Research Article

Strong Summability of Fourier Transforms at Lebesgue Points and Wiener Amalgam Spaces

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We characterize the set of functions for which strong summability holds at each Lebesgue point. More exactly, if \( f \) is in the Wiener amalgam space \( W(L_1, \ell_q)(\mathbb{R}) \) and \( f \) is almost everywhere locally bounded, or \( f \in W(L_p, \ell_q)(\mathbb{R}) \) \((1 < p < \infty, 1 \leq q < \infty)\), then strong \( \theta \)-summability holds at each Lebesgue point of \( f \). The analogous results are given for Fourier series, too.

1. Introduction

It was proved by Lebesgue [1] that the Fejér means [2] of the trigonometric Fourier series of an integrable function converge almost everywhere to the function; that is,

\[
\frac{1}{n+1} \sum_{k=0}^n (s_k f(x) - f(x)) \to 0 \quad \text{as } n \to \infty \quad (1)
\]

for almost every \( x \in \mathbb{T} \), where \( \mathbb{T} \) denotes the torus and \( s_k f \) the \( k \)th partial sum of the Fourier series of the one-dimensional function \( f \). The set of convergence is characterized as the Lebesgue points of \( f \).

Hardy and Littlewood [3] considered the so-called strong summability and verified that the strong means

\[
\frac{1}{n+1} \sum_{k=0}^n |s_k f(x) - f(x)|^2 \to 0 \quad \text{as } n \to \infty \quad (2)
\]

for almost every \( x \in \mathbb{T} \). This result does not hold for \( p = 1 \) (see Hardy and Littlewood [5]). However, the strong means tend to 0 almost everywhere for all \( f \in L_p(\mathbb{T}) \). This is because of Marcinkiewicz [6] for \( q = 2 \) and Zygmund [7] for all \( q > 0 \) (see also Bary [8]). Later Gabisoniya [9] characterized the set of convergence as the so-called Gabisoniya points. Strong summability with lacunary partial sums and Lebesgue points are investigated by Belinsky et al. [10–13].

In a general method of summation, the so-called \( \theta \)-summation method, which is generated by a single function \( \theta \) and which includes the well-known Fejér, Riesz, Weierstrass, and Abel summability methods, is studied intensively in the literature (see, e.g., Butzer and Nessel [14], Trigub and Belinsky [15–17], Liflyand [18], and Weisz [19, 20]). In this paper we generalize some of the above-mentioned results for strong \( \theta \)-summability of Fourier transforms and for Wiener amalgam spaces. We characterize the set of functions for which strong summability holds at each Lebesgue point.

More exactly, we will show that

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T \theta^r \left( \frac{1}{T} \right) |s_k f(x) - f(x)|^r \, dx = 0 \quad (3)
\]

at each Lebesgue point \( x \) of \( f \in W(L_1, \ell_q)(\mathbb{R}) \supset L_q(\mathbb{R}) \) \((1 \leq q < \infty)\) when \( f \) is locally bounded at \( x \), where \( r > 0 \). Moreover, the convergence holds at each Lebesgue point of \( f \) if \( f \in W(L_p, \ell_q)(\mathbb{R}) \supset L_p(\mathbb{R}) \) \((1 < p < \infty, 1 \leq q < \infty)\). Here \( W(L_p, \ell_q)(\mathbb{R}) \) denotes the Wiener amalgam spaces. Gabisoniya’s result was generalized in [21]. The analogous results are given for Fourier series, too.

2. Wiener Amalgam Spaces and Lebesgue Points

Let us fix \( d \geq 1, d \in \mathbb{N} \). For a set \( \mathbb{Y} \neq \emptyset \) let \( \mathbb{Y}^d \) be its Cartesian product \( \mathbb{Y} \times \cdots \times \mathbb{Y} \) taken with itself \( d \)-times. We
briefly write \( L_p(\mathbb{R}^d) \) instead of the \( L_p(\mathbb{R}^d, \lambda) \) space equipped with the norm
\[
\|f\|_p := \left( \int_{\mathbb{R}^d} |f(x)|^p d\lambda(x) \right)^{1/p} \quad (1 \leq p < \infty),
\]
with the usual modification for \( p = \infty \), where \( \lambda \) is the Lebesgue measure. \( L^\infty(\mathbb{R}^d) \) \( (1 \leq p < \infty) \) denotes the space of measurable functions \( f \) for which \( |f|^p \) is locally integrable. We say that \( f \) is locally bounded at \( x \) if there exists a neighborhood of \( x \) such that \( f \) is bounded on this neighborhood.

Now we generalize the \( L_p \) spaces. A measurable function \( f \) belongs to the Wiener amalgam space \( W(L_p, \ell_q)(\mathbb{R}^d) \) \( (1 \leq p, q \leq \infty) \) if
\[
\|f\|_{W(L_p, \ell_q)} := \left( \sum_{k \in \mathbb{Z}^d} \|f(\cdot + k)\|_{L^p}^q \right)^{1/q} < \infty,
\]
with the obvious modification for \( q = \infty \). It is easy to see that \( W(L_p, \ell_q)(\mathbb{R}^d) = L_p(\mathbb{R}^d) \) and the following continuous embeddings hold true:
\[
W(L_{p_1}, \ell_{q_1})(\mathbb{R}^d) \supset W(L_{p_2}, \ell_{q_2})(\mathbb{R}^d) \quad (p_1 \leq p_2),
\]
\[
W(L_{p_1}, \ell_{q_1})(\mathbb{R}^d) \subset W(L_{p_2}, \ell_{q_2})(\mathbb{R}^d) \quad (q_1 \leq q_2),
\]
\[1 \leq p, q \leq \infty \). Thus
\[
W(L_{1\infty}, \ell_1)(\mathbb{R}^d) \subset W(L_1, \ell_1)(\mathbb{R}^d) \subset W(L_1, \ell_\infty)(\mathbb{R}^d) \subset W(L_\infty, \ell_\infty)(\mathbb{R}^d)
\]
\[1 \leq p \leq \infty \)

A point \( x \in \mathbb{R}^d \) is called a \( p \)-Lebesgue point (or a Lebesgue point of order \( p \)) of \( f \in L^p_{1\infty}(\mathbb{R}^d) \) if
\[
\lim_{h \to 0} \left( \frac{1}{(2h)^d} \int_{-h}^h \cdots \int_{-h}^h |f(x-s) - f(x)|^p ds \right)^{1/p} = 0.
\]
It was proved by Feichtinger and Weisz [22, 23] that almost every point \( x \in \mathbb{R}^d \) is a \( p \)-Lebesgue point of \( f \in W(L_p, \ell_\infty)(\mathbb{R}^d) \) \( (1 \leq p < \infty) \). In context of Lebesgue points of \( L_p \) functions we call also for the earlier papers of Belinsky et al. [12, 13].

