Research Article

Hardy-Littlewood-Sobolev Inequalities on p -Adic Central Morrey Spaces

Qing Yan Wu and Zun Wei Fu

Department of Mathematics, Linyi University, Linyi, Shandong 276005, China

Correspondence should be addressed to Zun Wei Fu; zwfu@mail.bnu.edu.cn

Received 21 October 2014; Accepted 15 December 2014

Academic Editor: Yoshihiro Sawano

Copyright © 2015 Q. Y. Wu and Z. W. Fu. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We establish the Hardy-Littlewood-Sobolev inequalities on p -adic central Morrey spaces. Furthermore, we obtain the λ -central BMO estimates for commutators of p -adic Riesz potential on p -adic central Morrey spaces.

1. Introduction

Let $0 < \alpha < n$. The Riesz potential operator I_α is defined by setting, for all locally integrable functions f on $\mathbb{R}^n$,

$$I_\alpha f(x) = \frac{1}{\gamma_n(\alpha)} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy, \quad (1)$$

where $\gamma_n(\alpha) = \pi^{n/2} 2^\alpha \Gamma(\alpha/2) / \Gamma((n-\alpha)/2)$. It is closely related to the Laplacian operator of fractional degree. When $n > 2$ and $\alpha = 2$, $I_\alpha f$ is a solution of Poisson equation $-\Delta u = f$. The importance of Riesz potentials is owing to the fact that they are smooth operators and have been extensively used in various areas such as potential analysis, harmonic analysis, and partial differential equations. For more details about Riesz potentials one can refer to [1].

This paper focuses on the Riesz potentials on p -adic field. In the last 20 years, the field of p -adic numbers $\mathbb{Q}_p$ has been intensively used in theoretical and mathematical physics (cf. [2–12]). And it has already penetrated intensively into several areas of mathematics and its applications, among which harmonic analysis on p -adic field has been drawing more and more concern (see [13–22] and references therein).

For a prime number p , the field of p -adic numbers $\mathbb{Q}_p$ is defined as the completion of the field of rational numbers $\mathbb{Q}$ with respect to the non-Archimedean p -adic norm $|\cdot|_p$, which satisfies $|x|_p = 0$ if and only if $x = 0$; $|xy|_p = |x|_p |y|_p$; $|x+y|_p \leq \max\{|x|_p, |y|_p\}$. Moreover, if $|x|_p \neq |y|_p$, then $|x \pm y|_p = \max\{|x|_p, |y|_p\}$. It is well-known that $\mathbb{Q}_p$

is a typical model of non-Archimedean local fields. If any nonzero rational number x is represented as $x = p^\gamma(m/n)$, where $\gamma = \gamma(x) \in \mathbb{Z}$ and integers m, n are indivisible by p , then $|x|_p = p^{-\gamma}$.

The space $\mathbb{Q}_p^n = \mathbb{Q}_p \times \mathbb{Q}_p \times \cdots \times \mathbb{Q}_p$ consists of points $x = (x_1, x_2, \dots, x_n)$, where $x_j \in \mathbb{Q}_p$, $j = 1, 2, \dots, n$. The p -adic norm on $\mathbb{Q}_p^n$ is

$$|x|_p := \max_{1 \leq j \leq n} |x_j|_p, \quad x \in \mathbb{Q}_p^n. \quad (2)$$

Denote by

$$B_\gamma(a) = \{x \in \mathbb{Q}_p^n : |x-a|_p \leq p^\gamma\} \quad (3)$$

the ball of radius p^γ with center at $a \in \mathbb{Q}_p^n$ and by

$$S_\gamma(a) = B_\gamma(a) \setminus B_{\gamma-1}(a) = \{x \in \mathbb{Q}_p^n : |x-a|_p = p^\gamma\} \quad (4)$$

the sphere of radius p^γ with center at $a \in \mathbb{Q}_p^n$, where $\gamma \in \mathbb{Z}$. It is clear that

$$B_\gamma(a) = \bigcup_{k \leq \gamma} S_k(a). \quad (5)$$

It is well-known that $\mathbb{Q}_p^n$ is a classical kind of locally compact Vilenkin groups. A locally compact Vilenkin group G is a locally compact Abelian group containing a strictly decreasing sequence of compact open subgroups $\{G_n\}_{n=-\infty}^\infty$

such that (1) $\bigcup_{n=-\infty}^{\infty} G_n = G$ and $\bigcap_{n=-\infty}^{\infty} G_n = 0$ and (2) $\sup\{\text{order}(G_n/G_{n+1}) : n \in \mathbb{Z}\} < \infty$. For several decades, parallel to the p -adic harmonic analysis, a development was under way of the harmonic analysis on locally compact Vilenkin groups (cf. [23–25] and references therein).

Since $\mathbb{Q}_p^n$ is a locally compact commutative group under addition, it follows from the standard analysis that there exists a Haar measure dx on $\mathbb{Q}_p^n$, which is unique up to a positive constant factor and is translation invariant. We normalize the measure dx by the equality

$$\int_{B_0(0)} dx = |B_0(0)|_H = 1, \quad (6)$$

where $|E|_H$ denotes the Haar measure of a measurable subset E of $\mathbb{Q}_p^n$. By simple calculation, we can obtain that

$$\begin{aligned} |B_\gamma(a)|_H &= p^{\gamma n}, \\ |S_\gamma(a)|_H &= p^{\gamma n} (1 - p^{-n}) \end{aligned} \quad (7)$$

for any $a \in \mathbb{Q}_p^n$. We should mention that the Haar measure takes value in $\mathbb{R}$; there also exist p -adic valued measures (cf. [26, 27]). For a more complete introduction to the p -adic field, one can refer to [22] or [10].

