Research Article

Boundedness of Marcinkiewicz Integrals and Their Commutators on Generalized Weighted Morrey Spaces

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We will study the boundedness of Marcinkiewicz integrals with rough kernel on the generalized weighted Morrey spaces. We will also prove that the commutator operators formed by a $BMO(\mathbb{R}^n)$ function and Marcinkiewicz integrals are also bounded on the generalized weighted Morrey spaces.

1. Introduction and Results

Suppose that $\mathbb{S}^{n-1}$ is the unit sphere in $\mathbb{R}^n$ ($n \geq 2$) equipped with the normalized Lebesgue measure $d\sigma$. Let $\Omega \in L^q(\mathbb{S}^{n-1})$ with $1 < q \leq \infty$ be homogeneous of degree zero and satisfy the cancellation condition

$$\int_{\mathbb{S}^{n-1}} \Omega(x') d\sigma(x') = 0, \quad (1)$$

where $x' = x/|x|$ for any $x \neq 0$. The Marcinkiewicz integral of higher dimension μ_Ω is defined by

$$\mu_\Omega(f)(x) = \left(\int_0^\infty |F_{\Omega,t}(x)|^2 \frac{dt}{t^3} \right)^{1/2}, \quad (2)$$

where

$$F_{\Omega,t}(x) = \int_{|x-y| \leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y) dy. \quad (3)$$

We will also consider the commutator generated by Marcinkiewicz integral μ_Ω and b is defined as follows:

$$\begin{aligned} [b, \mu_\Omega](f)(x) &= \left(\int_0^\infty \left| \int_{|x-y| \leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} \right. \right. \\ &\quad \left. \left. \cdot (b(x) - b(y)) f(y) dy \right|^2 \frac{dt}{t^3} \right)^{1/2}. \end{aligned} \quad (4)$$

The following results concerning the boundedness of Marcinkiewicz integrals and their commutators on weighted L^p space are known.

Theorem 1 (see [1]). Suppose that $\Omega \in L^q(\mathbb{S}^{n-1})$ with $1 < q \leq \infty$. Then, for every $q' < p < \infty$ and $\omega \in A_{p/q'}$, there is a constant C independent of f such that

$$\|\mu_\Omega(f)\|_{L^p(\omega, \mathbb{R}^n)} \leq C \|f\|_{L^p(\omega, \mathbb{R}^n)}. \quad (5)$$

Theorem 2 (see [2]). Suppose that $\Omega \in L^q(\mathbb{S}^{n-1})$ with $1 < q \leq \infty$ and $b \in BMO(\mathbb{R}^n)$. Then, for every $q' < p < \infty$ and $\omega \in A_{p/q'}$, there is a constant C independent of f such that

$$\|[b, \mu_\Omega](f)\|_{L^p(\omega, \mathbb{R}^n)} \leq C \|b\|_* \|f\|_{L^p(\omega, \mathbb{R}^n)}. \quad (6)$$

The classical Morrey spaces, introduced by Morrey [3] in 1938, have been studied intensively by various authors and together with weighted Lebesgue spaces play an important role in the theory of partial differential equations; they appeared to be quite useful in the study of local behavior of the solutions of elliptic differential equations and describe local regularity more precisely than Lebesgue spaces. See [4–7] for details.

Given $f \in L^p_{\text{loc}}(\mathbb{R}^n)$, $1 \leq p < \infty$ and $0 \leq \lambda \leq n$. Morrey spaces are defined by

$$\|f\|_{L^{p,\lambda}(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n, r > 0} r^{-\lambda/p} \|f\|_{L^p(B(x,r))} < \infty. \quad (7)$$

Note that $L^{p,0}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ and $L^{p,n}(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$. If $\lambda < 0$ or $\lambda > n$, then $L^{p,\lambda}(\mathbb{R}^n) = \emptyset$, where $\emptyset$ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

In [8] Chiarenza and Frasca show the boundedness of the Hardy-Littlewood maximal operator in $L^{p,\lambda}(\mathbb{R}^n)$ that allows them to prove continuity of fractional and classical Calderón-Zygmund operators in these spaces. Recall that integral operators of that kind appear in the representation formulae of the solutions of elliptic/parabolic equations and systems. Thus the continuity of the Calderón-Zygmund integrals implies regularity of the solutions in the corresponding spaces.

Let $\Phi(r)$, $r > 0$, be a growth function, that is, a positive increasing function in $(0, \infty)$, and satisfy doubling condition

$$\Phi(2r) \leq D\Phi(r), \quad \forall r > 0, \quad (8)$$

where $D = D(\Phi) \geq 1$ is a doubling constant independent of r . In [9] Mizuhara gave generalization Morrey spaces $L^{p,\Phi}(\mathbb{R}^n)$ considering $\Phi(r)$ instead of r^λ in (7). He studied also a continuity in $L^{p,\Phi}(\mathbb{R}^n)$ of some classical integral operators.

Komori and Shirai [10] introduced a version of the weighted Morrey space $L^{p,\kappa}(\omega, \mathbb{R}^n)$, which is a natural generalization of the weighted Lebesgue space $L^p(\omega, \mathbb{R}^n)$. Let $1 \leq p < \infty$, $0 < \kappa < 1$ and let ω be a weight function. Then the spaces $L^{p,\kappa}(\omega, \mathbb{R}^n)$ are defined by

$$\begin{aligned} \|f\|_{L^{p,\kappa}(\omega, \mathbb{R}^n)} &= \sup_{x \in \mathbb{R}^n, r > 0} \left(\frac{1}{\omega(B(x,r))^\kappa} \int_{B(x,r)} |f(y)|^p \omega(y) dy \right)^{1/p} \\ &< \infty. \end{aligned} \quad (9)$$

Recently, Aliev and Guliev in [11] introduced another generalization of the Morrey spaces. Let $1 \leq p < \infty$ and let φ be a positive measurable function on $\mathbb{R}^n \times (0, \infty)$. We denote by $M^p_\varphi(\mathbb{R}^n)$ the generalized Morrey space, the space of all functions $f \in L^p_{\text{loc}}(\mathbb{R}^n)$ with finite norm

$$\|f\|_{M^p_\varphi(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n, r > 0} \frac{1}{\varphi(x,r)} \left(\frac{1}{|B(x,r)|} \|f\|_{L^p(B(x,r))}^p \right)^{1/p}. \quad (10)$$

They also studied in [11] the boundedness of μ_Ω and their commutator on generalized Morrey space $M^p_\varphi(\mathbb{R}^n)$ when $\Omega \in L^\infty(\mathbb{S}^{n-1})$.

