$ and $0 < p_k \leq \sup_k p_k = H$.

Proof. Clearly $g(x) = g(-x)$ for $x \in V_{\mathcal{F}}^I[\Delta_v^m, \lambda, A, u, p, \|\cdot, \dots, \cdot\|]_0$. It is trivial that $u_k \Delta_v^m x_k = 0$ for $x = 0$. Since $f_k(0) = 0$, we get $g(x) = 0$ for $x = 0$. Since $p_k/G \leq 1$, using Minkowski's inequality, we have

$$\begin{aligned} & \left(\frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} [f_k(\|u_k \Delta_v^m (x_k + y_k), z_1, \dots, z_{n-1}\|)]^{p_k} \right)^{1/G} \\ & \leq \left(\frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} [f_k(\|u_k \Delta_v^m x_k, z_1, \dots, z_{n-1}\| + \|u_k \Delta_v^m y_k, z_1, \dots, z_{n-1}\|)]^{p_k} \right)^{1/G} \\ & \leq \left(\frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} [f_k(\|u_k \Delta_v^m x_k, z_1, \dots, z_{n-1}\|)]^{p_k} \right)^{1/G} \\ & \quad + \left(\frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} [f_k(\|u_k \Delta_v^m y_k, z_1, \dots, z_{n-1}\|)]^{p_k} \right)^{1/G}. \end{aligned} \quad (10)$$

Hence $g(x + y) \leq g(x) + g(y)$. Finally, to check the continuity of scalar multiplication, let α be any complex number; therefore by definition

$$\begin{aligned} g(\alpha x) &= \sup_n \left(\frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} [f_k(\|u_k \Delta_v^m \alpha x_k, z_1, \dots, z_{n-1}\|)]^{p_k} \right)^{1/G} \\ &\leq C_{\alpha}^{H/G} g(x), \end{aligned} \quad (11)$$

where C_α is a positive integer such that $\alpha \leq C_\alpha$. Let $\alpha \rightarrow 0$ for any fixed x with $g(x) \neq 0$. By definition, for $|\alpha| < 1$, we have

$$\frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} [f_k(\|u_k \Delta_v^m \alpha x_k, z_1, \dots, z_{n-1}\|)]^{p_k} < \epsilon, \quad (12)$$

for $n > N(\epsilon)$.

Also, for $1 \leq n \leq N$, taking α small enough, since f_k is continuous, we have

$$\frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} [f_k(\|u_k \Delta_v^m \alpha x_k, z_1, \dots, z_{n-1}\|)]^{p_k} < \epsilon. \quad (13)$$

Now (12) and (13) imply that $g(\alpha x) \rightarrow 0$ as $\alpha \rightarrow 0$. This completes the proof. $\square$

Theorem 8. Let $u = (u_k)$ be a sequence of positive real numbers. Then for $m \geq 1$ the following inclusions:

$$\begin{aligned} \text{(i)} \quad V_{\mathcal{F}}^I[\Delta_v^{m-1}, \lambda, A, u, \|\cdot, \dots, \cdot\|_0] &\subset V_{\mathcal{F}}^I[\Delta_v^m, \lambda, A, u, \|\cdot, \dots, \cdot\|_0], \\ \text{(ii)} \quad V_{\mathcal{F}}^I[\Delta_v^{m-1}, \lambda, A, u, \|\cdot, \dots, \cdot\|_\infty] &\subset V_{\mathcal{F}}^I[\Delta_v^m, \lambda, A, u, \|\cdot, \dots, \cdot\|_\infty], \end{aligned}$$

are strict.

Proof. We will prove the result for $V_{\mathcal{F}}^I[\Delta_v^{m-1}, \lambda, A, u, \|\cdot, \dots, \cdot\|_0]$ only. The others can be proved similarly.