On p -adic field, the p -adic Riesz potential I_α^p [22] is defined by

$$I_\alpha^p f(x) = \frac{1}{\Gamma_n(\alpha)} \int_{\mathbb{Q}_p^n} \frac{f(y)}{|x-y|_p^{n-\alpha}} dy, \quad (8)$$

where $\Gamma_n(\alpha) = (1 - p^{\alpha-n})/(1 - p^{-\alpha})$, $\alpha \in \mathbb{C}$, $\alpha \neq 0$. When $n = 1$, Haran [4, 28] obtained the explicit formula of Riesz potentials on $\mathbb{Q}_p$ and developed analytical potential theory on $\mathbb{Q}_p$. Taibleson [22] gave the fundamental analytic properties of the Riesz potentials on local fields including $\mathbb{Q}_p^n$, as well as the classical Hardy-Littlewood-Sobolev inequalities. Kim [18] gave a simple proof of these inequalities by using the p -adic version of the Calderón-Zygmund decomposition technique. Volosivets [29] investigated the boundedness for Riesz potentials on generalized Morrey spaces. Like on Euclidean spaces, using the Riesz potential with $n > 2$ and $\alpha = 2$, one can introduce the p -adic Laplacians [13].

In this paper, we will consider the Riesz potentials and their commutators with p -adic central BMO functions on p -adic central Morrey spaces. Alvarez et al. [30] studied the relationship between central BMO spaces and Morrey spaces. Furthermore, they introduced λ -central BMO spaces and central Morrey spaces, respectively. In [31], we introduce their p -adic versions.

Definition 1. Let $\lambda \in \mathbb{R}$ and $1 < q < \infty$. The p -adic central Morrey space $\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)$ is defined by

$$\|f\|_{\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)} := \sup_{\gamma \in \mathbb{Z}} \left(\frac{1}{|B_\gamma|_H^{1+\lambda q}} \int_{B_\gamma} |f(x)|^q dx \right)^{1/q} < \infty, \quad (9)$$

where $B_\gamma = B_\gamma(0)$.

Remark 2. It is clear that

$$\begin{aligned} L^{q,\lambda}(\mathbb{Q}_p^n) &\subset \dot{B}^{q,\lambda}(\mathbb{Q}_p^n), \\ \dot{B}^{q,-1/q}(\mathbb{Q}_p^n) &= L^q(\mathbb{Q}_p^n). \end{aligned} \quad (10)$$

When $\lambda < -1/q$, the space $\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)$ reduces to $\{0\}$; therefore, we can only consider the case $\lambda \geq -1/q$. If $1 \leq q_1 < q_2 < \infty$, by Hölder's inequality,

$$\dot{B}^{q_2,\lambda}(\mathbb{Q}_p^n) \subset \dot{B}^{q_1,\lambda}(\mathbb{Q}_p^n) \quad (11)$$

for $\lambda \in \mathbb{R}$.

Definition 3. Let $\lambda < 1/n$ and $1 < q < \infty$. The space $\text{CBMO}^{q,\lambda}(\mathbb{Q}_p^n)$ is defined by the condition

$$\begin{aligned} &\|f\|_{\text{CBMO}^{q,\lambda}(\mathbb{Q}_p^n)} \\ &:= \sup_{\gamma \in \mathbb{Z}} \left(\frac{1}{|B_\gamma|_H^{1+\lambda q}} \int_{B_\gamma} |f(x) - f_{B_\gamma}|^q dx \right)^{1/q} < \infty. \end{aligned} \quad (12)$$

Remark 4. When $\lambda = 0$, the space $\text{CBMO}^{q,\lambda}(\mathbb{Q}_p^n)$ is just $\text{CBMO}^q(\mathbb{Q}_p^n)$, which is defined in [32]. If $1 \leq q_1 < q_2 < \infty$, by Hölder's inequality,

$$\text{CBMO}^{q_2,\lambda}(\mathbb{Q}_p^n) \subset \text{CBMO}^{q_1,\lambda}(\mathbb{Q}_p^n) \quad (13)$$

for $\lambda \in \mathbb{R}$. By the standard proof as that in $\mathbb{R}^n$, we can see that

$$\begin{aligned} &\|f\|_{\text{CBMO}^{q,\lambda}(\mathbb{Q}_p^n)} \\ &\sim \sup_{\gamma \in \mathbb{Z}} \inf_{c \in \mathbb{C}} \left(\frac{1}{|B_\gamma|_H^{1+\lambda q}} \int_{B_\gamma} |f(x) - c|^q dx \right)^{1/q}. \end{aligned} \quad (14)$$

Remark 5. Formulas (9) and (12) yield that $\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)$ is a Banach space continuously included in $\text{CBMO}^{q,\lambda}(\mathbb{Q}_p^n)$.

Here we introduce the p -adic weak central Morrey spaces.

Definition 6. Let $\lambda \in \mathbb{R}$ and $1 < q < \infty$. The p -adic weak central Morrey space $W\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)$ is defined by

$$\begin{aligned} &\|f\|_{W\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)} \\ &:= \sup_{\gamma \in \mathbb{Z}} \left(\frac{\sup_{t>0} t^q |\{x \in B_\gamma : |f(x)| > t\}|_H}{|B_\gamma|_H^{1+\lambda q}} \right)^{1/q} \\ &< \infty, \end{aligned} \quad (15)$$

where $B_\gamma = B_\gamma(0)$.

In Section 2, we will get the Hardy-Littlewood-Sobolev inequalities on p -adic central Morrey spaces. Namely, under

some conditions for indexes, I_α^p is bounded from $\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)$ to $\dot{B}^{r,\mu}(\mathbb{Q}_p^n)$ and is also bounded from $\dot{B}^{1,\lambda}(\mathbb{Q}_p^n)$ to $W\dot{B}^{r,\mu}(\mathbb{Q}_p^n)$. In Section 3, we establish the boundedness for commutators generated by I_α^p and λ -central BMO functions on p -adic central Morrey spaces.