Let $1 \leq p < \infty$ and let φ be a positive measurable function on $\mathbb{R}^n \times (0, \infty)$ and let ω be a nonnegative measurable function on $\mathbb{R}^n$. Following [12], we denote by $M^p_\varphi(\omega, \mathbb{R}^n)$ the generalized weighted Morrey space, the space of all functions $f \in L^p_{\text{loc}}(\omega, \mathbb{R}^n)$ with finite norm

$$\begin{aligned} \|f\|_{M^p_\varphi(\omega, \mathbb{R}^n)} &= \sup_{x \in \mathbb{R}^n, r > 0} \frac{1}{\varphi(x,r)} \left(\frac{1}{\omega(B(x,r))} \|f\|_{L^p(\omega, B(x,r))}^p \right)^{1/p}, \end{aligned} \quad (11)$$

where

$$\|f\|_{L^p(\omega, B(x,r))} = \left(\int_{B(x,r)} |f(y)|^p \omega(y) dy \right)^{1/p}. \quad (12)$$

If $\omega = 1$ and $\varphi(x,r) = r^{(\lambda-n)/p}$ with $0 \leq \lambda \leq n$, then $M^p_\varphi(\omega, \mathbb{R}^n) = L^{p,\lambda}(\mathbb{R}^n)$ is the classical Morrey space; if $\varphi(x,r) = \omega(B(x,r))^{(\kappa-1)/p}$, $0 < \kappa < 1$, then $M^p_\varphi(\omega, \mathbb{R}^n) = L^{p,\kappa}(\omega, \mathbb{R}^n)$ is the weighted Morrey space; if $\omega = 1$, $\varphi(x,r) = \Phi(r)r^{-n}$, then $M^p_\varphi(\omega, \mathbb{R}^n) = L^{p,\Phi}(\mathbb{R}^n)$. If $\omega = 1$, then $M^p_\varphi(\omega, \mathbb{R}^n) = M^p_\varphi(\mathbb{R}^n)$.

The purpose of this paper is to discuss the boundedness properties of Marcinkiewicz integrals with rough kernel and their commutators on the generalized weighted Morrey spaces $M^p_\varphi(\omega, \mathbb{R}^n)$. Our main results can be formulated as follows.

Theorem 3. Suppose that $\Omega \in L^q(\mathbb{S}^{n-1})$ with $1 < q \leq \infty$. Let $q' < p < \infty$, $\omega \in A_{p/q'}$, and (φ_1, φ_2) satisfy the condition

$$\int_s^\infty \frac{\text{ess inf}_{r < t < \infty} \varphi_1(x,t) \omega(B(x,t))^{1/p}}{\omega(B(x,r))^{1/p}} \frac{dr}{r} \leq C \varphi_2(x,s), \quad (13)$$

where C does not depend on x or r . Then there is a constant $C > 0$ independent of f such that

$$\|\mu_\Omega f\|_{M^p_{\varphi_2}(\omega, \mathbb{R}^n)} \leq C \|f\|_{M^p_{\varphi_1}(\omega, \mathbb{R}^n)}. \quad (14)$$

Theorem 4. Suppose that $\Omega \in L^q(\mathbb{S}^{n-1})$ with $1 < q \leq \infty$. Let $q' < p < \infty$, $\omega \in A_{p/q'}$, and (φ_1, φ_2) satisfy the condition

$$\begin{aligned} \int_s^\infty \left(1 + \ln \frac{r}{s} \right) \frac{\text{ess inf}_{r < t < \infty} \varphi_1(x,t) \omega(B(x,t))^{1/p}}{\omega(B(x,r))^{1/p}} \frac{dr}{r} \\ \leq C \varphi_2(x,s). \end{aligned} \quad (15)$$

If $b \in BMO(\mathbb{R}^n)$, then there is a constant $C > 0$ independent of f such that

$$\|[b, \mu_\Omega] f\|_{M^p_{\varphi_2}(\omega, \mathbb{R}^n)} \leq C \|b\|_* \|f\|_{M^p_{\varphi_1}(\omega, \mathbb{R}^n)}. \quad (16)$$

Corollary 5. Let $\Omega \in L^q(\mathbb{S}^{n-1})$ with $1 < q \leq \infty$ and $q' < p < \infty$.

(i) Suppose (φ_1, φ_2) satisfy the condition

$$\int_s^\infty \frac{\text{ess inf}_{r < t < \infty} \varphi_1(x,t) t^{n/p}}{r^{1+(n/p)}} dr \leq C \varphi_2(x,s). \quad (17)$$

Then there is a constant $C > 0$ independent of f such that

$$\|\mu_\Omega f\|_{M_{\varphi_2}^p(\mathbb{R}^n)} \leq C \|f\|_{M_{\varphi_2}^p(\mathbb{R}^n)}. \quad (18)$$

(ii) Suppose (φ_1, φ_2) satisfy the condition

$$\int_s^\infty \left(1 + \ln \frac{r}{s}\right) \frac{\text{ess inf}_{r < t < \infty} \varphi_1(x, t) t^{n/p}}{r^{1+(n/p)}} dr \leq C \varphi_2(x, s), \quad (19)$$

and $b \in BMO(\mathbb{R}^n)$. Then there is a constant $C > 0$ independent of f such that

$$\|[b, \mu_\Omega] f\|_{M_{\varphi_2}^p(\mathbb{R}^n)} \leq C \|b\|_* \|f\|_{M_{\varphi_2}^p(\mathbb{R}^n)}. \quad (20)$$

Remark 6. (i) Let $\varphi_1(x, t) = \varphi_2(x, t) = \omega(B(x, t))^{(k-1)/p}$, $0 < \kappa < 1$ and $\omega \in A_p(\mathbb{R}^n)$ ($1 \leq p < \infty$); then (φ_1, φ_2) satisfy conditions (13) and (15).

(ii) Let $\varphi_1(x, t) = \varphi_2(x, t) = (\Phi(t)t^{-n})^{1/p}$. If $1 \leq D(\Phi) \leq 2^n$, then (φ_1, φ_2) also satisfy conditions (13) and (15).