Suppose $x \in V_{\mathcal{F}}^I[\Delta_v^{m-1}, \lambda, A, u, \|\cdot, \dots, \cdot\|_0]$; by definition, for every $\delta > 0$ and $z_1, \dots, z_{n-1} \in X$, we have

$$\left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} [f_k(\|u_k \Delta_v^{m-1} x_k, z_1, \dots, z_{n-1}\|)] \geq \delta \right\} \in I. \quad (14)$$

By the property of modulus function, we have

$$\begin{aligned} &\frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} [f_k(\|u_k \Delta_v^m x_k, z_1, \dots, z_{n-1}\|)] \\ &\leq \frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} [f_k(\|u_k \Delta_v^{m-1} x_k, z_1, \dots, z_{n-1}\|) \\ &\quad + f_k(\|u_k \Delta_v^{m-1} x_{k+1}, z_1, \dots, z_{n-1}\|)] \\ &\leq D \frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} [f_k(\|u_k \Delta_v^{m-1} x_k, z_1, \dots, z_{n-1}\|)] \\ &\quad + D \frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} [f_k(\|u_k \Delta_v^{m-1} x_{k+1}, z_1, \dots, z_{n-1}\|)] \quad \text{by (7)}. \end{aligned} \quad (15)$$

Now, for given $\delta > 0$, we have

$$\begin{aligned} &\left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} [f_k(\|u_k \Delta_v^m x_k, z_1, \dots, z_{n-1}\|)] \geq \delta \right\} \\ &\subseteq \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} [f_k(\|u_k \Delta_v^{m-1} x_k, z_1, \dots, z_{n-1}\|)] \geq \frac{\delta}{2D} \right\} \\ &\cup \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} [f_k(\|u_k \Delta_v^{m-1} x_{k+1}, z_1, \dots, z_{n-1}\|)] \geq \frac{\delta}{2D} \right\}, \end{aligned} \quad (16)$$

for each $z_1, \dots, z_{n-1} \in X$. Since $x \in V_{\mathcal{F}}^I[\Delta_v^{m-1}, \lambda, A, u, \|\cdot, \dots, \cdot\|_0]$, it follows that the sets on the right-hand side in the above containment belong to I . Hence $x \in V_{\mathcal{F}}^I[\Delta_v^m, \lambda, A, u, \|\cdot, \dots, \cdot\|_0]$. To show that the inclusion is strict, we give the following example.

We take $f_k(x) = x$, $\lambda_n = n$, and $A = (a_{nk}) = I$, for all $n, k \in \mathbb{N}$, $v = 1$, and consider a sequence $x = (x_k) = k^{m-1}$; then $x \in V_{\mathcal{F}}^I[\Delta_v^m, \lambda, A, u, \|\cdot, \dots, \cdot\|_0]$ but does not belong to $V_{\mathcal{F}}^I[\Delta_v^{m-1}, \lambda, A, u, \|\cdot, \dots, \cdot\|_0]$ for $u = (u_k) = 1, k \in \mathbb{N}$. This shows that the inclusion is strict. $\square$

Theorem 9. Let $\mathcal{F}' = (f'_k)$ and $\mathcal{F}'' = (f''_k)$ be sequences of modulus functions. If $\limsup_{t \rightarrow \infty} (f'_k(t)/f''_k(t)) = P > 0$, then $V_{\mathcal{F}'}^I[\Delta_v^m, \lambda, A, u, p, \|\cdot, \dots, \cdot\|] \subset V_{\mathcal{F}''}^I[\Delta_v^m, \lambda, A, u, p, \|\cdot, \dots, \cdot\|]$.

Proof. Let $\limsup_{t \rightarrow \infty} (f'_k(t)/f''_k(t)) = P$; then there exists a constant $M > 0$ such that $f'_k(t) \geq M f''_k(t)$ for all $t \geq 0$. Therefore, for each $z_1, \dots, z_{n-1} \in X$, we have

$$\begin{aligned} &\frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} [f'_k(\|u_k \Delta_v^m x_k - L, z_1, \dots, z_{n-1}\|)]^{p_k} \\ &\geq (M)^H \frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} [f''_k(\|u_k \Delta_v^m x_k - L, z_1, \dots, z_{n-1}\|)]^{p_k}. \end{aligned} \quad (17)$$

Then, for every $\delta > 0$ and $z_1, \dots, z_{n-1} \in X$, we have the following relationship:

$$\left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} [f''_k(\|u_k \Delta_v^m x_k - L, z_1, \dots, z_{n-1}\|)]^{p_k} \geq \delta \right\}$$

$$\subseteq \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} \left[f'_k (\|u_k \Delta_v^m x_k - L, z_1, \dots, z_{n-1}\|) \right]^{p_k} \geq \delta (M)^H \right\}. \quad (18)$$