Throughout this paper the letter C will be used to denote various constants, and the various uses of the letter do not, however, denote the same constant.

2. Hardy-Littlewood-Sobolev Inequalities

We get the following Hardy-Littlewood-Sobolev inequalities on p -adic central Morrey spaces.

Theorem 7. *Let α be a complex number with $0 < \operatorname{Re} \alpha < n$ and let $1 \leq q < n/\operatorname{Re} \alpha$, $0 < 1/r = 1/q - \operatorname{Re} \alpha/n$, $\lambda < -\operatorname{Re} \alpha/n$, and $\mu = \lambda + \operatorname{Re} \alpha/n$.*

- (i) *If $q > 1$, then I_α^p is bounded from $\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)$ to $\dot{B}^{r,\mu}(\mathbb{Q}_p^n)$.*
- (ii) *If $q = 1$, then I_α^p is bounded from $\dot{B}^{1,\lambda}(\mathbb{Q}_p^n)$ to $W\dot{B}^{r,\mu}(\mathbb{Q}_p^n)$.*

In order to give the proof of this theorem, we need the following result.

Lemma 8 (see [22]). *Let α be a complex number with $0 < \operatorname{Re} \alpha < n$ and let $1 \leq q < r < \infty$ satisfy $1/r = 1/q - \operatorname{Re} \alpha/n$.*

- (i) *If $f \in L^q(\mathbb{Q}_p^n)$, $q > 1$, then*

$$\|I_\alpha^p f\|_{L^r(\mathbb{Q}_p^n)} \leq A_{qr} \|f\|_{L^q(\mathbb{Q}_p^n)}, \quad (16)$$

where A_{qr} is independent of f .

- (ii) *If $f \in L^1(\mathbb{Q}_p^n)$, $s > 0$, then*

$$\left| \left\{ x \in \mathbb{Q}_p^n : |I_\alpha^p f(x)| > s \right\} \right|_H \leq \left(A_r \frac{\|f\|_{L^1(\mathbb{Q}_p^n)}}{s} \right)^r, \quad (17)$$

where $A_r > 0$ is independent of f .

Proof of Theorem 7. Let f be a function in $\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)$. For fixed $\gamma \in \mathbb{Z}$, denote $B_\gamma(0)$ by B_γ .

- (i) If $q > 1$, write

$$\begin{aligned} & \left(\frac{1}{|B_\gamma|_H^{1+\mu r}} \int_{B_\gamma} |I_\alpha^p f(x)|^r dx \right)^{1/r} \\ & \leq \left(\frac{1}{|B_\gamma|_H^{1+\mu r}} \int_{B_\gamma} |I_\alpha^p(f\chi_{B_\gamma})(x)|^r dx \right)^{1/r} \\ & \quad + \left(\frac{1}{|B_\gamma|_H^{1+\mu r}} \int_{B_\gamma} |I_\alpha^p(f\chi_{B_\gamma^c})(x)|^r dx \right)^{1/r} \\ & := I + II. \end{aligned} \quad (18)$$

For I , since $1/r = 1/q - \operatorname{Re} \alpha/n$ and $\mu = \lambda + \operatorname{Re} \alpha/n$, by Lemma 8,

$$\begin{aligned} I &= \left(\frac{1}{|B_\gamma|_H^{1+\mu r}} \int_{B_\gamma} |I_\alpha^p(f\chi_{B_\gamma})(x)|^r dx \right)^{1/r} \\ &\leq |B_\gamma|_H^{-1/r-\mu} \left(\int_{B_\gamma} |f\chi_{B_\gamma}(x)|^q dx \right)^{1/q} \\ &\leq \|f\|_{\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)}. \end{aligned} \quad (19)$$

For II , we firstly give the following estimate. For $x \in B_\gamma$, by Hölder's inequality, we have

$$\begin{aligned} & |I_\alpha^p(f\chi_{B_\gamma^c})(x)| \\ &= \left| \frac{1}{\Gamma_n(\alpha)} \int_{B_\gamma^c} \frac{f(y)}{|x-y|_p^{n-\alpha}} dy \right| \\ &\leq \frac{1}{\Gamma_n(\alpha)} \int_{B_\gamma^c} \frac{|f(y)|}{|x-y|_p^{n-\operatorname{Re} \alpha}} dy \\ &= \frac{1}{\Gamma_n(\alpha)} \sum_{k=\gamma+1}^{\infty} \int_{S_k} \frac{|f(y)|}{|x-y|_p^{n-\operatorname{Re} \alpha}} dy \\ &= \frac{1}{\Gamma_n(\alpha)} \sum_{k=\gamma+1}^{\infty} \int_{S_k} p^{-k(n-\operatorname{Re} \alpha)} |f(y)| dy \\ &\leq \frac{1}{\Gamma_n(\alpha)} \sum_{k=\gamma+1}^{\infty} p^{-k(n-\operatorname{Re} \alpha)} \left(\int_{B_k} |f(y)|^q dy \right)^{1/q} |B_k|_H^{1-1/q} \\ &\leq \frac{1}{\Gamma_n(\alpha)} \|f\|_{\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)} \sum_{k=\gamma+1}^{\infty} p^{-k(n-\operatorname{Re} \alpha)} |B_k|_H^{1+\lambda} \\ &\leq C |B_\gamma|_H^\mu \|f\|_{\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)}. \end{aligned} \quad (20)$$

The last inequality is due to the fact that $\lambda < -\operatorname{Re} \alpha/n$. Consequently,

$$\begin{aligned} II &= \left(\frac{1}{|B_\gamma|_H^{1+\mu r}} \int_{B_\gamma} |I_\alpha^p(f\chi_{B_\gamma^c})(x)|^r dx \right)^{1/r} \\ &\leq C \|f\|_{\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)}. \end{aligned} \quad (21)$$