We verify only (i). (ii) can be verified similarly. In fact, from (36) in Section 2 we have constant $\delta > 0$ such that

$$\omega(B(x, 2^k s)) \geq C 2^{\delta k} \omega(B(x, s)). \quad (21)$$

Then

$$\begin{aligned} & \int_s^\infty \frac{\text{ess inf}_{r < t < \infty} \varphi_1(x, t) \omega(B(x, t))^{1/p}}{\omega(B(x, r))^{1/p}} \frac{dr}{r} \\ & \leq \int_s^\infty \left(1 + \ln \frac{r}{s}\right) \frac{\text{ess inf}_{r < t < \infty} \varphi_1(x, t) \omega(B(x, t))^{1/p}}{\omega(B(x, r))^{1/p}} \frac{dr}{r} \\ & = \int_s^\infty \left(1 + \ln \frac{r}{s}\right) \omega(B(x, r))^{(k-1)/p} \frac{dr}{r} \\ & \leq \sum_{k=0}^\infty (1+k) \int_{2^k s}^{2^{k+1} s} \omega(B(x, r))^{(k-1)/p} \frac{dr}{r} \\ & \leq C \sum_{k=0}^\infty (1+k) \omega(B(x, 2^{k+1} s))^{(k-1)/p} \\ & \leq C \sum_{k=0}^\infty (1+k) 2^{-(k+1)\delta((1-\kappa)/p)} \omega(B(x, s))^{(k-1)/p} \\ & \leq C \omega(B(x, s))^{(k-1)/p} = C \varphi_2(x, s). \end{aligned} \quad (22)$$

Corollary 7. Suppose that $\Omega \in L^q(\mathbb{S}^{n-1})$ with $1 < q \leq \infty$ and $0 < \kappa < 1$. If $q' < p < \infty$, $\omega \in A_{p/q'}(\mathbb{R}^n)$, then there is a constant $C > 0$ independent of f such that

$$\|\mu_\Omega f\|_{L^{p,\kappa}(\omega, \mathbb{R}^n)} \leq C \|f\|_{L^{p,\kappa}(\omega, \mathbb{R}^n)}. \quad (23)$$

If $q' < p < \infty$, $\omega \in A_{p/q'}(\mathbb{R}^n)$, and $b \in BMO(\mathbb{R}^n)$, then there is a constant $C > 0$ independent of f such that

$$\|[b, \mu_\Omega] f\|_{L^{p,\kappa}(\omega, \mathbb{R}^n)} \leq C \|b\|_* \|f\|_{L^{p,\kappa}(\omega, \mathbb{R}^n)}. \quad (24)$$

Corollary 8. Suppose that $\Omega \in L^q(\mathbb{S}^{n-1})$ with $1 < q \leq \infty$. If $q' < p < \infty$, $1 \leq D(\Phi) \leq 2^n$, then there is a constant $C > 0$ independent of f such that

$$\|\mu_\Omega f\|_{L^{p,\Phi}(\mathbb{R}^n)} \leq C \|f\|_{L^{p,\Phi}(\mathbb{R}^n)}. \quad (25)$$

If $q' < p < \infty$, $1 \leq D(\Phi) \leq 2^n$, and $b \in BMO(\mathbb{R}^n)$, then there is a constant $C > 0$ independent of f such that

$$\|[b, \mu_\Omega] f\|_{L^{p,\Phi}(\mathbb{R}^n)} \leq C \|b\|_* \|f\|_{L^{p,\Phi}(\mathbb{R}^n)}. \quad (26)$$

2. Some Preliminaries

We begin with some properties of A_p weights which play a great role in the proofs of our main results.

A weight ω is a nonnegative, locally integrable function on $\mathbb{R}^n$. Let $B = B(x_0, r_B)$ denote the ball with the center x_0 and radius r_B . For a given weight function ω and a measurable set E , we also denote the Lebesgue measure of E by $|E|$ and set weighted measure $\omega(E) = \int_E \omega(x) dx$. For any given weight function ω on $\mathbb{R}^n$, $X \subseteq \mathbb{R}^n$, and $0 < p < \infty$, denote by $L^p(\omega, X)$ the space of all function f satisfying

$$\|f\|_{L^p(\omega, X)} = \left(\int_X |f(x)|^p \omega(x) dx \right)^{1/p} < \infty. \quad (27)$$

A weight ω is said to belong to $A_p(\mathbb{R}^n)$ for $1 < p < \infty$, if there exists a constant

$$\left(\frac{1}{|B|} \int_B \omega(x) dx \right) \left(\frac{1}{|B|} \int_B \omega(x)^{1-p'} dx \right)^{p-1} \leq C, \quad (28)$$

where p' is the dual of p such that $1/p + 1/p' = 1$. The class $A_1(\mathbb{R}^n)$ is defined by

$$\frac{1}{|B|} \int_B \omega(y) dy \leq C \cdot \text{ess inf}_{x \in B} \omega(x) \quad (29)$$

for every ball $B \subset \mathbb{R}^n$.

A weight ω is said to belong to $A_\infty(\mathbb{R}^n)$ if there are positive numbers C and δ so that

$$\frac{\omega(E)}{\omega(B)} \leq C \left(\frac{|E|}{|B|} \right)^\delta \quad (30)$$

for all balls B and all measurable $E \subset B$. It is well known that

$$A_\infty(\mathbb{R}^n) = \bigcup_{1 \leq p < \infty} A_p(\mathbb{R}^n). \quad (31)$$

By (28), we have

$$\left(\omega^{-(p'/p)}(B) \right)^{1/p'} = \left\| \omega^{-(1/p)} \right\|_{L^{p'}(B)} \leq C |B| (\omega(B))^{-(1/p)} \quad (32)$$

for $1 < p < \infty$. Note that

$$\left(\text{ess inf}_{x \in E} f(x) \right)^{-1} = \text{ess sup}_{x \in E} \frac{1}{f(x)} \quad (33)$$

is true for any real-valued nonnegative function f and measurable on E (see [13] page 143) and (29); we get

$$\begin{aligned} \|\omega^{-1}\|_{L^\infty(B)} &= \operatorname{ess\,sup}_{x \in B} \frac{1}{\omega(x)} \\ &= \frac{1}{\operatorname{ess\,inf}_{x \in B} \omega(x)} \leq C |B| \omega(B)^{-1}. \end{aligned} \quad (34)$$

The classical $A_p(\mathbb{R}^n)$ weight theory was first introduced by Muckenhoupt in the study of weighted L^p -boundedness of Hardy-Littlewood maximal function in [14].

