Research Article

Tricorns and Multicorns of S-Iteration Scheme

Shin Min Kang, Arif Rafiq, Abdul Latif, Abdul Aziz Shahid, and Young Chel Kwun

Department of Mathematics and RINS, Gyeongsang National University, Jinju 660-701, Republic of Korea
Department of Mathematics, Lahore Leads University, Lahore 54810, Pakistan
Department of Mathematics, King Abdulaziz University, Jeddah 21589, Saudi Arabia
Department of Mathematics, Dong-A University, Busan 604-714, Republic of Korea

Correspondence should be addressed to Young Chel Kwun; yckwun@dau.ac.kr

Received 12 September 2014; Accepted 12 January 2015

1. Introduction

In 1918, French mathematician Julia [1] investigated the iteration process of complex function and attained a Julia set. On the other hand, the object Mandelbrot set was given by Mandelbrot [2]. In 1989, Crowe et al. [3] considered formal analogy with Mandelbrot set and named it “Mandelbrot sets” and showed its feature bifurcations along arcs rather than at points. The word “tricorn” was coined by Milnor for the connectedness locus for antiholomorphic polynomials \( z^2 + c \), which plays an intermediate role between quadratic and cubic polynomials. Tricorn has many similarities with the Mandelbrot set due to a compact subset of \( \mathbb{C} \).

In this paper we introduce and visualize a new class of relative superior tricorns and relative superior multicorns using \( S \)-iteration scheme. This paper is organized as follows. In Section 2, some basic definitions are presented. Section 3 contains the escape criterion for relative superior tricorns and multicorns. In Section 4, we generate relative superior tricorns and multicorns of \( S \)-iteration scheme for quadratic, cubic, and biquadratic functions. At last, paper has been concluded in Section 5.

2. Preliminaries

Definition 1 (see [11], multicicorn). The multicorn \( A_c \) for the quadratic function \( A_c(z) = z^n + c \) is defined as the collection of all \( c \in \mathbb{C} \) for which the orbit of the point 0 is bounded; that is,

\[
A_c = \{ c \in \mathbb{C} : A_c^n(0) \text{ does not tend to } \infty \} ,
\]
where $C$ is a complex space and $A^n_c$ is the $n$th iterate of the function $A_c(z)$. An equivalent formulation is that the connectedness of loci for higher degree antiholomorphic polynomials $A_c(z) = z^n + c$ is called multicons.

Notice that, at $n = 2$, multicons reduce to tricorn. Moreover, the tricorn naturally lives in the real slice $d = \overline{d}$ in the two-dimensional parameter space of maps $z \rightarrow (z^2 + d)^2 + c$. They have $(n+1)$-fold rotational symmetries. Also, by dividing these symmetries, the resulting multicons are called unicorns [7].

Definition 2 (see [12], $S$-iteration scheme for relative superior tricorns and multicons). Let $X$ be a subset of real or complex numbers and $f : X \rightarrow X$. For $x_0 \in X$, one constructs the sequences $\{x_n\}$ and $\{y_n\}$ in $X$ in the following manner:

\[
x_n = (1 - s_{n-1}) f(x_{n-1}) + s_{n-1} f(y_{n-1}),
\]

\[
y_n = (1 - s_n') x_n + s_n' f(x_n),
\]

where $0 < s_n < 1$, $0 < s_n' < 1$, and $s_n$, $s_n'$ both are convergent to nonzero number.

The sequences $\{x_n\}$ and $\{y_n\}$ constructed above are called $S$-iteration scheme sequences of iterations or relative superior sequences of iterates. We denote it by $RSO(x_0, s_n, s_n', t)$.

Definition 3 (Mandelbrot set). The Mandelbrot set $M$ consists of all parameters $c$ for which the filled Julia set of $Q_c$ is connected; that is

\[
M = \{ c \in C : K(Q_c) \text{ is connected} \}.
\]

In fact, $M$ contains an enormous amount of information about the structure of Julia sets. The Mandelbrot set $M$ for the quadratic $Q_c(z) = z^2 + c$ is defined as the collection of all $c \in C$ for which the orbit of the point $0$ is bounded; that is

\[
M = \{ c \in C : |Q_c^n(0)| \text{ is bounded} \}.
\]

We choose the initial point $0$, as $0$ is the only critical point of $Q_c$ [11].