The above estimates imply that

$$\|I_\alpha^p f\|_{\dot{B}^{r,\mu}(\mathbb{Q}_p^n)} \leq C \|f\|_{\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)}. \quad (22)$$

(ii) If $q = 1$, set $f_1 = f\chi_{B_\gamma}$ and $f_2 = f - f_1$; by Lemma 8, we have

$$\begin{aligned} & \left| \left\{ x \in B_\gamma : |I_\alpha^p f_1(x)| > t \right\} \right|_H \\ & \leq C \left(\frac{\|f_1\|_{L^1(\mathbb{Q}_p^n)}}{t} \right)^r = Ct^{-r} \left(\int_{B_\gamma} |f(x)| dx \right)^r \\ & \leq Ct^{-r} |B_\gamma|_H^{(1+\lambda)r} \|f\|_{\dot{B}^{1,\lambda}(\mathbb{Q}_p^n)}^r \\ & = Ct^{-r} |B_\gamma|_H^{1+\mu r} \|f\|_{\dot{B}^{1,\lambda}(\mathbb{Q}_p^n)}^r. \end{aligned} \quad (23)$$

On the other hand, by the same estimate as (30), we have

$$|I_\alpha^p f_2(x)| \leq C |B_\gamma|_H^\mu \|f_2\|_{\dot{B}^{1,\lambda}(\mathbb{Q}_p^n)}. \quad (24)$$

Then using Chebyshev's inequality, we obtain

$$\begin{aligned} & \left| \left\{ x \in B_\gamma : |I_\alpha^p f_2(x)| > t \right\} \right|_H \leq t^{-r} \int_{B_\gamma} |I_\alpha^p f_2(x)|^r dx \\ & \leq Ct^{-r} |B_\gamma|_H^{1+\mu r} \|f_2\|_{\dot{B}^{1,\lambda}(\mathbb{Q}_p^n)}^r \\ & \leq Ct^{-r} |B_\gamma|_H^{1+\mu r} \|f\|_{\dot{B}^{1,\lambda}(\mathbb{Q}_p^n)}^r. \end{aligned} \quad (25)$$

Since

$$|I_\alpha^p f(x)| \leq |I_\alpha^p f_1(x)| + |I_\alpha^p f_2(x)|, \quad (26)$$

we get

$$\begin{aligned} & \left| \left\{ x \in B_\gamma : |I_\alpha^p f(x)| > t \right\} \right|_H \leq \left| \left\{ x \in B_\gamma : |I_\alpha^p f_1(x)| > \frac{t}{2} \right\} \right|_H \\ & \quad + \left| \left\{ x \in B_\gamma : |I_\alpha^p f_2(x)| > \frac{t}{2} \right\} \right|_H \\ & \leq Ct^{-r} |B_\gamma|_H^{1+\mu r} \|f\|_{\dot{B}^{1,\lambda}(\mathbb{Q}_p^n)}^r. \end{aligned} \quad (27)$$

Therefore,

$$\left(\frac{t^r \left| \left\{ x \in B_\gamma : |I_\alpha^p f(x)| > t \right\} \right|_H}{|B_\gamma|_H^{1+\mu r}} \right)^{1/r} \leq C \|f\|_{\dot{B}^{1,\lambda}(\mathbb{Q}_p^n)}, \quad (28)$$

for any $t > 0$ and $\gamma \in \mathbb{Z}$. This completes the proof. $\square$

For application, we now introduce a pseudo-differential operator D^α defined by Vladimirov in [33].

The operator $D^\alpha : \psi \rightarrow D^\alpha \psi$ is defined as convolution of generalized functions $f_{-\alpha}$ and ψ :

$$D^\alpha \psi = f_{-\alpha} * \psi, \quad \alpha \neq -1, \quad (29)$$

where $f_\alpha = |x|_p^{\alpha-1}/\Gamma(\alpha)$ and $\Gamma(\alpha) = (1-p^{\alpha-1})/(1-p^{-\alpha})$.

Let us consider the equation

$$D^\alpha \psi = g, \quad g \in \mathcal{E}', \quad (30)$$

where $\mathcal{E}'$ is the space of linear continuous functionals on $\mathcal{E}$ and here $\mathcal{E}$ denotes the set of locally constant functions on $\mathbb{Q}_p$. A complex-valued function $f(x)$ defined on $\mathbb{Q}_p$ is called locally constant if for any point $x \in \mathbb{Q}_p$ there exists an integer $l(x) \in \mathbb{Z}$ such that

$$\begin{aligned} f(x+x') &= f(x), \\ |x'|_p &\leq p^{l(x)}. \end{aligned} \quad (31)$$

The following lemma (page 154 in [10]) gives solutions of (30).

Lemma 9. For $\alpha > 0$ any solution of (30) is expressed by the formula

$$\psi = D^{-\alpha} g + C, \quad (32)$$

where C is an arbitrary constant; for $\alpha < 0$ a solution of (30) is unique and it is expressed by formula (32) for $C = 0$.

Combining with Theorem 7, we obtain the following regular property of the solution.

Corollary 10. Let $0 < \alpha < 1$ and let $1 \leq q < 1/\alpha$, $0 < 1/r = 1/q - \alpha$, $\lambda < -\alpha$, and $\mu = \lambda + \alpha$. If $g \in \mathcal{E}' \cap \dot{B}^{q,\lambda}(\mathbb{Q}_p^n)$, then

- (i) when $q > 1$, (30) has a solution in $\dot{B}^{r,\mu}(\mathbb{Q}_p^n)$,
- (ii) when $q = 1$, (30) has a solution in $W\dot{B}^{r,\mu}(\mathbb{Q}_p^n)$.