Lemma 9 (see [14, 15]). *Suppose $\omega \in A_p(\mathbb{R}^n)$ and the following statements hold.*

(i) *For any $1 \leq p < \infty$, there is a positive number C such that*

$$\frac{\omega(B_k)}{\omega(B_j)} \leq C 2^{np(k-j)} \quad \text{for } k > j. \quad (35)$$

(ii) *For any $1 \leq p < \infty$, there are positive numbers C and δ such that*

$$\frac{\omega(B_k)}{\omega(B_j)} \geq C 2^{\delta(k-j)} \quad \text{for } k > j. \quad (36)$$

(iii) *For any $1 < p < \infty$, one has $\omega^{1-p'} \in A_{p'}(\mathbb{R}^n)$.*

Following [16], a locally integrable function b is said to be in $BMO(\mathbb{R}^n)$ if

$$\sup_{B \subset \mathbb{R}^n} \frac{1}{|B|} \int_B |b(x) - b_B| dx = \|b\|_* < \infty, \quad (37)$$

where

$$b_B = \frac{1}{|B|} \int_B b(y) dy. \quad (38)$$

Lemma 10 (see [12]). *Suppose $\omega \in A_\infty(\mathbb{R}^n)$ and $b \in BMO(\mathbb{R}^n)$. Then, for any $1 \leq p < \infty$ and $r_1, r_2 > 0$, one has*

$$\begin{aligned} &\left(\frac{1}{\omega(B(x_0, r_1))} \int_{B(x_0, r_1)} |b(x) - b_{B(x_0, r_2)}|^p \omega(x) dx \right)^{1/p} \\ &\leq C \left(1 + \left| \ln \frac{r_1}{r_2} \right| \right) \|b\|_*. \end{aligned} \quad (39)$$

3. Proof of Theorem 3

We first prove the following conclusions.

Theorem 11. *Suppose that $\Omega \in L^q(\mathbb{S}^{n-1})$, $1 < q \leq \infty$. Then, for every $q' < p < \infty$, $\omega \in A_{p/q'}$, there is a constant C independent of f such that*

$$\begin{aligned} &\|\mu_\Omega(f)\|_{L^p(\omega, B(x_0, s))} \\ &\leq C \omega(B(x_0, s))^{1/p} \\ &\quad \cdot \int_{2s}^\infty \|f\|_{L^p(\omega, B(x_0, r))} \omega(B(x_0, r))^{-(1/p)} \frac{dr}{r}. \end{aligned} \quad (40)$$

Proof. We represent f as $f = f_1 + f_2$, where $f_1(y) = f(y)\chi_{B(x_0, 2s)}(y)$, $\chi_{B(x_0, 2s)}$ denotes the characteristic function of $B(x_0, 2s)$. Then

$$\begin{aligned} &\|\mu_\Omega(f)\|_{L^p(\omega, B(x_0, s))} \\ &\leq \|\mu_\Omega(f_1)\|_{L^p(\omega, B(x_0, s))} + \|\mu_\Omega(f_2)\|_{L^p(\omega, B(x_0, s))}. \end{aligned} \quad (41)$$

Since $f_1 \in L^p(\omega, \mathbb{R}^n)$ and from the boundedness of μ_Ω on $L^p(\omega, \mathbb{R}^n)$ ($q' < p < \infty$) (Theorem 1) it follows that

$$\begin{aligned} \|\mu_\Omega(f_1)\|_{L^p(\omega, B(x_0, s))} &\leq \|\mu_\Omega(f_1)\|_{L^p(\omega, \mathbb{R}^n)} \\ &\leq C \|f_1\|_{L^p(\omega, \mathbb{R}^n)} = C \|f\|_{L^p(\omega, B(x_0, 2s))}. \end{aligned} \quad (42)$$

By Hölder inequality,

$$|B(x_0, s)| \leq C \omega(B(x_0, s))^{1/p} \|\omega^{-(1/p)}\|_{L^{p'}(B(x_0, s))}. \quad (43)$$

Then, for $q' < p < \infty$,

$$\begin{aligned} &\|\mu_\Omega(f_1)\|_{L^p(\omega, B(x_0, s))} \\ &\leq C |B(x_0, s)| \|f\|_{L^p(\omega, B(x_0, 2s))} \int_{2s}^\infty \frac{dr}{r^{n+1}} \\ &\leq C |B(x_0, s)| \int_{2s}^\infty \|f\|_{L^p(\omega, B(x_0, r))} \frac{dr}{r^{n+1}} \\ &\leq C \omega(B(x_0, s))^{1/p} \|\omega^{-(1/p)}\|_{L^{p'}(B(x_0, s))} \\ &\quad \cdot \int_{2s}^\infty \|f\|_{L^p(\omega, B(x_0, r))} \frac{dr}{r^{n+1}} \\ &\leq C \omega(B(x_0, s))^{1/p} \\ &\quad \cdot \int_{2s}^\infty \|f\|_{L^p(\omega, B(x_0, r))} \|\omega^{-(1/p)}\|_{L^{p'}(B(x_0, r))} \frac{dr}{r^{n+1}}. \end{aligned} \quad (44)$$

By (32), we get

$$\begin{aligned} &\|\mu_\Omega(f_1)\|_{L^p(\omega, B(x_0, s))} \leq C \omega(B(x_0, s))^{1/p} \\ &\quad \cdot \int_{2s}^\infty \|f\|_{L^p(\omega, B(x_0, r))} \omega(B(x_0, r))^{-(1/p)} \frac{dr}{r}. \end{aligned} \quad (45)$$