Since $x \in V_{\mathcal{F}'}^I[\Delta_v^m, \lambda, A, u, p, \|\cdot, \dots, \cdot\|]$, it follows that the set on left-side of the above containment belongs to I , which gives $x \in V_{\mathcal{F}''}^I[\Delta_v^m, \lambda, A, u, p, \|\cdot, \dots, \cdot\|]$. $\square$

Theorem 10. Let $\mathcal{F} = (f_k)$, $\mathcal{F}' = (f'_k)$, and $\mathcal{F}'' = (f''_k)$ be sequences of modulus functions. Then

- (i) $V_{\mathcal{F}'}^I[\Delta_v^m, \lambda, A, u, p, \|\cdot, \dots, \cdot\|] \subset V_{\mathcal{F} \circ \mathcal{F}'}^I[\Delta_v^m, \lambda, A, u, p, \|\cdot, \dots, \cdot\|]$,
- (ii) $V_{\mathcal{F}'}^I[\Delta_v^m, \lambda, A, u, p, \|\cdot, \dots, \cdot\|] \cup V_{\mathcal{F}''}^I[\Delta_v^m, \lambda, A, u, p, \|\cdot, \dots, \cdot\|] \subset V_{\mathcal{F}' + \mathcal{F}''}^I[\Delta_v^m, \lambda, A, u, p, \|\cdot, \dots, \cdot\|]$.

Proof. (i) Let $x = (x_k) \in V_{\mathcal{F}'}^I[\Delta_v^m, \lambda, A, u, p, \|\cdot, \dots, \cdot\|]$, for every $\varepsilon > 0$ and for some $L > 0$ such that

$$\left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} \left[f'_k (\|u_k \Delta_v^m x_k - L, z_1, \dots, z_{n-1}\|) \right]^{p_k} \geq \varepsilon \right\} \in I, \quad (19)$$

for each $z_1, \dots, z_{n-1} \in X$. For given $\varepsilon > 0$, we choose $\delta \in (0, 1)$ such that $f'_k(t) < \varepsilon$ for all $0 < t < \delta$. On the other hand, we have

$$\begin{aligned} & \frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} \left[(f_k \circ f'_k) (\|u_k \Delta_v^m x_k - L, z_1, \dots, z_{n-1}\|) \right]^{p_k} \\ &= \frac{1}{\lambda_n} \sum_{k \in I_n, [f'_k(\|u_k \Delta_v^m x_k - L, z_1, \dots, z_{n-1}\|)]^{p_k} < \delta} a_{nk} \left[(f_k \circ f'_k) \right. \\ & \quad \cdot (\|u_k \Delta_v^m x_k - L, z_1, \dots, z_{n-1}\|) \left. \right]^{p_k} \\ &+ \frac{1}{\lambda_n} \sum_{k \in I_n, [f'_k(\|u_k \Delta_v^m x_k - L, z_1, \dots, z_{n-1}\|)]^{p_k} \geq \delta} a_{nk} \left[(f_k \circ f'_k) \right. \\ & \quad \cdot (\|u_k \Delta_v^m x_k - L, z_1, \dots, z_{n-1}\|) \left. \right]^{p_k} \end{aligned}$$

$$\begin{aligned} & \leq (\varepsilon)^H + \max \left(1, \left(2 \frac{f_k(1)}{\delta} \right)^H \right) \frac{1}{\lambda_n} \\ & \quad \cdot \sum_{k \in I_n} a_{nk} \left[f'_k (\|u_k \Delta_v^m x_k - L, z_1, \dots, z_{n-1}\|) \right]^{p_k}, \end{aligned} \quad (20)$$

by Lemma 5.

By using (19), we obtain $x \in V_{\mathcal{F} \circ \mathcal{F}'}^I[\Delta_v^m, \lambda, A, u, p, \|\cdot, \dots, \cdot\|]$.