3. Commutators of p -Adic Riesz Potential

In this section, we will establish the λ -central BMO estimates for commutators $I_\alpha^{p,b}$ of p -adic Riesz potential which is defined by

$$I_\alpha^{p,b} f = b I_\alpha^p f - I_\alpha^p (bf), \quad (33)$$

for some suitable functions f .

Theorem 11. Suppose $0 < \operatorname{Re} \alpha < n$, $1 < q_1 < n/\operatorname{Re} \alpha$, $q'_1 < q_2 < \infty$, and $1/q = 1/q_1 + 1/q_2 - \operatorname{Re} \alpha/n$. Let $0 \leq \lambda_2 < 1/n$, λ_1 satisfies $\lambda_1 < -\lambda_2 - \operatorname{Re} \alpha/n$, and $\lambda = \lambda_1 + \lambda_2 + \operatorname{Re} \alpha/n$. If $b \in CBMO^{q_2,\lambda_2}(\mathbb{Q}_p^n)$, then $I_\alpha^{p,b}$ is bounded from $\dot{B}^{q_1,\lambda_1}(\mathbb{Q}_p^n)$ to $\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)$, and the following inequality holds:

$$\|I_\alpha^{p,b} f\|_{\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)} \leq C \|b\|_{CBMO^{q_2,\lambda_2}(\mathbb{Q}_p^n)} \|f\|_{\dot{B}^{q_1,\lambda_1}(\mathbb{Q}_p^n)}. \quad (34)$$

Before proving this theorem, we need the following result.

Lemma 12 (see [31]). Suppose that $b \in CBMO^{q,\lambda}(\mathbb{Q}_p^n)$ and $j, k \in \mathbb{Z}$, $\lambda \geq 0$. Then

$$\begin{aligned} & |b_{B_j} - b_{B_k}| \\ & \leq p^n |j - k| \|b\|_{CBMO^{q,\lambda}(\mathbb{Q}_p^n)} \max \left\{ |B_j|_H^\lambda, |B_k|_H^\lambda \right\}. \end{aligned} \quad (35)$$

Proof of Theorem 11. Suppose that f is a function in $\dot{B}^{q_1, \lambda_1}(\mathbb{Q}_p^n)$. For fixed $\gamma \in \mathbb{Z}$, denote $B_\gamma(0)$ by B_γ . We write

$$\begin{aligned} & \left(\frac{1}{|B_\gamma|_H} \int_{B_\gamma} |I_\alpha^{p,b} f(x)|^q dx \right)^{1/q} \\ & \leq \left(\frac{1}{|B_\gamma|_H} \int_{B_\gamma} |(b(x) - b_{B_\gamma})(I_\alpha^p f \chi_{B_\gamma})(x)|^q dx \right)^{1/q} \\ & \quad + \left(\frac{1}{|B_\gamma|_H} \int_{B_\gamma} |(b(x) - b_{B_\gamma})(I_\alpha^p f \chi_{B_\gamma^c})(x)|^q dx \right)^{1/q} \\ & \quad + \left(\frac{1}{|B_\gamma|_H} \int_{B_\gamma} |I_\alpha^p((b - b_{B_\gamma}) f \chi_{B_\gamma})(x)|^q dx \right)^{1/q} \\ & \quad + \left(\frac{1}{|B_\gamma|_H} \int_{B_\gamma} |I_\alpha^p((b - b_{B_\gamma}) f \chi_{B_\gamma^c})(x)|^q dx \right)^{1/q} \\ & := J_1 + J_2 + J_3 + J_4. \end{aligned} \quad (36)$$

Set $1/r = 1/q_1 - \text{Re } \alpha/n$; then $1/q = 1/q_2 + 1/r$; by Lemma 8 and Hölder's inequality, we have

$$\begin{aligned} J_1 &= \left(\frac{1}{|B_\gamma|_H} \int_{B_\gamma} |(b(x) - b_{B_\gamma})(I_\alpha^p f \chi_{B_\gamma})(x)|^q dx \right)^{1/q} \\ &\leq |B_\gamma|_H^{-1/q} \left(\int_{B_\gamma} |b(x) - b_{B_\gamma}|^{q_2} dx \right)^{1/q_2} \\ &\quad \cdot \left(\int_{B_\gamma} |I_\alpha^p(f \chi_{B_\gamma})(x)|^r dx \right)^{1/r} \\ &\leq C |B_\gamma|_H^{-1/r+\lambda_2} \|b\|_{\text{CBMO}^{q_2, \lambda_2}(\mathbb{Q}_p^n)} \\ &\quad \cdot \left(\int_{B_\gamma} |f \chi_{B_\gamma}(x)|^{q_1} dx \right)^{1/q_1} \\ &\leq C |B_\gamma|_H^\lambda \|b\|_{\text{CBMO}^{q_2, \lambda_2}(\mathbb{Q}_p^n)} \|f\|_{\dot{B}^{q_1, \lambda_1}(\mathbb{Q}_p^n)}. \end{aligned} \quad (37)$$

Similarly, denote $1/l = 1/q_1 + 1/q_2$; then $1/q = 1/l - \text{Re } \alpha/n$, and by Hölder's inequality and Lemma 8, we get

$$\begin{aligned} J_3 &= \left(\frac{1}{|B_\gamma|_H} \int_{B_\gamma} |I_\alpha^p((b - b_{B_\gamma}) f \chi_{B_\gamma})(x)|^q dx \right)^{1/q} \\ &\leq C |B_\gamma|_H^{-1/q} \left(\int_{B_\gamma} |(b(x) - b_{B_\gamma}) f(x)|^l dx \right)^{1/l} \\ &\leq C |B_\gamma|_H^{-1/q} \left(\int_{B_\gamma} |b(x) - b_{B_\gamma}|^{q_2} dx \right)^{1/q_2} \\ &\quad \cdot \left(\int_{B_\gamma} |f(x)|^{q_1} dx \right)^{1/q_1} \\ &\leq C |B_\gamma|_H^\lambda \|b\|_{\text{CBMO}^{q_2, \lambda_2}(\mathbb{Q}_p^n)} \|f\|_{\dot{B}^{q_1, \lambda_1}(\mathbb{Q}_p^n)}. \end{aligned} \quad (38)$$