To estimate $\|\mu_\Omega(f_2)\|_{L^p(\omega, B(x_0, s))}$, observe that when $x \in B(x_0, s)$ and $y \in B(x_0, 2^{j+1}s) \setminus B(x_0, 2^j s)$ ($j \geq 1$),

$$t \geq |x - y| \geq |y - x_0| - |x - x_0| \geq 2^{j-1}s. \quad (46)$$

Therefore, by Minkowski's inequality, we have

$$\begin{aligned}
& \mu_{\Omega}(f_2)(x) \\
&= \left(\int_0^{\infty} \left| \int_{B(x_0, 2s)^c \cap \{y: |x-y| \leq t\}} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y) dy \right|^2 \frac{dt}{t^3} \right)^{1/2} \\
&= \left(\int_0^{\infty} \left| \sum_{j=1}^{\infty} \int_{(B(x_0, 2^{j+1}s) \setminus B(x_0, 2^j s))^c \cap \{y: |x-y| \leq t\}} \frac{\Omega(x-y)}{|x-y|^{n-1}} \right. \right. \\
&\quad \cdot \left. \left. f(y) dy \right|^2 \frac{dt}{t^3} \right)^{1/2} \\
&\leq C \sum_{j=1}^{\infty} \left(\int_{B(x_0, 2^{j+1}s) \setminus B(x_0, 2^j s)} \frac{|\Omega(x-y)|}{|x-y|^{n-1}} |f(y)| dy \right) \\
&\quad \cdot \left(\int_{2^{j-1}s}^{\infty} \frac{dt}{t^3} \right)^{1/2} \\
&\leq C \sum_{j=1}^{\infty} (2^{j+1}s)^{-1} \int_{B(x_0, 2^{j+1}s) \setminus B(x_0, 2^j s)} \frac{|\Omega(x-y)|}{|x-y|^{n-1}} |f(y)| dy.
\end{aligned} \tag{47}$$

When $\Omega \in L^{\infty}(\mathbb{S}^{n-1})$, then by assumption we have $\omega \in A_p$, $1 < p < \infty$. It follows from the Hölder inequality and (32) that

$$\begin{aligned}
& \sup_{x \in B(x_0, s)} |\mu_{\Omega}(f_2)(x)| \\
&\leq C \|\Omega\|_{L^{\infty}(\mathbb{S}^{n-1})} \sum_{j=1}^{\infty} (2^{j+1}s)^{-n} \int_{B(x_0, 2^{j+1}s)} |f(y)| dy \\
&\leq C \sum_{j=1}^{\infty} (2^{j+1}s)^{-n} \|f\|_{L^p(\omega, B(x_0, 2^{j+1}s))} \\
&\quad \cdot \|\omega^{-(1/p)}\|_{L^{p'}(B(x_0, 2^{j+1}s))} \\
&\leq C \sum_{j=1}^{\infty} \int_{2^{j+1}s}^{2^{j+2}s} (2^{j+1}s)^{-(1+n)} \|f\|_{L^p(\omega, B(x_0, r))} \\
&\quad \cdot \|\omega^{-(1/p)}\|_{L^{p'}(B(x_0, r))} dr \\
&\leq C \int_{2s}^{\infty} \|f\|_{L^p(\omega, B(x_0, r))} \|\omega^{-(1/p)}\|_{L^{p'}(B(x_0, r))} \frac{dr}{r^{1+n}} \\
&\leq C \int_{2s}^{\infty} \|f\|_{L^p(\omega, B(x_0, r))} \omega(B(x_0, r))^{-(1/p)} \frac{dr}{r}.
\end{aligned} \tag{48}$$

Then

$$\begin{aligned}
& \|\mu_{\Omega}(f_2)\|_{L^p(\omega, B(x_0, s))} \\
&\leq C \omega(B(x_0, s))^{1/p} \int_{2s}^{\infty} \|f\|_{L^p(\omega, B(x_0, r))} \\
&\quad \cdot \omega(B(x_0, r))^{-(1/p)} \frac{dr}{r}.
\end{aligned} \tag{49}$$

When $\Omega \in L^q(\mathbb{S}^{n-1})$, $1 < q < \infty$. It follows from the Hölder inequality that

$$\begin{aligned}
& \int_{B(x_0, 2^{j+1}s) \setminus B(x_0, 2^j s)} \frac{|\Omega(x-y)|}{|x-y|^{n-1}} |f(y)| dy \\
&\leq C \left(\int_{B(x_0, 2^{j+1}s) \setminus B(x_0, 2^j s)} |\Omega((x-y)')|^q dy \right)^{1/q} \\
&\quad \cdot \left(\int_{B(x_0, 2^{j+1}s) \setminus B(x_0, 2^j s)} \frac{|f(y)|^{q'}}{|x-y|^{(n-1)q'}} dy \right)^{1/q'}.
\end{aligned} \tag{50}$$

When $x \in B(x_0, s)$ and $y \in B(x_0, 2^{j+1}s) \setminus B(x_0, 2^j s)$, then by a direct calculation we can see that $2^{j-1}s \leq |y-x| < 2^{j+1}s$. Hence

$$\begin{aligned}
& \left(\int_{B(x_0, 2^{j+1}s) \setminus B(x_0, 2^j s)} |\Omega((x-y)')|^q dy \right)^{1/q} \\
&\leq C \|\Omega\|_{L^q(\mathbb{S}^{n-1})} |B(x_0, 2^{j+1}s)|^{1/q}.
\end{aligned} \tag{51}$$

We also note that if $x \in B(x_0, s)$, $y \in B(x_0, 2s)^c$, then $|y-x| \sim |y-x_0|$. Consequently

$$\begin{aligned}
& \left(\int_{B(x_0, 2^{j+1}s) \setminus B(x_0, 2^j s)} \frac{|f(y)|^{q'}}{|x-y|^{(n-1)q'}} dy \right)^{1/q'} \\
&\leq (2^{j+1}s)^{1-n} \left(\int_{B(x_0, 2^{j+1}s)} |f(y)|^{q'} dy \right)^{1/q'}.
\end{aligned} \tag{52}$$

It follows from (47), (50), (51), and (52) that

$$\begin{aligned}
& \sup_{x \in B(x_0, s)} |\mu_{\Omega}(f_2)(x)| \\
&\leq C \sum_{j=1}^{\infty} (2^{j+1}s)^{-(n/q')} \left(\int_{B(x_0, 2^{j+1}s)} |f(y)|^{q'} dy \right)^{1/q'}.
\end{aligned} \tag{53}$$