In this paper the constants \( C \) and \( C_p \) may vary from line to line and the constants \( C_p \) are depending only on \( p \).

### 3. The Kernel Functions

The Fourier transform of \( f \in L_1(\mathbb{R}) \) is given by
\[
\hat{f}(x) = \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} f(u) e^{ixu} du \quad (x \in \mathbb{R}),
\]
where \( i = \sqrt{-1} \). Suppose first that \( f \in L_p(\mathbb{R}) \) for some \( 1 \leq p \leq 2 \). The Fourier inversion formula
\[
f(x) = \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} \hat{f}(u) e^{ixu} du \quad (x \in \mathbb{R}, \hat{f} \in L_1(\mathbb{R})),
\]
motivates the definition of the Dirichlet integral \( s_t f (t > 0) \) introduced by
\[
s_t f(x) := \frac{1}{(2\pi)^{1/2}} \int_{-t}^t \hat{f}(u) e^{ixu} du
\]
\[
= \frac{1}{2\pi} \int_{\mathbb{R}} f(x-u) D_t(u) du,
\]
where the Dirichlet kernel is defined by
\[
D_t(x) := \int_{-t}^t e^{ixu} du = \frac{2\sin(tx)}{x}.
\]
Obviously, \( |D_t| \leq C_t \).

It is easy to see that, with the help of the integral in (11), the definition of \( s_t f \) can be extended to all \( f \in W(L_1, \ell_\infty)(\mathbb{R}^d) \) with \( 1 \leq q < \infty \). Note that \( W(L_1, \ell_\infty)(\mathbb{R}^d) \supset L_\infty(\mathbb{R}^d) \), where \( 1 \leq p < \infty \). It is known (see, e.g., Grafakos [24] or [20]) that, for \( f \in L_p(\mathbb{R}), 1 < p < \infty \),
\[
\lim_{t \to \infty} s_t f = f \quad \text{in the } L_p(\mathbb{R}) \text{-norm and a.e.}
\]

This convergence does not hold for \( p = 1 \). However, using a summability method, we can generalize these results. We may take a general summability method, the so-called \( \theta \)-summation defined by a function \( \theta : \mathbb{R}_+ \to \mathbb{R} \). This summation contains all well-known summability methods, such as the Marcinkiewicz-Fejér, Riesz, Weierstrass, Abel, Picard, and Bessel summations.

Suppose that \( \theta \) is continuous on \( \mathbb{R}_+ \); the support of \( \theta \) is \([0, c]\) for some \( 0 < c \leq \infty \) and \( \theta \) is differentiable on \((0, c)\). Suppose further that
\[
\theta(0) = 1,
\]
\[
\int_0^\infty (t \vee 1)^d |\theta'(t)| \, dt < \infty,
\]
\[
\lim_{t \to \infty} t^d \theta(t) = 0,
\]
where \( d \in \mathbb{N}, \vee \) denotes the maximum, and \( \wedge \) denotes the minimum.

For \( T > 0 \) the \( \theta \)-means of a function \( f \in L_p(\mathbb{R}) \) \( (1 \leq p \leq 2) \) are defined by
\[
\sigma_T^\theta f(x) := \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} \theta \left( \frac{|u|}{T} \right) \hat{f}(u) e^{ixu} du.
\]
It is easy to see that
\[
\sigma_T^\theta f(x) = \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} f(x-u) K_T^\theta(u) du.
\]
Note that this formula is well defined for all $f \in W(L_1, \ell_\infty)(\mathbb{R})$. Here
\[
K_T^\theta(x) := \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} \theta\left(\frac{|t|}{T}\right)e^{iux}dt
\]
\[
= \frac{-1}{(2\pi)^{1/2}T} \int_{\mathbb{R}} \theta\left(\frac{t}{T}\right)\left(t\right)^i e^{iux}dt
\]
\[
= \frac{-1}{(2\pi)^{1/2}T} \int_{\mathbb{R}} \theta\left(\frac{t}{T}\right)D_t(x) dt
\]
denotes the $\theta$-kernel. Thus
\[
\sigma_T^\theta f(x) = \frac{1}{T} \int_0^T s_t f(x) dt
\]
for all $f \in W(L_1, \ell_q)(\mathbb{R})$ with $1 \leq q < \infty$. Note that for the Fejér means (i.e., for $\theta(t) = \max((1-|t|), 0)$) we get the usual definition
\[
\sigma_T^\theta f(x) = \frac{1}{T} \int_0^T s_t f(x) dt.
\]
In Feichtinger and Weisz [22, 23] we have proved that, under conditions (14) and (25) with $d = 1$,
\[
\lim_{T \to \infty} \sigma_T^\theta f(x) = f(x)
\]
for all Lebesgue points of $f \in W(L_1, \ell_\infty)(\mathbb{R})$. In this paper, we investigate the problem of the strong summability, that is, whether the convergence
\[
\lim_{T \to \infty} \frac{-1}{T} \int_0^T \theta\left(\frac{t}{T}\right)\left(s_t f(x) - f(x)\right)^r dt = 0
\]
holds for Lebesgue points and some $r > 0$. Usually $\theta$ is increasing; then we can take the absolute value of $\theta'$ in the integral.
To this end we have to introduce some $d$-dimensional definitions. In the $d$-dimensional case we define the Dirichlet kernel by
\[
D_t(x) := \prod_{i=1}^d D_t(x_i) = 2^d \prod_{i=1}^d \sin\left(\frac{tx_i}{x_i}\right) (t > 0)
\]
and the so called Marcinkiewicz-$\theta$-kernel by
\[
K_T^\theta(x) = \frac{-1}{(2\pi)^{d/2}T} \int_0^\infty \theta\left(\frac{t}{T}\right)D_t(x) dt (T > 0),
\]
where $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$. In [21], we have seen that we may suppose that $x_1 > x_2 > \cdots > x_d > 0$ and $x_1 - \sum_{j=2}^d x_j > 0$ and we proved the next lemma. Denote by
\[
\text{soc } t := \begin{cases} \cos t, & \text{if } d \text{ is even;} \\ \sin t, & \text{if } d \text{ is odd.} \end{cases}
\]