To estimate J_2 and J_4 , we firstly give the following estimates. For $x \in B_\gamma$, by Hölder's inequality, we obtain

$$\begin{aligned} & |I_\alpha^p(f \chi_{B_\gamma^c})(x)| \\ &= \left| \frac{1}{\Gamma_n(\alpha)} \int_{B_\gamma^c} \frac{f(y)}{|x-y|_p^{n-\alpha}} dy \right| \leq \frac{1}{\Gamma_n(\alpha)} \int_{B_\gamma^c} \frac{|f(y)|}{|x-y|_p^{n-\text{Re } \alpha}} dy \\ &= \frac{1}{\Gamma_n(\alpha)} \sum_{k=\gamma+1}^{\infty} \int_{S_k} |f(y)| p^{-k(n-\text{Re } \alpha)} dy \\ &\leq \frac{1}{\Gamma_n(\alpha)} \sum_{k=\gamma+1}^{\infty} p^{-k(n-\text{Re } \alpha)} |B_k|_H^{1-1/q_1} \left(\int_{S_k} |f(y)|^{q_1} dy \right)^{1/q_1} \\ &\leq \frac{1}{\Gamma_n(\alpha)} \|f\|_{\dot{B}^{q_1, \lambda_1}(\mathbb{Q}_p^n)} \sum_{k=\gamma+1}^{\infty} p^{-k(n-\text{Re } \alpha)} |B_k|_H^{1+\lambda_1} \\ &= \frac{1}{\Gamma_n(\alpha)} \|f\|_{\dot{B}^{q_1, \lambda_1}(\mathbb{Q}_p^n)} \frac{p^{(\gamma+1)(n\lambda_1+\text{Re } \alpha)}}{1-p^{n\lambda_1+\text{Re } \alpha}} \\ &= C |B_\gamma|_H^{\lambda_1+\text{Re } \alpha/n} \|f\|_{\dot{B}^{q_1, \lambda_1}(\mathbb{Q}_p^n)}, \end{aligned} \quad (39)$$

where the penultimate “=” is due to the fact that $\lambda_1 + \text{Re } \alpha/n < -\lambda_2 \leq 0$. Similarly,

$$\begin{aligned} & |I_\alpha^p((b - b_{B_\gamma}) f \chi_{B_\gamma^c})(x)| \\ &= \left| \frac{1}{\Gamma_n(\alpha)} \int_{B_\gamma^c} \frac{(b(y) - b_{B_\gamma}) f(y)}{|x-y|_p^{n-\alpha}} dy \right| \\ &\leq \frac{1}{\Gamma_n(\alpha)} \int_{B_\gamma^c} \frac{|b(y) - b_{B_\gamma}| |f(y)|}{|x-y|_p^{n-\text{Re } \alpha}} dy \\ &= \frac{1}{\Gamma_n(\alpha)} \sum_{k=\gamma+1}^{\infty} \int_{S_k} |b(y) - b_{B_\gamma}| |f(y)| p^{-k(n-\text{Re } \alpha)} dy \\ &= \frac{1}{\Gamma_n(\alpha)} \sum_{k=\gamma+1}^{\infty} p^{-k(n-\text{Re } \alpha)} |B_k|_H^{1-1/q_1-1/q_2} \\ &\quad \cdot \left(\int_{S_k} |f(y)|^{q_1} dy \right)^{1/q_1} \left(\int_{S_k} |b(y) - b_{B_\gamma}|^{q_2} dy \right)^{1/q_2} \\ &\leq \frac{1}{\Gamma_n(\alpha)} \|f\|_{\dot{B}^{q_1, \lambda_1}(\mathbb{Q}_p^n)} \sum_{k=\gamma+1}^{\infty} p^{-k(n-\text{Re } \alpha)} |B_k|_H^{1-1/q_2+\lambda_1} \\ &\quad \cdot \left(\int_{B_k} |b(y) - b_{B_\gamma}|^{q_2} dy \right)^{1/q_2} \\ &\leq \frac{1}{\Gamma_n(\alpha)} \|f\|_{\dot{B}^{q_1, \lambda_1}(\mathbb{Q}_p^n)} \sum_{k=\gamma+1}^{\infty} p^{-k(n-\text{Re } \alpha)} |B_k|_H^{1-1/q_2+\lambda_1} \\ &\quad \times \left[\left(\int_{B_k} |b(y) - b_{B_\gamma}|^{q_2} dy \right)^{1/q_2} + |b_k - b_{B_\gamma}| |B_k|_H^{1/q_2} \right]. \end{aligned} \quad (40)$$

Since $k \geq \gamma + 1$, by Lemma 12, we have

$$|b_{B_k} - b_{B_\gamma}| \leq p^n (k - \gamma) \|b\|_{\text{CBMO}^{q_2, \lambda_2}(\mathbb{Q}_p^n)} |B_k|_H^{\lambda_2}. \quad (41)$$