Since $q' < p < \infty$, $\nu = p/q' > 1$. Thus from the Hölder inequality we get that

$$\begin{aligned}
& \sup_{x \in B(x_0, s)} |\mu_{\Omega}(f_2)(x)| \\
&\leq C \sum_{j=1}^{\infty} (2^{j+1}s)^{-n/q'} \|f\|_{L^p(\omega, B(x_0, 2^{j+1}s))} \\
&\quad \cdot \|\omega^{-(1/p)}\|_{L^{q' \nu'}(B(x_0, 2^{j+1}s))} \\
&\leq C \sum_{j=1}^{\infty} \int_{2^{j+1}s}^{2^{j+2}s} (2^{j+1}s)^{-(1+n/q')} \\
&\quad \cdot \|f\|_{L^p(\omega, B(x_0, r))} \|\omega^{-(1/p)}\|_{L^{q' \nu'}(B(x_0, r))} dr \\
&\leq \int_{2s}^{\infty} \|f\|_{L^p(\omega, B(x_0, r))} \\
&\quad \cdot \|\omega^{-(1/p)}\|_{L^{q' \nu'}(B(x_0, r))} \frac{dr}{r^{1+n/q'}}.
\end{aligned} \tag{54}$$

Note that $\omega \in A_{\gamma}$. It follows from the A_{γ} condition that

$$\omega(B(x_0, r)) \left(\omega^{1-\gamma'}(B(x_0, r)) \right)^{\gamma-1} \leq C. \quad (55)$$

That is,

$$\left\| \omega^{-(1/p)} \right\|_{L^{q', \gamma'}(B(x_0, r))} \leq r^{n/q'} \omega(B(x_0, r))^{-(1/p)}. \quad (56)$$

Then

$$\begin{aligned} & \sup_{x \in B(x_0, s)} |\mu_{\Omega}(f_2)(x)| \\ & \leq C \int_{2s}^{\infty} \|f\|_{L^p(\omega, B(x_0, r))} \omega(B(x_0, r))^{-(1/p)} \frac{dr}{r}. \end{aligned} \quad (57)$$

Therefore

$$\begin{aligned} & \|\mu_{\Omega}(f_2)\|_{L^p(\omega, B(x_0, s))} \\ & \leq C \omega(B(x_0, s))^{1/p} \\ & \cdot \int_{2s}^{\infty} \|f\|_{L^p(\omega, B(x_0, r))} \omega(B(x_0, r))^{-(1/p)} \frac{dr}{r}. \end{aligned} \quad (58)$$

Combining (45), (49), and (58), the proof of Theorem 11 is completed. $\square$

Proof of Theorem 3. Since $f \in M_{\varphi_1}^p(\omega, \mathbb{R}^n)$, from (33) and the fact $\|f\|_{L^p(\omega, B(x_0, r))}$ is a nondecreasing function of r , we get

$$\begin{aligned} & \frac{\|f\|_{L^p(\omega, B(x_0, r))}}{\text{ess inf}_{0 < r < t < \infty} \varphi_1(x_0, t) \omega(B(x_0, t))^{1/p}} \\ & \leq \text{ess sup}_{0 < r < t < \infty} \frac{\|f\|_{L^p(\omega, B(x_0, r))}}{\varphi_1(x_0, t) \omega(B(x_0, t))^{1/p}} \\ & \leq \text{ess sup}_{0 < t < \infty} \frac{\|f\|_{L^p(\omega, B(x_0, t))}}{\varphi_1(x_0, t) \omega(B(x_0, t))^{1/p}} \\ & \leq \|f\|_{M_{\varphi_1}^p(\omega, \mathbb{R}^n)}. \end{aligned} \quad (59)$$

For $q' < p < \infty$, since (φ_1, φ_2) satisfy (13), we have

$$\begin{aligned} & \int_s^{\infty} \|f\|_{L^p(\omega, B(x_0, r))} \omega(B(x_0, r))^{-(1/p)} \frac{dr}{r} \\ & \leq \int_s^{\infty} \frac{\|f\|_{L^p(\omega, B(x_0, r))}}{\text{ess inf}_{r < t < \infty} \varphi_1(x_0, t) \omega(B(x_0, t))^{1/p}} \\ & \cdot \frac{\text{ess inf}_{r < t < \infty} \varphi_1(x_0, t) \omega(B(x_0, t))^{1/p}}{\omega(B(x_0, r))^{1/p}} \frac{dr}{r} \\ & \leq C \|f\|_{M_{\varphi_1}^p(\omega, \mathbb{R}^n)} \\ & \cdot \int_s^{\infty} \frac{\text{ess inf}_{r < t < \infty} \varphi_1(x_0, t) \omega(B(x_0, t))^{1/p}}{\omega(B(x_0, r))^{1/p}} \frac{dr}{r} \\ & \leq C \|f\|_{M_{\varphi_1}^p(\omega, \mathbb{R}^n)} \varphi_2(x_0, s). \end{aligned} \quad (60)$$

Then by (40), we get

$$\begin{aligned} & \|\mu_{\Omega}(f)\|_{M_{\varphi_2}^p(\omega, \mathbb{R}^n)} \\ & \leq C \sup_{x_0 \in \mathbb{R}^n, s > 0} \frac{1}{\varphi_2(x_0, s)} \\ & \cdot \left(\frac{1}{\omega(B(x_0, s))} \int_{B(x_0, s)} |\mu_{\Omega}(f)(y)|^p \omega(y) dy \right)^{1/p} \\ & \leq C \sup_{x_0 \in \mathbb{R}^n, s > 0} \frac{1}{\varphi_2(x_0, s)} \\ & \cdot \int_s^{\infty} \|f\|_{L^p(\omega, B(x_0, r))} \omega(B(x_0, r))^{-(1/p)} \frac{dr}{r} \\ & \leq C \|f\|_{M_{\varphi_1}^p(\omega, \mathbb{R}^n)}. \end{aligned} \quad (61)$$

This completes the proof of Theorem 3. $\square$

4. Proof of Theorem 4

As in the proof of Theorem 3, it suffices to prove the following result.