**Lemma 1.** Let
\[
\left| \int_0^\infty \theta\left(\frac{t}{T}\right) t^i (\text{soc } tu) dt \right| \leq Cu^{-\alpha} (i = 0, \ldots, d - 1)
\]
for some $0 < \alpha < \infty$. Then
\[
\left| K_T^\theta(x) \right| \leq CT^{-\alpha}j^{-1} \cdots j^{-1} (j = 0, \ldots, d).
\]
If in addition $x_1 - \sum_{j=2}^d x_j > 1/T$ and $x_{j+1} < 1/T$, where $x_{d+1} = 0$, then
\[
\left| K_T^\theta(x) \right| \leq CT^{-\alpha}d^{-j} \cdots j^{-1} \left( x_1 - \sum_{j=2}^d x_j \right)^{-\alpha} (j = 1, \ldots, d).
\]
The next lemma is due to the author [25].

**Lemma 2.** If (14) and (25) are satisfied for some $d \in \mathbb{N}$ and $0 < \alpha < \infty$, then
\[
\int_{\mathbb{R}^d} |K_T^\theta| d\lambda \leq C (T \in \mathbb{R}_+).
\]

**4. Strong Summability of Fourier Transforms**

In this section we characterize a wide set of functions for which strong summability holds at each Lebesgue point. For the convergence of $f \in W(L_p, \ell_q)(\mathbb{R}) (1 < p < \infty, 1 \leq q < \infty)$ at $p$-Lebesgue points we proved the following result in [21]. Note that $W(L_p, \ell_q)(\mathbb{R}) = L_p(\mathbb{R})$.

**Theorem 3.** Suppose that (14) and (25) hold for some $d \in \mathbb{N}$ and $0 < \alpha < \infty$. Let $f \in W(L_p, \ell_q)(\mathbb{R})$ for some $1 < p < \infty$ and $1 \leq q < \infty$. If $x_j$ is a $p$-Lebesgue point of $f$ for all $j = 1, \ldots, d$, then
\[
\lim_{T \to \infty} \frac{-1}{T} \int_0^\infty \theta\left(\frac{t}{T}\right) \prod_{j=1}^d (s_t f(x_j) - f(x_j)) dt = 0.
\]
If all $x_j$ $(j = 1, \ldots, d)$ are equal, then we obtain the following.

**Corollary 4.** Suppose that (14) and (25) hold for some even $d \in \mathbb{N}$ and $0 < \alpha < \infty$. Let $f \in W(L_p, \ell_q)(\mathbb{R})$ for some $1 < p < \infty$ and $1 \leq q < \infty$. If $x \in \mathbb{R}$ is a $p$-Lebesgue point of $f$, then
\[
\lim_{T \to \infty} \frac{-1}{T} \int_0^\infty \theta\left(\frac{t}{T}\right) |s_t f(x) - f(x)| d\lambda = 0.
\]
Obviously, the convergence holds almost everywhere. Corollary 4 does not hold for $p = 1$ (see Hardy and Littlewood [5]). However, we [21] extended it for $p = 1$, but for much more specialized points than the Lebesgue points, for the so-called Gabisoniya points, which were introduced in [9]. In the next theorem we generalize Theorem 3 and Corollary 4 for $p = 1$ and for a subspace of $W(L_1, \ell_q)(\mathbb{R})$. 
Theorem 5. Suppose that (14) and (25) hold for some $d \in \mathbb{N}$ and $0 < \alpha < \infty$. Let $f \in W(L_1, \ell_q)(\mathbb{R})$ for some $1 \leq q < \infty$. If $x_1$ is a Lebesgue point of $f$ and $f$ is locally bounded at $x_j$ for all $j = 1, \ldots, d$, then

\[
\lim_{T \to \infty} -\frac{1}{T} \int_{0}^{\infty} \theta^\prime \left( \frac{t}{T} \right) \prod_{j=1}^{d} \left( s_j f(x_j) - f(x_j) \right) dt = 0. \tag{30}
\]

Proof. It is easy to see that

\[
-\frac{1}{T} \int_{0}^{\infty} \theta^\prime \left( \frac{t}{T} \right) \prod_{j=1}^{d} \left( s_j f(x_j) - f(x_j) \right) dt = \frac{1}{T} \cdot \int_{0}^{\infty} \theta^\prime \left( \frac{t}{T} \right) \prod_{j=1}^{d} \left( \frac{1}{2\pi} \int_{\mathbb{R}} f(x_j - s_j) D_t(s_j) ds_j \right) dt
\]

\[
- f(x_j) dt = -\frac{1}{T} \int_{0}^{\infty} \theta^\prime \left( \frac{t}{T} \right) \prod_{j=1}^{d} \lim_{n \to \infty} \frac{1}{2\pi} \int_{\mathbb{R}} f(x_j - s_j) D_t(s_j) ds_j dt.
\]