3. Escape Criterion for Relative Superior Tricorns and Multicons

The escape criterion plays an important role in the generation and analysis of relative superior tricorns and multicons. We now obtain a general escape criterion for polynomials of the form $G_c(z) = z^n + c$.

Theorem 4. For general function $G_c(z) = z^n + c$, $n = 1, 2, 3, \ldots$, suppose that $|z| \geq |c| > (2/s)^{1/n-1}$ and $|z| \geq |c| > (2/s')^{1/n-1}$, where $0 < s, s' < 1$, and $c$ is a complex number. Define

\[
z_1 = (1 - s) (z^n + c) + s G_c(z),
\]

\[
z_n = (1 - s) (z_{n-1}^n + c) + s G_c(z_{n-1}),
\]

Then $|z_n| \rightarrow \infty$ as $n \rightarrow \infty$. Thus the general escape criterion is $\max(|c|, (2/s)^{1/n-1}, (2/s')^{1/n-1})$.

Proof. We will use induction. For $n = 1$, we get $G_c(z) = z + c$, so the escape criterion is $|c|$, which is obvious; that is, $|z| > \max(1, |c|, 0, 0)$. For $n = 2$, we get $G_c(z) = z^2 + c$ so the escape criterion is $|z| > \max(|c|, 2/s, 2/s')$. For $n = 3$, we get $G_c(z) = z^3 + c$ so the escape criterion is $|z| > \max(|c|, (2/s)^{1/2}, (2/s')^{1/2})$.

Now suppose that theorem is true for any $n$. Let $G_c(z) = z^{n+1} + c$ and $|z| \geq |c| > (2/s)^{1/n}$ and $|z| \geq |c| > (2/s')^{1/n}$ exist. Then for $G_c(z) = z^{n+1} + c$, consider

\[
|G_c(z)| = |(1 - s') z + s' G_c(z)|
\]

\[
= |(1 - s') z + s' (z^{n+1} + c)|
\]

\[
= |(1 - s') z + s' z^n + s' c|
\]

\[
\geq |s' z^{n+1} + (1 - s') z| - |s' c|
\]

\[
\geq |s' z^{n+1} - 1| - |s' z^n|
\]

Also, for

\[
z_n = (1 - s) f(z_{n-1}) + s G_c(z),
\]

we obtain

\[
|z_n| = |(1 - s) (z^{n+1} + c) + s z| (\text{\textit{conclusion}})
\]

\[
= |(1 - s) z^{n+1} + (1 - s) c + s z| \leq |z| s' z^n - s |z|
\]

\[
\geq |s' z^{n+1} - 1| - |(1 - s) z^n|
\]

\[
\geq |s' z^{n+1} - 1| |s' z^n|
\]

\[
\geq |s' z^{n+1} - 1| - |z| + s |z|
\]

Since $|z| > (2/s)^{1/n}$ and $|z| > (2/s')^{1/n}$ it follows that $|z^n| > 2/s$ and $|z^n| > 2/s'$. It can be easily seen that

\[
|z^n| > \frac{2}{(ss')^{1/n}} > \frac{2}{(ss' - s + 1)},
\]

which implies that

\[
(ss' - s + 1) |z^n| - 1 > 1.
\]
Hence there exists $\lambda > 0$ such that $(ss' - s + 1)z^n - 1 > 1 + \lambda$. Consequently

$$|z_1| > (1 + \lambda)|z|,$$

$$\vdots$$

$$|z_n| > (1 + \lambda)^n|z|.$$  

Hence $|z_n| \to \infty$ as $n \to \infty$. So $|z| > \max(|c|, (2/s)^{1/n}, (2/s')^{1/n})$ is the escape criterion. This completes the proof.

**Corollary 5.** Suppose that $|c| > (2/s)^{1/n-1}$ and $|c| > (2/s')^{1/n-1}$ exist. Then the relative superior orbit of $S$-iteration scheme $RSO(G_c, 0, s, s')$ escapes to infinity.