Theorem 12. Assume that $\Omega \in L^q(\mathbb{S}^{n-1})$ with $1 < q \leq \infty$ and $b \in BMO(\mathbb{R}^n)$. Then, for every $q' < p < \infty$, $\omega \in A_{p/q'}$, there is a constant C independent of f such that

$$\begin{aligned} & \|[b, \mu_{\Omega}](f)\|_{L^p(\omega, B(x_0, s))} \\ & \leq C \|b\|_* \omega(B(x_0, s))^{1/p} \\ & \cdot \int_{2s}^{\infty} \left(1 + \ln \frac{r}{s} \right) \|f\|_{L^p(\omega, B(x_0, r))} \omega(B(x_0, r))^{-(1/p)} \frac{dr}{r}. \end{aligned} \quad (62)$$

Proof. We represent f as

$$\begin{aligned} f(y) &= f_1 + f_2, \\ f_1(y) &= f(y) \chi_{B(x_0, 2s)}(y), \\ f_2(y) &= f(y) \chi_{(B(x_0, 2s))^c}(y). \end{aligned} \quad (63)$$

Then

$$\begin{aligned} & \|[b, \mu_{\Omega}](f)\|_{L^p(\omega, B(x_0, s))} \\ & \leq \|[b, \mu_{\Omega}](f_1)\|_{L^p(\omega, B(x_0, s))} \\ & \quad + \|[b, \mu_{\Omega}](f_2)\|_{L^p(\omega, B(x_0, s))}. \end{aligned} \quad (64)$$

By Theorem 2, we have

$$\|[b, \mu_{\Omega}](f_1)\|_{L^p(\omega, B(x_0, s))} \leq C \|b\|_* \|f\|_{L^p(\omega, B(x_0, 2s))}. \quad (65)$$

As the proof of (45), we get

$$\begin{aligned} & \| [b, T_\Omega] (f_1) \|_{L^p(\omega, B(x_0, s))} \\ & \leq C \|b\|_* \omega(B(x_0, s))^{1/p} \\ & \cdot \int_{2s}^{\infty} \|f\|_{L^p(\omega, B(x_0, r))} \omega(B(x_0, r))^{-(1/p)} \frac{dr}{r}. \end{aligned} \quad (66)$$

We now turn to deal with the term $\| [b, \mu_\Omega] (f_2) \|_{L^p(\omega, B(x_0, s))}$. For any given $x \in B(x_0, s)$, we have

$$\begin{aligned} & | [b, \mu_\Omega] (f_2) (x) | \\ & \leq C |b(x) - b_{B(x_0, s)}| |\mu_\Omega(f_2)(x)| \\ & \quad + C |\mu_\Omega((b - b_{B(x_0, s)}) f_2)(x)| \\ & = I_1 + I_2. \end{aligned} \quad (67)$$

By (48) and (57), for any $\Omega \in L^q(\mathbb{S}^{n-1})$, $1 < q \leq \infty$, we have

$$\begin{aligned} I_1 & \leq C |b(y) - b_{B(x_0, s)}| \\ & \cdot \int_{2s}^{\infty} \|f\|_{L^p(\omega, B(x_0, r))} \omega(B(x_0, r))^{-(1/p)} \frac{dr}{r}. \end{aligned} \quad (68)$$

Then from Lemma 10 we get

$$\begin{aligned} \|I_1\|_{L^p(\omega, B(x_0, s))} & \leq C \|b\|_* \omega(B(x_0, s))^{1/p} \\ & \cdot \int_{2s}^{\infty} \|f\|_{L^p(\omega, B(x_0, r))} \omega(B(x_0, r))^{-(1/p)} \frac{dr}{r}. \end{aligned} \quad (69)$$

When $\Omega \in L^\infty(\mathbb{S}^{n-1})$, by assumption, we have $\omega \in A_p$, $1 < p < \infty$. It follows from the Hölder inequality that

$$\begin{aligned} & \sup_{x \in B(x_0, s)} |\mu_\Omega((b - b_{B(x_0, s)}) f_2)(x)| \\ & \leq C \|\Omega\|_{L^\infty(\mathbb{S}^{n-1})} \sum_{j=1}^{\infty} (2^{j+1}s)^{-n} \\ & \quad \cdot \int_{B(x_0, 2^{j+1}s)} |b(y) - b_{B(x_0, s)}| |f(y)| dy \\ & \leq C \sum_{j=1}^{\infty} (2^{j+1}s)^{-n} \|f\|_{L^p(\omega, B(x_0, 2^{j+1}s))} \\ & \quad \cdot \|b(\cdot) - b_{B(x_0, s)}\|_{L^{p'}(\omega^{1-p'}, B(x_0, 2^{j+1}s))} \\ & \leq C \sum_{j=1}^{\infty} \int_{2^{j+1}s}^{2^{j+2}s} (2^{j+1}s)^{-(1+n)} \|f\|_{L^p(\omega, B(x_0, r))} \\ & \quad \cdot \|b(\cdot) - b_{B(x_0, s)}\|_{L^{p'}(\omega^{1-p'}, B(x_0, r))} dr \\ & \leq C \int_{2s}^{\infty} \|f\|_{L^p(\omega, B(x_0, r))} \\ & \quad \cdot \|b(\cdot) - b_{B(x_0, s)}\|_{L^{p'}(\omega^{1-p'}, B(x_0, r))} \frac{dr}{r^{1+n}}. \end{aligned} \quad (70)$$

By Lemma 9, we have $\omega^{1-p'} \in A_{p'}(\mathbb{R}^n)$. Then by Lemma 10 and (32),

$$\begin{aligned} & \|b(\cdot) - b_{B(x_0, s)}\|_{L^{p'}(\omega^{1-p'}, B(x_0, r))} \\ & \leq C \left(\int_{B(x_0, r)} |b(z) - b_{B(x_0, s)}|^{p'} \omega^{1-p'}(z) dz \right)^{1/p'} \\ & \leq C \|b\|_* \left(1 + \left| \ln \frac{r}{s} \right| \right) (\omega^{1-p'}(B(x_0, r)))^{1/p'} \\ & = C \|b\|_* \left(1 + \left| \ln \frac{r}{s} \right| \right) \|\omega^{-(1/p)}\|_{L^{p'}(B(x_0, r))} \\ & \leq C \|b\|_* r^n \left(1 + \left| \ln \frac{r}{s} \right| \right) \omega(B(x_0, r))^{-(1/p)}. \end{aligned} \quad (71)$$

Then

$$\begin{aligned} & \sup_{x \in B(x_0, s)} |\mu_\Omega((b - b_{B(x_0, s)}) f_2)(x)| \\ & \leq C \int_{2s}^{\infty} \left(1 + \left| \ln \frac{r}{s} \right| \right) \|f\|_{L^p(\omega, B(x_0, r))} \omega(B(x_0, r))^{-(1/p)} dr. \end{aligned} \quad (72)$$

Therefore,

$$\begin{aligned} & \|I_2\|_{L^p(\omega, B(x_0, s))} \\ & \leq C \|b\|_* \omega(B(x_0, s))^{1/p} \\ & \quad \cdot \int_{2s}^{\infty} \|f\|_{L^p(\omega, B(x_0, r))} \omega(B(x_0, r))^{-(1/p)} \frac{dr}{r}. \end{aligned} \quad (73)$$