Since

\[
\left| \int_{-n}^{n} f(x_j - s_j) D_t(s_j) ds_j \right| \leq C_t \|f\| \|W(L_1, \ell_q)\|,
\]

\[
\left| \int_{-n}^{n} f(x_j) D_t(s_j) ds_j \right| = \left| f(x_j) \right| \left| \int_{-n}^{n} \frac{\sin(ts_j)}{s_j} ds_j \right|
\]

\[
= \left| f(x_j) \right| \left| \int_{-n}^{n} \frac{\sin u}{u} du \right| \leq C \left| f(x_j) \right|
\]

we obtain

\[
-\frac{1}{T} \int_{0}^{\infty} \theta^\prime \left( \frac{t}{T} \right) \prod_{j=1}^{d} \left( s_j f(x_j) - f(x_j) \right) dt = \lim_{n \to \infty} -\frac{1}{T} \cdot \int_{0}^{\infty} \theta^\prime \left( \frac{t}{T} \right) \prod_{j=1}^{d} \left( s_j f(x_j) - f(x_j) \right) dt
\]

\[
= \frac{1}{(2\pi)^d} \cdot \frac{1}{\lim_{n \to \infty} \left( \int_{-n}^{n} f(x_j - s_j) - f(x_j) \right) \cdot D_t(s) ds dt}
\]

\[
= \frac{1}{(2\pi)^d} \cdot \frac{1}{\lim_{n \to \infty} \left( \int_{-n}^{n} f(x_j - s_j) - f(x_j) \right) \cdot D_t(s) ds dt}
\]

\[
- f(x_j) K_T^\alpha(s) ds.
\]

For simplicity we will prove the rest of the theorem for $d = 3$, only. It can be proved for higher dimensions similarly. As we mentioned earlier, we may suppose that $s_1 > s_2 > s_3 > 0$ and $s_1 - s_2 - s_3 > 0$. Let us fix a small $r > 0$ and denote the square $[0, r/2]^2$ by $S_{r/2}$. We can prove in the same way as we did in Theorem 4 of [21] that

\[
\int_{S_{r/2}} \prod_{j=1}^{d} \left| f(x_j - s_j) - f(x_j) \right| K_T^\alpha(s) ds < e \tag{34}
\]

if $r$ is small enough and $T$ is large enough. The estimation of this integral on the set $S_{r/2}'$ of that proof does not work now, because there we used a modified maximal function which is not necessarily bounded in our case. So we have to show here that

\[
\lim_{T \to \infty} \int_{S_{r/2}} \prod_{j=1}^{d} \left| f(x_j - s_j) - f(x_j) \right| K_T^\alpha(s) ds = 0. \tag{35}
\]

To this end let us introduce the sets

\[
A_1 := \left\{ s : s_1 > \frac{r}{2}, 0 < s_3 < s_2 < \frac{1}{2} \right\},
\]

\[
A_2 := \left\{ s : s_1 > \frac{r}{2}, 0 < s_3 < \frac{1}{2} < s_2 < \delta \right\},
\]

\[
A_3 := \left\{ s : s_1 > \frac{r}{2}, \frac{1}{2} < s_3 < s_2 < \delta \right\},
\]

\[
A_4 := \left\{ s : s_1 > \frac{r}{2}, 0 < s_3 < \frac{1}{2} < s_2 < \delta \right\},
\]

\[
A_5 := \left\{ s : s_1 > \frac{r}{2}, \frac{1}{2} < s_3 < \delta < s_2 \right\},
\]

\[
A_6 := \left\{ s : s_1 > \frac{r}{2}, \delta < s_3 < s_2 \right\},
\]

\[
B_1 := \left\{ s : 0 < s_1 - s_2 - s_3 < \frac{1}{2} \right\},
\]

\[
B_2 := \left\{ s : \frac{1}{T} < s_1 - s_2 - s_3 < \delta \right\},
\]

\[
B_3 := \left\{ s : 0 < s_1 - s_2 - s_3 < \frac{s_1 - s_2}{2} \right\},
\]

\[
B_4 := \left\{ s : \frac{(s_1 - s_2)}{2} < \delta < s_1 - s_2 - s_3 < s_1 - s_2 - \delta \right\},
\]

\[
B_5 := \left\{ s : (s_1 - s_2 - \delta) < \delta < s_1 - s_2 - s_3 < s_1 - s_2 \right\},
\]

\[
C_1 := \left\{ s : 0 < s_1 - s_2 < \delta \right\},
\]

\[
C_2 := \left\{ s : \delta < s_1 - s_2 < \frac{s_1}{2} \right\},
\]

\[
C_3 := \left\{ s : \frac{s_1}{2} < s_1 - s_2 < s_1 - \delta \right\},
\]

\[
C_4 := \left\{ s : s_1 - \delta < s_1 - s_2 < s_1 \right\}
\]
for a given small $\delta > 0$ and large $T$. Then we have to estimate the integral in (35) for $A_i, i = 1, \ldots, 6$. First of all observe that $A_i \cap B_j = \emptyset$ for $i = 1, 2, 3$ and $j = 1, 2$. On the set $A_1$ we have $s_1, s_2 < 1/T$ and so $s_1 - s_2 - s_3 > s_1/2$. Hence

$$|K_T^\theta(s)| \leq C T^{2-\alpha} s_1^{-1} (s_1 - s_2 - s_3)^{-\alpha} \leq C T^{2-\alpha} s_1^{-1-\alpha}. $$  (37)

Since $f$ is locally bounded at $x_1$, we get by (37) that

$$\int_{A_1 \cap \bigcup ( B_i \cup B_j \cup B_k )} \frac{3}{\prod_{j=1}^3} \left| f \left( x_j - s_j \right) - f \left( x_j \right) \right| |K_T^\theta(s)| ds
\leq C T^{2-\alpha} \int_{s_{1/2}}^{\infty} \int_0^{1/T} \int_0^{1/T} s_1^{-\alpha} s_2^{-\alpha} s_3^{-\alpha} \left| f \left( x_j - s_j \right) \right| ds
\leq C T^{-\alpha} \int_{s_{1/2}}^{\infty} \int_0^{1/T} \int_0^{1/T} s_1^{-\alpha} s_2^{-\alpha} s_3^{-\alpha} \left| f \left( x_j - s_j \right) \right| ds$$