**Corollary 6.** Assume that $|z_k| > \max(|c|, (2/s)^{1/k-1}, (2/s')^{1/k-1})$ for some $k \geq 0$. Then $|z_{k+1}| > \gamma^n|z_k|$ and $|z_n| \to \infty$ as $n \to \infty$.

This corollary provides an algorithm for computing the relative superior Mandelbrot sets for the functions of the form $G_c(z) = z^n + c$ and $n = 2, 3, \ldots$ also gives escape criterion to generate relative superior tricorns and multicorns.

**4. Generation of Relative Superior Tricorns and Multicorns**

We generate relative superior tricorns and multicorns of $S$-iteration scheme for quadratic, cubic, and biquadratic functions using software MAPLE.

**4.1. Relative Superior Tricorns for Quadratic Functions.** In case of quadratic antipolynomial, relative superior tricorns maintain the symmetry along $x$-axis (Figures 1–6).

**4.2. Relative Superior Multicorns for Cubic Functions.** In case of cubic antipolynomial, relative superior multicorns maintain the symmetry along $x$-axis and $y$-axis (Figures 7–12).

**4.3. Relative Superior Multicorns for Biquadratic Functions.** In case of biquadratic antipolynomial, relative superior multicorns maintain the symmetry along $x$-axis (Figures 13–18).


**5. Conclusions**

In this paper relative superior antifractal has been visualized with respect to relative superior orbit and analyzed the pattern of symmetry among them. In the dynamics of antipolynomials $z \to z^n + c$ for $n \geq 2$, we obtained many relative superior tricorns and multicorns for the same value of $n$ by using different values of $s$ and $s'$ in $S$-iteration scheme. We found that the number of branches and main ovoids attached to the branches of the relative superior tricorns and
Figure 4: Relative superior tricorn for $s = 0.6$ and $s' = 0.5$.

Figure 5: Relative superior tricorn for $s = 0.4$ and $s' = 0.7$.

Figure 6: Relative superior tricorn for $s = 0.3$ and $s' = 0.7$.

Figure 7: Relative superior multicorn for $s = 1.0$ and $s' = 1.0$.

Figure 8: Relative superior multicorn for $s = 0.3$ and $s' = 0.5$.

Figure 9: Relative superior multicorn for $s = 0.2$ and $s' = 0.6$. 
Figure 10: Relative superior multicorn for $s = 0.6$ and $s' = 0.5$.

Figure 11: Relative superior multicorn for $s = 0.4$ and $s' = 0.3$.

Figure 12: Relative superior multicorn for $s = 0.3$ and $s' = 0.8$.

Figure 13: Relative superior multicorn for $s = 1.0$ and $s' = 1.0$.

Figure 14: Relative superior multicorn for $s = 0.5$ and $s' = 0.6$.

Figure 15: Relative superior multicorn for $s = 0.1$ and $s' = 0.8$. 
Figure 16: Relative superior multicorn for $s = 0.2$ and $s' = 0.6$.

Figure 17: Relative superior multicorn for $s = 0.7$ and $s' = 0.4$.

Figure 18: Relative superior multicorn for $s = 0.3$ and $s' = 0.8$.

Figure 19: Relative superior multicorn for $s = 0.01$, $s' = 0.4$, and $n = 5$.

Figure 20: Relative superior multicorn for $s = 0.3$, $s' = 0.4$, and $n = 7$.

Figure 21: Relative superior multicorn for $s = 0.6$, $s' = 0.5$, and $n = 8$. 
multicorns had been $n + 1$, where $n$ is the power of $\mathbb{Z}$. We also found that for $n$ is odd the symmetry of relative superior multicorn is about both $x$-axis and $y$-axis but for $n$ is even the symmetry is maintained only along $x$-axis. We believe that results of this paper will inspire those who are interested in generating automatically nicely looking graphics.

**Conflict of Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.

**Acknowledgment**

This study was supported by research funds from Dong-A University.

**References**


Submit your manuscripts at
http://www.hindawi.com