(ii) The result of this part is proved by using the following inequality:

$$\begin{aligned} & \frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} \left[(f'_k + f''_k) (\|u_k \Delta_v^m x_k - L, z_1, \dots, z_{n-1}\|) \right]^{p_k} \\ & \leq \frac{D}{\lambda_n} \sum_{k \in I_n} a_{nk} \left[f'_k (\|u_k \Delta_v^m x_k - L, z_1, \dots, z_{n-1}\|) \right]^{p_k} \\ & \quad + \frac{D}{\lambda_n} \sum_{k \in I_n} a_{nk} \left[f''_k (\|u_k \Delta_v^m x_k - L, z_1, \dots, z_{n-1}\|) \right]^{p_k}, \end{aligned} \quad (21)$$

where $\sup_k p_k = H$ and $D = \max(1, 2^{H-1})$. $\square$

Theorem 11. Let $\mathcal{F} = (f_k)$ be a sequence of modulus functions and $p = (p_k)$ a bounded sequence of strictly positive real numbers. Then $V^I[\Delta_v^m, \lambda, A, u, p, \|\cdot, \dots, \cdot\|] \subseteq V_{\mathcal{F}}^I[\Delta_v^m, \lambda, A, u, p, \|\cdot, \dots, \cdot\|]$.

Proof. This can be proved by using the same techniques as in Theorem 10. $\square$

Theorem 12. Let $\mathcal{F} = (f_k)$ be a sequence of modulus functions. If $\limsup_{t \rightarrow \infty} (f_k(t)/t) = M > 0$, then $V_{\mathcal{F}}^I[\Delta_v^m, \lambda, A, u, p, \|\cdot, \dots, \cdot\|] \subseteq V^I[\Delta_v^m, \lambda, A, u, p, \|\cdot, \dots, \cdot\|]$.

Proof. Suppose $x = (x_k) \in V_{\mathcal{F}}^I[\Delta_v^m, \lambda, A, u, p, \|\cdot, \dots, \cdot\|]$ and $\limsup_{t \rightarrow \infty} (f_k(t)/t) = M > 0$, then there exists a constant $K > 0$ such that $f_k(t) \geq Kt$ for all $t \geq 0$. Thus, we have

$$\begin{aligned} & \frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} \left[f_k (\|u_k \Delta_v^m x_k - L, z_1, \dots, z_{n-1}\|) \right]^{p_k} \\ & \geq (K)^H \frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} \left[(\|u_k \Delta_v^m x_k - L, z_1, \dots, z_{n-1}\|) \right]^{p_k}, \end{aligned} \quad (22)$$

for each $z_1, \dots, z_{n-1} \in X$. This completes the proof. $\square$

Theorem 13. If $0 < p_k \leq q_k$ and (q_k/p_k) be bounded. Then $V_{\mathcal{F}}^I[\Delta_v^m, \lambda, A, u, q, \|\cdot, \dots, \cdot\|] \subset V_{\mathcal{F}}^I[\Delta_v^m, \lambda, A, u, p, \|\cdot, \dots, \cdot\|]$.

Proof. It is easy to prove so we omit the detail. $\square$

3. Statistical Convergence

The notion of statistical convergence introduced by Fast [30] in 1951 and later developed by Fridy [31], Šalát [32] and many

others. Furthermore, Kostyrko et al. [33] presented a very interesting generalization of statistical convergence called as I -convergence. Some recent developments in this regard can be found in [34–37] and many others.

In this section, we define a new class of generalized statistical convergent sequences with the help of an ideal, modulus functions and infinite matrix. We also made an effort to establish a strong connection between this convergence and the sequence space $V_{\mathcal{F}}^I[\Delta_v^m, \lambda, A, u, p, \|\cdot, \dots, \cdot\|]$.

Definition 14. Let $I \subseteq P(\mathbb{N})$ be a non-trivial ideal and $\lambda = (\lambda_n)$ be a non-decreasing sequence. A sequence $x = (x_k) \in X$ is said to be $S_{\lambda}^{\Delta_v^m}(I, u, A, \|\cdot, \dots, \cdot\|)$ -convergent to a number L provided that for every $\varepsilon > 0$, $\delta > 0$ and $z_1, \dots, z_{n-1} \in X$, the set

$$\left\{n \in \mathbb{N} : \frac{1}{\lambda_n} |\{k \in I_n : a_{nk} (\|u_k \Delta_v^m x_k - L, z_1, \dots, z_{n-1}\|) \geq \varepsilon\}| \geq \delta\right\} \in I. \quad (23)$$