(38)

and so

$$\int_{A_2 \cap \bigcup ( B_i \cup B_j \cup B_k )} \frac{3}{\prod_{j=1}^3} \left| f \left( x_j - s_j \right) - f \left( x_j \right) \right| |K_T^\theta(s)| ds
\leq C T^{1-\alpha} \int_{s_{1/2}}^{\infty} \int_0^{1/T} \int_0^{1/T} s_1^{-\alpha} s_2^{-\alpha} s_3^{-\alpha} \left| f \left( x_j - s_j \right) \right| ds
\leq C T^{-\alpha} \ln T \int_{s_{1/2}}^{\infty} \int_0^{1/T} \int_0^{1/T} s_1^{-\alpha} s_2^{-\alpha} s_3^{-\alpha} \left| f \left( x_j - s_j \right) \right| ds$$

(40)

Similarly,

$$\int_{A_3 \cap \bigcup ( B_i \cup B_j \cup B_k )} \frac{3}{\prod_{j=1}^3} \left| f \left( x_j - s_j \right) - f \left( x_j \right) \right| |K_T^\theta(s)| ds
\leq C T^{-\alpha} \int_{s_{1/2}}^{\infty} \int_0^{1/T} \int_0^{1/T} s_1^{-\alpha} s_2^{-\alpha} s_3^{-\alpha} \left| f \left( x_j - s_j \right) \right| ds$$

(41)

On the set $A_4 \cap B_1$ we have $s_1 - s_2 < 2/T < \delta$ if $T$ is large enough and so $s_2 > s_1/2$. Then

$$\left|K_T^\theta(s)\right| \leq C T s_1^{-1} s_2^{-1} s_3^{-1}. $$  (42)

$$\int_{A_4 \cap \bigcup ( B_i \cup B_j \cup B_k )} \frac{3}{\prod_{j=1}^3} \left| f \left( x_j - s_j \right) - f \left( x_j \right) \right| |K_T^\theta(s)| ds
\leq C T \int_{s_{1/2}}^{\infty} \int_0^{1/T} \int_0^{1/T} s_1^{-\alpha} s_2^{-\alpha} s_3^{-\alpha} \left| f \left( x_j - s_j \right) \right| ds$$

(43)

and so

$$\int_{A_2 \cap \bigcup ( B_i \cup B_j \cup B_k )} \frac{3}{\prod_{j=1}^3} \left| f \left( x_j - s_j \right) - f \left( x_j \right) \right| |K_T^\theta(s)| ds
\leq C T^{1-\alpha} \int_{s_{1/2}}^{\infty} \int_0^{1/T} \int_0^{1/T} s_1^{-\alpha} s_2^{-\alpha} s_3^{-\alpha} \left| f \left( x_j - s_j \right) \right| ds
\leq C T^{-\alpha} \ln T \int_{s_{1/2}}^{\infty} \int_0^{1/T} \int_0^{1/T} s_1^{-\alpha} s_2^{-\alpha} s_3^{-\alpha} \left| f \left( x_j - s_j \right) \right| ds$$

On the set $A_4 \cap B_1$ we have $s_1 - s_2 < 2/T < \delta$ if $T$ is large enough and so $s_2 > s_1/2$. Then

$$\left|K_T^\theta(s)\right| \leq C T s_1^{-1} s_2^{-1} s_3^{-1}. $$  (42)

$$\int_{A_4 \cap \bigcup ( B_i \cup B_j \cup B_k )} \frac{3}{\prod_{j=1}^3} \left| f \left( x_j - s_j \right) - f \left( x_j \right) \right| |K_T^\theta(s)| ds
\leq C T \int_{s_{1/2}}^{\infty} \int_0^{1/T} \int_0^{1/T} s_1^{-\alpha} s_2^{-\alpha} s_3^{-\alpha} \left| f \left( x_j - s_j \right) \right| ds$$

(43)

The second term is less than $\epsilon$ if $N_0$ is large enough and the first term is less than $\epsilon$ if $\delta$ is small enough. The set $A_4 \cap B_2$ can be handled in the same way.

On $A_4 \cap ( B_i \cup B_j \cup B_k )$ we have $s_1 - s_2 > \delta$. Thus $(B_i \cup B_j \cup B_k) \cap C_1 = \emptyset$ and $s_1 - s_2 - s_3 > s_1 - s_2 - 1/T > (s_1 - s_2)/2$. Then

$$\left|K_T^\theta(s)\right| \leq C T^{-1} s_1^{-1} s_2^{-1} s_3^{-1} \left( s_1 - s_2 - s_3 \right)^{-\alpha}. $$  (44)

On $C_2$ we have

$$\left|K_T^\theta(s)\right| \leq C T^{-1} s_1^{-2} s_2^{-1} s_3^{-1} \left( s_1 - s_2 \right)^{-\alpha}. $$

(45)

where $\eta$ is chosen such that $0 < \eta < 1$ and $\alpha$. Hence

$$\int_{A_4 \cap \bigcup ( B_i \cup B_j \cup B_k )} \frac{3}{\prod_{j=1}^3} \left| f \left( x_j - s_j \right) - f \left( x_j \right) \right| |K_T^\theta(s)| ds
\leq C T^{2-\alpha} \int_{s_{1/2}}^{\infty} \int_0^{1/T} \int_0^{1/T} s_1^{-\alpha} s_2^{-\alpha} s_3^{-\alpha} \left| f \left( x_j - s_j \right) \right| ds$$