Thus

$$\begin{aligned} & \left| I_\alpha^p \left((b - b_{B_\gamma}) f \chi_{B_\gamma^c} \right) (x) \right| \\ & \leq \frac{1}{\Gamma_n(\alpha)} \|b\|_{\text{CBMO}^{q_2, \lambda_2}(\mathbb{Q}_p^n)} \|f\|_{\dot{B}^{q_1, \lambda_1}(\mathbb{Q}_p^n)} \\ & \quad \times \sum_{k=\gamma+1}^{\infty} p^{-k(n-\text{Re } \alpha)} |B_k|_H^{1-1/q_2+\lambda_1} \\ & \quad \cdot \left[|B_k|_H^{1/q_2+\lambda_2} + p^n (k - \gamma) |B_k|_H^{1/q_2+\lambda_2} \right] \\ & \leq \frac{C}{\Gamma_n(\alpha)} \|b\|_{\text{CBMO}^{q_2, \lambda_2}(\mathbb{Q}_p^n)} \|f\|_{\dot{B}^{q_1, \lambda_1}(\mathbb{Q}_p^n)} \\ & \quad \cdot \sum_{k=\gamma+1}^{\infty} (k - \gamma) p^{-k(n-\text{Re } \alpha)} |B_k|_H^{1+\lambda_1+\lambda_2} \\ & \leq \frac{C}{\Gamma_n(\alpha)} \|b\|_{\text{CBMO}^{q_2, \lambda_2}(\mathbb{Q}_p^n)} \|f\|_{\dot{B}^{q_1, \lambda_1}(\mathbb{Q}_p^n)} \sum_{k=\gamma+1}^{\infty} (k - \gamma) p^{kn\lambda} \\ & = C |B_\gamma|_H^\lambda \|b\|_{\text{CBMO}^{q_2, \lambda_2}(\mathbb{Q}_p^n)} \|f\|_{\dot{B}^{q_1, \lambda_1}(\mathbb{Q}_p^n)}. \end{aligned} \quad (42)$$

Now by (39) and Hölder's inequality, we obtain

$$\begin{aligned} J_2 &= \left(\frac{1}{|B_\gamma|_H} \int_{B_\gamma} \left| (b(x) - b_{B_\gamma}) \left(I_\alpha^p f \chi_{B_\gamma^c} \right) (x) \right|^q dx \right)^{1/q} \\ &\leq C |B_\gamma|_H^{\lambda_1 + \text{Re } \alpha / n - 1/q} \|f\|_{\dot{B}^{q_1, \lambda_1}(\mathbb{Q}_p^n)} \\ &\quad \cdot \left(\int_{B_\gamma} |b(x) - b_{B_\gamma}|^q dx \right)^{1/q} \\ &\leq C |B_\gamma|_H^{\lambda_1 + \text{Re } \alpha / n - 1/q_2} \|f\|_{\dot{B}^{q_1, \lambda_1}(\mathbb{Q}_p^n)} \\ &\quad \cdot \left(\int_{B_\gamma} |b(x) - b_{B_\gamma}|^{q_2} dx \right)^{1/q_2} \\ &\leq C |B_\gamma|_H^\lambda \|b\|_{\text{CBMO}^{q_2, \lambda_2}(\mathbb{Q}_p^n)} \|f\|_{\dot{B}^{q_1, \lambda_1}(\mathbb{Q}_p^n)}. \end{aligned} \quad (43)$$

It follows from (42) that

$$\begin{aligned} J_4 &= \left(\frac{1}{|B_\gamma|_H} \int_{B_\gamma} \left| I_\alpha^p \left((b - b_{B_\gamma}) f \chi_{B_\gamma^c} \right) (x) \right|^q dx \right)^{1/q} \\ &\leq C |B_\gamma|_H^\lambda \|b\|_{\text{CBMO}^{q_2, \lambda_2}(\mathbb{Q}_p^n)} \|f\|_{\dot{B}^{q_1, \lambda_1}(\mathbb{Q}_p^n)}. \end{aligned} \quad (44)$$

The above estimates imply that

$$\|I_\alpha^{p,b} f\|_{\dot{B}^{q, \lambda}(\mathbb{Q}_p^n)} \leq C \|b\|_{\text{CBMO}^{q_2, \lambda_2}(\mathbb{Q}_p^n)} \|f\|_{\dot{B}^{q_1, \lambda_1}(\mathbb{Q}_p^n)}. \quad (45)$$

This completes the proof of the theorem. $\square$

Remark 13. Since p -adic field is a kind of locally compact Vilenkin groups, we can further consider the Hardy-Littlewood-Sobolev inequalities on such groups, which is more complicated and will appear elsewhere.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This work was partially supported by NSF of China (Grant nos. 11271175, 11171345, and 11301248) and AMEP (DYSP) of Linyi University and Macao Science and Technology Development Fund, MSAR (Ref. 018/2014/A1).