In this case, we write $S_{\lambda}^{\Delta_v^m}(I, u, A) - \lim_{k \rightarrow \infty} \|u_k \Delta_v^m x_k, z_1, \dots, z_{n-1}\| = \|L/p, z_1, \dots, z_{n-1}\|$. Let $S_{\lambda}^{\Delta_v^m}(I, A, u, \|\cdot, \dots, \cdot\|)$ denotes the set of all $S_{\lambda}^{\Delta_v^m}(I, u, A, \|\cdot, \dots, \cdot\|)$ -convergent sequences in X .

Theorem 15. Let $\mathcal{F} = (f_k)$ be a sequence of modulus functions and $0 < \inf_k p_k = h \leq p_k \leq \sup_k p_k = H < \infty$. Then $V_{\mathcal{F}}^I[\Delta_v^m, \lambda, A, u, p, \|\cdot, \dots, \cdot\|] \subset S_{\lambda}^{\Delta_v^m}[I, A, u, \|\cdot, \dots, \cdot\|]$.

Proof. Suppose $x = (x_k) \in V_{\mathcal{F}}^I[\Delta_v^m, \lambda, A, u, p, \|\cdot, \dots, \cdot\|]$ and $\varepsilon > 0$ be given. Then for each $z_1, \dots, z_{n-1} \in X$, we obtain

$$\begin{aligned} & \frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} [f_k (\|u_k \Delta_v^m x_k - L, z_1, \dots, z_{n-1}\|)]^{p_k} \\ &= \frac{1}{\lambda_n} \sum_{k \in I_n, \|u_k \Delta_v^m x_k - L, z_1, \dots, z_{n-1}\| \geq \varepsilon} a_{nk} [f_k (\|u_k \Delta_v^m x_k - L, \\ & \quad z_1, \dots, z_{n-1}\|)]^{p_k} \\ & \quad + \frac{1}{\lambda_n} \sum_{k \in I_n, \|u_k \Delta_v^m x_k - L, z_1, \dots, z_{n-1}\| < \varepsilon} a_{nk} [f_k (\|u_k \Delta_v^m x_k - L, \\ & \quad z_1, \dots, z_{n-1}\|)]^{p_k} \\ &\geq \frac{1}{\lambda_n} \sum_{k \in I_n, \|u_k \Delta_v^m x_k - L, z_1, \dots, z_{n-1}\| \geq \varepsilon} a_{nk} [f_k (\|u_k \Delta_v^m x_k - L, \\ & \quad z_1, \dots, z_{n-1}\|)]^{p_k} \\ &\geq \frac{1}{\lambda_n} \sum_{k \in I_n} [f_k(\varepsilon)]^{p_k} \end{aligned}$$

$$\begin{aligned} &\geq \sum_{k \in I_n} \min([f_k(\varepsilon)]^h, [f_k(\varepsilon)]^H) \\ &\geq K \frac{1}{\lambda_n} |\{k \in I_n : a_{nk} (\|u_k \Delta_v^m x_k - L, z_1, \dots, z_{n-1}\|) \geq \varepsilon\}|, \end{aligned} \quad (24)$$

where $K = \min([f_k(\varepsilon)]^h, [f_k(\varepsilon)]^H)$. Then for every $\delta > 0$ and $z_1, \dots, z_{n-1} \in X$, we have

$$\begin{aligned} &\left\{n \in \mathbb{N} : \frac{1}{\lambda_n} |\{k \in I_n : a_{nk} (\|u_k \Delta_v^m x_k - L, z_1, \dots, z_{n-1}\|) \geq \varepsilon\}| \geq \delta\right\} \\ &\subseteq \left\{n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} [f_k (\|u_k \Delta_v^m x_k - L, z_1, \dots, z_{n-1}\|)]^{p_k} \geq K\delta\right\}. \end{aligned} \quad (25)$$