(46)
On $C_3$, (44) implies that

$$|K^\Theta_T(s)| \leq CT^{-\alpha} s_1^{-1-\alpha} s_2^{-1-\alpha} \leq CT^{-\alpha} s_1^{-1-\alpha} s_2^{-1-\alpha},$$

$$\int_{A_0 \cap (B_3 \cup B_4 \cup B_5 \cap C_3)} \frac{3}{2} \left| f(x_j - s_j) - f(x_j) \right| |K^\Theta_T(s)| \, ds$$

$$\leq CT^{-\alpha} \int_{j_0}^{s_1} \int_{s_2}^{s_1} \int_{s_3}^{s_2} s_1^{-1-\alpha} s_2^{-1-\alpha} s_3^{-1-\alpha} \frac{3}{2} \left| f(x_j - s_j) \right| \, ds$$

$$\leq CT^{-\alpha} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (j \vee 1)^{-1-\eta} \sum_{\{i,j\} \subseteq \{0,1,2\}} (j \vee 1)^{-1-\eta} \cdot \sum_{j=0}^{\infty} (j \vee 1)^{-1-\eta} \int_{j_0}^{s_1} \int_{s_2}^{s_1} \int_{s_3}^{s_2} \left| f(x_j - s_j) \right| \, ds$$

$$\leq CT^{-\alpha} \int_{j_0}^{s_1} \int_{s_2}^{s_1} \int_{s_3}^{s_2} \left| f(x_j - s_j) \right| \, ds$$

$$\leq C \int_{j_0}^{s_1} \int_{s_2}^{s_1} \int_{s_3}^{s_2} \left| f(x_j - s_j) \right| \, ds$$

$$\leq \epsilon.$$  

(47)

Observe that $s_2 < \delta$ contradicts $C_4$.

Consider the set $A_2 \cap B_1$. If $s_1 - s_2 > 2i$, then $s_1 - s_2 - 1/T > (s_1 - s_2)/2$ and if $s_1 - s_2 < 2i$, then $s_1 > 1/T > (s_1 - s_2)/2$. Observe that $s_1 - s_2 < 1/T + \delta < 2\delta$. Hence

$$|K^\Theta_T(s)| \leq CS_1^{-1} s_2^{-1-\alpha} \leq CS_1^{-2} (s_1 - s_2)^{-1/2}. \tag{48}$$

Since $s_3 < \delta$, we can integrate in $s_3$ to obtain

$$\int_{A_2 \cap B_1} \frac{3}{2} \left| f(x_j - s_j) - f(x_j) \right| |K^\Theta_T(s)| \, ds$$

$$\leq C \int_{j_0}^{s_1} \int_{s_2}^{s_1} \int_{s_3}^{s_2} s_1^{-1/2} (s_1 - s_2)^{-1/2} \sum_{j=0}^{\infty} (j \vee 1)^{-1-\eta} \int_{j_0}^{s_1} \int_{s_2}^{s_1} \int_{s_3}^{s_2} \left| f(x_j - s_j) \right| \, ds$$

$$\leq C \int_{j_0}^{s_1} \int_{s_2}^{s_1} \int_{s_3}^{s_2} \left| f(x_j - s_j) \right| \, ds$$

$$\leq C \int_{j_0}^{s_1} \int_{s_2}^{s_1} \int_{s_3}^{s_2} \left| f(x_j - s_j) \right| \, ds$$

$$\leq \epsilon.$$  

(49)

As in (43). Similarly, $s_1 - s_2 < 2\delta$ holds as well on $A_5 \cap B_2$. If $s_1 - s_2 - 3s_3 > s_3$, then

$$|K^\Theta_T(s)| \leq CT^{-\alpha} s_1^{-1-\alpha} s_2^{-1-\alpha} s_3^{-1-\alpha}$$

$$\leq CT^{-\alpha} S_1^{-2} S_2^{-2} S_3^{-1-\alpha} \tag{50}$$

and so

$$\int_{A_5 \cap B_2} \frac{3}{2} \left| f(x_j - s_j) - f(x_j) \right| |K^\Theta_T(s)| \, ds$$

$$\leq CT^{-\alpha} \int_{j_0}^{s_1} \int_{s_2}^{s_1} \int_{s_3}^{s_2} s_1^{-2-\alpha} s_2^{-2-\alpha} s_3^{-1-\alpha} \sum_{j=0}^{\infty} (j \vee 1)^{-1-\eta} \int_{j_0}^{s_1} \int_{s_2}^{s_1} \int_{s_3}^{s_2} \left| f(x_j - s_j) \right| \, ds$$

$$\leq CT^{-\alpha} \int_{j_0}^{s_1} \int_{s_2}^{s_1} \int_{s_3}^{s_2} \left| f(x_j - s_j) \right| \, ds$$

$$\leq \epsilon.$$  

(51)

as before. If $s_3 > s_1 - s_2 - s_3$, then

$$|K^\Theta_T(s)| \leq CT^{-\alpha} s_1^{-2} (s_1 - s_2 - s_3)^{-1-\alpha},$$

$$\int_{A_6 \cap B_3} \frac{3}{2} \left| f(x_j - s_j) - f(x_j) \right| |K^\Theta_T(s)| \, ds$$

$$\leq CT^{-\alpha} \int_{j_0}^{s_1} \int_{s_2}^{s_1} \int_{s_3}^{s_2} s_1^{-2} (s_1 - s_2 - s_3)^{-1-\alpha} \sum_{j=0}^{\infty} (j \vee 1)^{-1-\eta} \int_{j_0}^{s_1} \int_{s_2}^{s_1} \int_{s_3}^{s_2} \left| f(x_j - s_j) \right| \, ds$$

$$\leq CT^{-\alpha} \int_{j_0}^{s_1} \int_{s_2}^{s_1} \int_{s_3}^{s_2} \left| f(x_j - s_j) \right| \, ds$$

$$\leq \epsilon.$$  

(52)

Moreover, $B'_5$ contradicts $A_5$, more exactly, $s_3 > 1/T$.