References

- [1] L. Grafakos, *Modern Fourier Analysis*, vol. 250 of *Graduate Texts in Mathematics*, Springer, New York, NY, USA, 2nd edition, 2008.
- [2] S. Albeverio and W. Karwowski, "A random walk on p -adics—the generator and its spectrum," *Stochastic Processes and their Applications*, vol. 53, no. 1, pp. 1–22, 1994.
- [3] V. A. Avetisov, A. H. Bikulov, S. V. Kozyrev, and V. A. Osipov, " p -adic models of ultrametric diffusion constrained by hierarchical energy landscapes," *Journal of Physics. A. Mathematical and General*, vol. 35, no. 2, pp. 177–189, 2002.
- [4] S. Haran, "Riesz potentials and explicit sums in arithmetic," *Inventiones Mathematicae*, vol. 101, no. 3, pp. 697–703, 1990.
- [5] A. Khrennikov, *p -Adic Valued Distributions in Mathematical Physics*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1994.
- [6] A. Khrennikov, *Non-Archimedean Analysis: Quantum Paradoxes, Dynamical Systems and Biological Models*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1997.
- [7] A. N. Kochubei, "A non-Archimedean wave equation," *Pacific Journal of Mathematics*, vol. 235, no. 2, pp. 245–261, 2008.
- [8] V. S. Varadarajan, "Path integrals for a class of p -adic Schrödinger equations," *Letters in Mathematical Physics*, vol. 39, no. 2, pp. 97–106, 1997.
- [9] V. S. Vladimirov and I. V. Volovich, " p -adic quantum mechanics," *Communications in Mathematical Physics*, vol. 123, no. 4, pp. 659–676, 1989.
- [10] V. S. Vladimirov, I. V. Volovich, and E. I. Zelenov, *p -Adic Analysis and Mathematical Physics. Volume I*, Series on Soviet and East European Mathematics, World Scientific, Singapore, 1992.
- [11] I. V. Volovich, " p -adic space-time and string theory," *Akademiya Nauk SSSR: Teoreticheskaya i Matematicheskaya Fizika*, vol. 71, no. 3, pp. 337–340, 1987.
- [12] I. V. Volovich, " p -adic string," *Classical and Quantum Gravity*, vol. 4, no. 4, pp. L83–L87, 1987.
- [13] S. Albeverio, A. Yu. Khrennikov, and V. M. Shelkovich, "Harmonic analysis in the p -adic Lizorkin spaces: fractional operators, pseudo-differential equations, p -adic wavelets, Tauberian theorems," *Journal of Fourier Analysis and Applications*, vol. 12, pp. 393–425, 2006.

- [14] N. M. Chuong and H. D. Hung, “Maximal functions and weighted norm inequalities on local fields,” *Applied and Computational Harmonic Analysis*, vol. 29, no. 3, pp. 272–286, 2010.
- [15] N. M. Chuong, Y. V. Egorov, A. Khrennikov, Y. Meyer, and D. Mumford, *Harmonic, Wavelet and p -Adic Analysis*, World Scientific Publishers, Singapore, 2007.
- [16] Y.-C. Kim, “Carleson measures and the BMO space on the p -adic vector space,” *Mathematische Nachrichten*, vol. 282, no. 9, pp. 1278–1304, 2009.
- [17] Y.-C. Kim, “Weak type estimates of square functions associated with quasiradial Bochner-Riesz means on certain Hardy spaces,” *Journal of Mathematical Analysis and Applications*, vol. 339, no. 1, pp. 266–280, 2008.
- [18] Y.-C. Kim, “A simple proof of the p -adic version of the Sobolev embedding theorem,” *Communications of the Korean Mathematical Society*, vol. 25, no. 1, pp. 27–36, 2010.
- [19] S. Z. Lu and D. C. Yang, “The decomposition of Herz spaces on local fields and its applications,” *Journal of Mathematical Analysis and Applications*, vol. 196, no. 1, pp. 296–313, 1995.
- [20] K. M. Rogers, “A van der Corput lemma for the p -adic numbers,” *Proceedings of the American Mathematical Society*, vol. 133, no. 12, pp. 3525–3534, 2005.
- [21] K. M. Rogers, “Maximal averages along curves over the p -adic numbers,” *Bulletin of the Australian Mathematical Society*, vol. 70, no. 3, pp. 357–375, 2004.
- [22] M. H. Taibleson, *Fourier Analysis on Local Fields*, Princeton University Press, Princeton, NJ, USA, University of Tokyo Press, Tokyo, Japan, 1975.
- [23] S.-h. Lan, “The commutators on Herz spaces over locally compact Vilenkin groups,” *Advances in Mathematics*, vol. 35, no. 5, pp. 539–550, 2006.
- [24] C. Tang, “The boundedness of multilinear commutators on locally compact Vilenkin groups,” *Journal of Function Spaces and Applications*, vol. 4, no. 3, pp. 261–273, 2006.
- [25] J. Wu, “Boundedness of commutators on homogeneous Morrey-Herz spaces over locally compact Vilenkin groups,” *Analysis in Theory and Applications*, vol. 25, no. 3, pp. 283–296, 2009.
- [26] A. Khrennikov, “ p -adic valued probability measures,” *Indagationes Mathematicae*, vol. 7, no. 3, pp. 311–330, 1996.
- [27] A. Khrennikov and M. Nilsson, *p -Adic Deterministic and Random Dynamical Systems*, Kluwer, Dordrecht, The Netherlands, 2004.
- [28] S. Haran, “Analytic potential theory over the p -adics,” *Annales de l’institut Fourier*, vol. 43, no. 4, pp. 905–944, 1993.
- [29] S. S. Volosivets, “Maximal function and Riesz potential on p -adic linear spaces,” *p -Adic Numbers, Ultrametric Analysis, and Applications*, vol. 5, no. 3, pp. 226–234, 2013.
- [30] J. Alvarez, J. Lakey, and M. Guzmán-Partida, “Spaces of bounded λ -central mean oscillation, Morrey spaces, and λ -central Carleson measures,” *Universitat de Barcelona. Collectanea Mathematica*, vol. 51, no. 1, pp. 1–47, 2000.
- [31] Q. Y. Wu, L. Mi, and Z. W. Fu, “Boundedness of p -adic Hardy operators and their commutators on p -adic central Morrey and BMO spaces,” *Journal of Function Spaces and Applications*, vol. 2013, Article ID 359193, 10 pages, 2013.
- [32] Z. W. Fu, Q. Y. Wu, and S. Z. Lu, “Sharp estimates of p -adic hardy and Hardy-Littlewood-Pólya operators,” *Acta Mathematica Sinica*, vol. 29, no. 1, pp. 137–150, 2013.
- [33] V. S. Vladimirov, “Generalized functions over p -adic number field,” *Uspekhi Matematicheskikh Nauk*, vol. 43, pp. 17–53, 1988.