Since $x_k \rightarrow L$ in $V_{\mathcal{F}}^I[\Delta_v^m, \lambda, A, u, p, \|\cdot, \dots, \cdot\|]$ so that $S_{\lambda}^{\Delta_v^m}(I, u, A) - \lim_{k \rightarrow \infty} \|u_k \Delta_v^m x_k, z_1, \dots, z_{n-1}\| = \|L, z_1, \dots, z_{n-1}\|$. $\square$

Theorem 16. Let $\mathcal{F} = (f_k)$ be a sequence of modulus functions and $p = (p_k)$ be a bounded sequence of strictly positive real numbers. If $0 < \inf_k p_k = h \leq p_k \leq \sup_k p_k = H < \infty$ then $S_{\lambda}^{\Delta_v^m}[I, A, u, \|\cdot, \dots, \cdot\|] \subset V_{\mathcal{F}}^I[\Delta_v^m, \lambda, A, u, p, \|\cdot, \dots, \cdot\|]$.

Proof. By using ([11, Theorem 3.5]) we can prove easily. $\square$

Theorem 17. Let $\mathcal{F} = (f_k)$ be a bounded sequence of modulus functions and $p = (p_k)$ be a bounded sequence of strictly positive real numbers. If $0 < \inf_k p_k = h \leq p_k \leq \sup_k p_k = H < \infty$. Then $S_{\lambda}^{\Delta_v^m}[I, A, u, \|\cdot, \dots, \cdot\|] = V_{\mathcal{F}}^I[\Delta_v^m, \lambda, A, u, p, \|\cdot, \dots, \cdot\|]$ if and only if $\mathcal{F} = (f_k)$ is a bounded.

Proof. This part can be obtained by combining Theorems 15 and 16. Conversely suppose $\mathcal{F} = (f_k)$ be unbounded defined by $f_k(k) = k$ for all $k \in \mathbb{N}$. We take a fixed set $B \in I$, where I is an admissible ideal and define $x = (x_k)$ as follows

$$x_k = \begin{cases} k^{m+1}, & \text{for } n - \lfloor \sqrt{\lambda_n} \rfloor + 1 \leq k \leq n, \quad n \notin B \\ k^{m+1}, & \text{for } n - \lfloor \sqrt{\lambda_n} \rfloor + 1 \leq k \leq n, \quad n \in B \\ 0, & \text{otherwise.} \end{cases} \quad (26)$$

For given $\varepsilon > 0$ and for each $z_1, \dots, z_{n-1} \in X$ we have,

$$\begin{aligned} &\lim_{n \rightarrow \infty} \frac{1}{\lambda_n} |\{k \in I_n : a_{nk} (\|u_k \Delta_v^m x_k - 0, z_1, \dots, z_{n-1}\|) \geq \varepsilon\}| \\ &< \frac{\lfloor \sqrt{\lambda_n} \rfloor}{\lambda_n} \rightarrow 0, \end{aligned} \quad (27)$$

for $n \notin B$. Hence for $\delta > 0$, there exists a positive integer n_0 such that $(1/\lambda_n)|\{k \in I_n : a_{nk}\|u_k\Delta_v^m x_k - 0, z_1, \dots, z_{n-1}\| \geq \varepsilon\}| < \delta$ for $n \notin B$ and $n \geq n_0$. Now we have $(1/\lambda_n)|\{k \in I_n : a_{nk}\|u_k\Delta_v^m x_k - 0, z_1, \dots, z_{n-1}\| \geq \varepsilon\}| \geq \delta \subset \{B \cup (1, 2, \dots, n_0 - 1)\}$. Since I be an admissible ideal it follows that $S_\lambda^{\Delta_v^m}(I, u, A) - \lim_{k \rightarrow \infty} \|u_k x_k, z_1, \dots, z_{n-1}\| \rightarrow 0$ for each $z_1, \dots, z_{n-1} \in X$.

On the other hand, if we take $p = (p_k) = 1$, for all $k \in \mathbb{N}$ then $x_k \notin V_{\mathcal{F}}^I[\Delta_v^m, \lambda, A, u, p, \|\cdot, \dots, \cdot\|]$. This contradicts the fact $S_\lambda^{\Delta_v^m}[I, A, u, \|\cdot, \dots, \cdot\|] = V_{\mathcal{F}}^I[\Delta_v^m, \lambda, A, u, p, \|\cdot, \dots, \cdot\|]$, so our supposition is wrong. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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