On $A_6$, $s_3 > \delta$ and $s_1 - s_2 - s_3 > 0$, thus $s_1 - s_2 > \delta$. Similar to (48),

$$|K^\Theta_T(s)| \leq CS_1^{-2} S_2^{-2} S_3^{-1} \tag{54}$$

$$\leq CS_1^{-2} (s_1 - s_2)^{-1}.$$
on $A_6 \cap B_1 \cap C_2$. Then

\[
\int_{A_6 \cap B_1 \cap C_2} \left| \frac{3}{\prod_{j=1}^{3} f(x_j - s_j) - f(x_j)} \right| ds \\
\leq C \int_{r/2}^{\infty} \frac{r^{s_1-\delta}}{s_1-s_2-1/T} s_1^{-2} (s_1 - s_2)^{-1}
\]

\[
\cdot \sum_{0 \leq j \leq (i+1)/2} \int_{s_1-s_j}^{s_1-s_j-1/T} f(x_1 - s_j) \left| f(x_2 - s_2) \right| ds ds \leq C \left\| f \times f \right\|_{W(L_1,E_m)}
\]

\[
+ C N_0^{1/2} \left\| f \times f \times f \right\|_{W(L_1,E_m)}
\]

which is small enough if $N_0$ is large and $\delta$ is small enough. On $A_6 \cap B_1 \cap C_3$ we use the estimation

\[
\left| K_\alpha^\theta (s) \right| \leq C s_1^{-1} s_2^{-1} s_3^{-1} \leq C s_1^{-2} s_2^{-1}
\]

(56)

to obtain

\[
\int_{A_6 \cap B_1 \cap C_2} \left| \frac{3}{\prod_{j=1}^{3} f(x_j - s_j) - f(x_j)} \right| ds \\
\leq C \int_{r/2}^{\infty} \frac{r^{s_1-\delta}}{s_1-s_2-1/T} s_1^{-2} \prod_{j=1}^{3} f(x_j - s_j) ds
\]

\[
\leq C \left\| f \times f \times f \right\|_{W(L_1,E_m)}
\]

(57)

as just before.

On $A_6 \cap B_2$ we get

\[
\left| K_\alpha^\theta (s) \right| \leq C T^{-\alpha} s_1^{-1} s_2^{-1} s_3^{-1} (s_1 - s_2 - s_3)^{-\alpha}
\]

\[
\leq C T^{-\alpha} s_1^{-1} s_2^{-1} (s_1 - s_2 - s_3)^{-\alpha}
\]

(58)

which implies that

\[
\int_{A_6 \cap B_1 \cap C_2} \left| \frac{3}{\prod_{j=1}^{3} f(x_j - s_j) - f(x_j)} \right| ds \\
\leq C T^{-\alpha} \int_{r/2}^{\infty} \frac{r^{s_1-\delta}}{s_1-s_2-1/T} s_1^{-2} (s_1 - s_2)^{-1}
\]

\[
\cdot \prod_{j=1}^{3} f(x_j - s_j) ds \leq C T^{-\alpha} \sum_{i=0}^{\infty} (i \vee 1)^{-1-\mu}
\]

(59)

where $0 < \mu < 1/2 \land \alpha/2$. Similarly,

\[
\int_{A_6 \cap B_1 \cap C_3} \left| \frac{3}{\prod_{j=1}^{3} f(x_j - s_j) - f(x_j)} \right| ds \\
\leq C T^{-\alpha} \int_{r/2}^{\infty} \frac{r^{s_1-\delta}}{s_1-s_2-1/T} s_1^{-2} (s_1 - s_2 - s_3)^{-\alpha}
\]

\[
\cdot \prod_{j=1}^{3} f(x_j - s_j) ds \leq C T^{-\alpha} \sum_{i=0}^{\infty} (i \vee 1)^{-1-\mu}
\]

(60)

Moreover, on $A_6 \cap B_4$

\[
\left| K_\alpha^\theta (s) \right| \leq C T^{-\alpha} s_1^{-1} s_2^{-1} s_3^{-1} (s_1 - s_2 - s_3)^{-\alpha}
\]

\[
\leq C T^{-\alpha} s_1^{-1} s_2^{-1} (s_1 - s_2 - s_3)^{-\alpha}
\]

(59)
\[ \sum_{0 \leq j < (i+1)/2} (j \lor 1) -1 - \alpha + 2 \mu \sum_{0 \leq k < (j+1)/2} (k \lor 1) \sum_{0 \leq j < (i+1)/2} f(x_j - s_j) ds_j ds_2 ds_3 \leq C T^{-\alpha} \| f \|_W^{3} \rightarrow 0. \]
\[
\int_{A \cap B \cap C} 1 \sum_{0 \leq j < (i+1)/2} |f(x_j - s_j) - f(x_j)| \| K_\alpha^\rho (\sigma) \| ds \leq C T^{-\alpha} \int_{0}^{\infty} \int_{0}^{s_1} \int_{0}^{s_2 - s_1} \int_{0}^{s_3 - s_2} \int_{0}^{s_4 - s_3} |f(x_j - s_j)| ds_4 ds_3 ds_2 ds_1 \leq C T^{-\alpha} \| f \|_W^{3} \rightarrow 0. \]

Since \( A_\alpha \cap B = \emptyset \), the proof of the theorem is complete. \( \square \)

**Corollary 6.** Suppose that (14) and (25) hold for some even \( d \in \mathbb{N} \) and \( 0 < \alpha < \infty \). Let \( f \in W(L_1, \ell_q)(\mathbb{R}) \) for some \( 1 \leq q < \infty \). If \( x \in \mathbb{R} \) is a Lebesgue point of \( f \) and \( f \) is locally bounded at \( x \), then
\[
\lim_{T \to \infty} \frac{1}{T} \int_{-T}^{T} |f(t)| dt = 0. \]

If \( f \) is almost everywhere locally bounded, then the corollary holds almost everywhere. Unfortunately, it is not true that an integrable function is almost everywhere locally bounded (see [21]). The strong summability can be extended to all exponents in the usual way (see [21]).

**Corollary 7.** Suppose that (14) holds for all \( d \in \mathbb{N} \), \( \theta \) is nonincreasing, and \( r > 0 \). Under the same conditions as in Corollaries 4 or 6, respectively, one gets that
\[
\lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} \left| s_1 f(x) - f(x) \right|^r dt = 0. \]

Note that under the same conditions we get for the Fejér summation that
\[
\lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} \left| s_1 f(x) - f(x) \right|^r dt = 0. \]

### 5. Strong Summability of Fourier Series

In this section we formulate the above results for Fourier series. For an integrable function \( f \) the \( k \)th Fourier coefficient is defined by
\[
\hat{f}(k) = \frac{1}{2\pi} \int_{T} f(x) e^{-ikx} dx \quad (k \in \mathbb{Z}),
\]
where \( T \) denotes the torus. For \( f \in L_1(T) \) and \( n \in \mathbb{N} \) the \( n \)th partial sum \( s_n f \) is introduced by
\[
s_n f(x) := \sum_{k=-n}^{n} \hat{f}(k) e^{ikx} = \frac{1}{2\pi} \int_{T} f(x - u) D_n(u) du,
\]
where
\[
D_n(u) := \sum_{k=-n}^{n} e^{iku} = \frac{2 \sin((n + 1/2)x)}{\sin(x/2)} \quad (n \in \mathbb{N})
\]
is the Dirichlet kernel. Let
\[
\theta = (\theta(k,n), \ k \in \mathbb{N}, \ n \in \mathbb{N}_+) \quad (70)
\]
be a two-parameter sequence of real numbers satisfying
\[
\lim_{n \to \infty} \theta(k,n) = 1,
\]
\[
\lim_{k \to \infty} K_\theta^\beta \theta(k,n) = 0,
\]
where
\[
\sum_{k=0}^{\infty} (k + 1)^\beta |\Delta_1 \theta(k,n)| \leq Cn^\beta \quad \text{for} \ \beta = 0, d,
\]

The \( \theta \)-means of \( f \in L_1(T) \) are defined by
\[
\sigma_{n,\theta}^\beta f(x) := \sum_{k=-\infty}^{\infty} \theta(|k|,n) \hat{f}(k) e^{ikx} = \frac{1}{2\pi} \int_{T} f(x - u) K_\theta^\beta(u) du,
\]
where the \( \theta \)-kernel is given by
\[
K_\theta^\beta(x) := \sum_{j=0}^{\infty} \Delta_1 \theta(j,n) D_j(x).
\]

Hence
\[
\sigma_{n,\theta}^\beta f(x) = \sum_{j=0}^{\infty} \Delta_1 \theta(j,n) s_j f(x).
\]

### Example 8 (Cesàro summation)

For \( k \in \mathbb{N}, \alpha \neq -1, -2, \ldots, \) let
\[
A_k^\alpha := \left( \frac{k + \alpha}{k} \right) = \frac{(\alpha + 1) (\alpha + 2) \cdots (\alpha + k)}{k!} \quad (k \in \mathbb{N})
\]
be the Cesàro summation. The first type of examples is the Cesàro summation, which is generated by a sequence \( \theta \).
Let \( \theta(k, n) = \begin{cases} A_{n-k-1}^{\alpha} & \text{if } |k| \leq n-1 \\ A_{n-1}^{\alpha} & \text{if } |k| \geq n \end{cases} \) for some \( 0 < \alpha < \infty \). Since \( \Delta_1 \theta(k, n) = A_{n-k-1}^{\alpha-1}/A_{n-1}^{\alpha} \), the Cesàro operators can be given by

\[
\sigma_{\alpha}^d f(x) = \frac{1}{A_{n-1}^{\alpha}} \sum_{k=0}^{n-1} A_{n-1-k}^{\alpha-1} \delta_k f(x).
\]

(76)

If \( \alpha = 1 \), we get back the Fejér means.

The other type is generated by a function \( \theta : \mathbb{R} \rightarrow \mathbb{R} \) (see all other examples in this paper). Let \( \theta \) be continuous on \( \mathbb{R} \), and

\[
\theta(k, n) := \theta \left( \frac{k}{n} \right) \quad (k \in \mathbb{N}, n \in \mathbb{N}_+).
\]

(77)

Suppose that

\[
\theta(0) = 1, \quad \lim_{t \to \infty} t^d \theta(t) = 0,
\]

(78)

\[
\sum_{k=0}^{\infty} (k + 1)^\beta \Delta_1 \theta \left( \frac{k}{n} \right) \leq C n^\beta \quad \text{for } \beta = 0, d.
\]

(79)

Note that for the Fejér means we get the usual definition

\[
\sigma_{\alpha}^d f(x) = \frac{1}{n} \sum_{j=0}^{n-1} \delta_j f(x).
\]

(79)

All examples of this paper satisfy (26) and (27).

**Theorem 9.** Suppose that (26), (27), and (69) are satisfied for some \( d \in \mathbb{N} \) and \( 0 < \alpha < \infty \). Let \( f \in L^p(T) \) for some \( 1 < p < \infty \). If \( x_j \) is a \( p \)-Lebesgue point of \( f \) for all \( j = 1, \ldots, d \), then

\[
\lim_{n \to \infty} \sum_{k=0}^{\infty} \Delta_1 \theta(k, n) \prod_{j=1}^{d} \left( \| f(x_j) - f(x_j) \| \right) = 0.
\]

(80)

Moreover, if \( d \) is even, then

\[
\lim_{n \to \infty} \sum_{k=0}^{\infty} \Delta_1 \theta(k, n) \| f(x) - f(x) \| = 0.
\]

(81)

**Theorem 10.** Suppose that (26), (27), and (69) are satisfied for some \( d \in \mathbb{N} \) and \( 0 < \alpha < \infty \). If \( f \in L^1(T) \), \( x_j \) is a Lebesgue point of \( f \), and \( f \) is locally bounded at \( x_j \) for all \( j = 1, \ldots, d \), then (80) and (81) hold.

**Corollary 11.** Suppose that \( r > 0 \), \( \Delta_1 \theta(k, n) \leq 0 \), and (69) is satisfied for all \( d \). Under the same conditions as in Theorems 9 or 10, respectively, one gets that

\[
\lim_{n \to \infty} \sum_{k=0}^{\infty} \Delta_1 \theta(k, n) \| \delta_k f(x) - f(x) \| = 0.
\]

(82)

Note that under the same conditions we get for the Fejér summation that

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \| f(x) - f(x) \| = 0.
\]

(83)

Finally we note that the Weierstrass, Abel, Picard, Bessel, Fejér, de La Vallée-Poussin, Rogosinski, and Riesz summations can be considered as special cases of \( \theta \)-summation (see Weisz [21]).

**Conflict of Interests**

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