When $\Omega \in L^q(\mathbb{S}^{n-1})$, it follows from (50), (51), and (52) that

$$\begin{aligned} I_2 & \leq C \sum_{j=1}^{\infty} (2^{j+1}s)^{-(n/q')} \\ & \quad \cdot \left(\int_{B(x_0, 2^{j+1}s)} |(b(y) - b_{B(x_0, s)}) f(y)|^{q'} dy \right)^{1/q'}. \end{aligned} \quad (74)$$

Set $s = p/q'$. Since $\omega \in A_s$, from Lemma 9, we know $\omega^{1-s'} \in A_{s'}$. By Hölder inequality

$$\begin{aligned} & \left(\int_{B(x_0, 2^{j+1}s)} |b(y) - b_{B(x_0, s)}|^{q'} |f(y)|^{q'} dy \right)^{1/q'} \\ & \leq C \|f\|_{L^p(\omega, B(x_0, 2^{j+1}s))} \\ & \quad \cdot \|b(\cdot) - b_{B(x_0, s)}\|_{L^{q's'}(\omega^{1-s'}, B(x_0, 2^{j+1}s))}. \end{aligned} \quad (75)$$

Consequently,

$$\begin{aligned}
 I_2 &\leq \sum_{j=1}^{\infty} \int_{2^{j+1}s}^{2^{j+2}s} (2^{j+1}s)^{-(1+(n/q'))} \\
 &\quad \cdot \|f\|_{L^p(\omega, B(x_0, 2^{j+1}s))} \\
 &\quad \cdot \|b(\cdot) - b_{B(x_0, s)}\|_{L^{q's'}(\omega^{1-s'}, B(x_0, 2^{j+1}s))} dr \quad (76) \\
 &\leq C \int_{2s}^{\infty} \|f\|_{L^p(\omega, B(x_0, r))} \\
 &\quad \cdot \|b(\cdot) - b_{B(x_0, s)}\|_{L^{q's'}(\omega^{1-s'}, r)} \frac{dr}{r^{1+n/q'}}.
 \end{aligned}$$

Since $\omega^{1-s'} \in A_{s'}$, from (32), we know

$$(\omega^{1-s'} B(x_0, r))^{1/q's'} \leq C r^{n/q'} (B(x_0, r))^{-(1/p)}. \quad (77)$$

Using Lemma 10 and the fact that $\omega \in A_s$, we thus obtain

$$\begin{aligned}
 &\left(\int_{B(x_0, 2^{j+1}s)} |b(y) - b_{B(x_0, s)}|^{q's'} \omega^{1-s'}(y) dy \right)^{1/q's'} \\
 &\leq C \|b\|_* \left(1 + \ln \frac{r}{s} \right) (\omega^{1-s'}(B(x_0, r)))^{1/q's'} \quad (78) \\
 &\leq C \|b\|_* r^{n/q'} \left(1 + \ln \frac{r}{s} \right) \omega(B(x_0, r))^{-(1/p)}.
 \end{aligned}$$

Hence

$$\begin{aligned}
 I_2 &\leq C \|b\|_* \\
 &\quad \cdot \int_{2s}^{\infty} \left(1 + \ln \frac{r}{s} \right) \|f\|_{L^p(\omega, B(x_0, r))} \\
 &\quad \cdot \omega(B(x_0, r))^{-(1/p)} \frac{dr}{r}. \quad (79)
 \end{aligned}$$

Therefore

$$\begin{aligned}
 &\|I_2\|_{L^p(\omega, B(x_0, s))} \\
 &\leq C \|b\|_* \omega(B(x_0, s))^{1/p} \\
 &\quad \cdot \int_{2s}^{\infty} \left(1 + \ln \frac{r}{s} \right) \|f\|_{L^p(\omega, B(x_0, r))} \\
 &\quad \cdot \omega(B(x_0, r))^{-(1/p)} \frac{dr}{r}. \quad (80)
 \end{aligned}$$

□

Proof of Theorem 4. Since $f \in M_{\varphi_1}^p(\omega, \mathbb{R}^n)$ and (φ_1, φ_2) satisfy (15), for $q' < p < \infty$, we have

$$\begin{aligned}
 &\int_s^{\infty} \left(1 + \ln \frac{r}{s} \right) \|f\|_{L^p(\omega, B(x_0, r))} \omega(B(x_0, r))^{-(1/p)} \frac{dr}{r} \\
 &\leq \int_s^{\infty} \left(1 + \ln \frac{r}{s} \right) \frac{\|f\|_{L^p(\omega, B(x_0, r))}}{\operatorname{ess\,inf}_{r < t < \infty} \varphi_1(x_0, t) \omega(B(x_0, t))^{1/p}} \\
 &\quad \cdot \frac{\operatorname{ess\,inf}_{r < t < \infty} \varphi_1(x_0, t) \omega(B(x_0, t))^{1/p}}{\omega(B(x_0, r))^{1/p}} \frac{dr}{r} \\
 &\leq C \|f\|_{M_{\varphi_1}^p(\omega, \mathbb{R}^n)} \int_s^{\infty} \left(1 + \ln \frac{r}{s} \right) \\
 &\quad \cdot \frac{\operatorname{ess\,inf}_{r < t < \infty} \varphi_1(x_0, t) \omega(B(x_0, t))^{1/p}}{\omega(B(x_0, r))^{1/p}} \frac{dr}{r} \\
 &\leq C \|f\|_{M_{\varphi_1}^p(\omega, \mathbb{R}^n)} \varphi_2(x_0, s). \quad (81)
 \end{aligned}$$

Then from (62) we get

$$\begin{aligned}
 &\| [b, \mu_{\Omega}](f) \|_{M_{\varphi_2}^p(\omega, \mathbb{R}^n)} \\
 &\leq C \sup_{x_0 \in \mathbb{R}^n, s > 0} \frac{1}{\varphi_2(x_0, s)} \\
 &\quad \cdot \left(\frac{1}{\omega(B(x_0, s))} \int_{B(x_0, s)} |[b, \mu_{\Omega}](f)(y)|^p \omega(y) dy \right)^{1/p} \\
 &\leq C \|b\|_* \sup_{x_0 \in \mathbb{R}^n, s > 0} \frac{1}{\varphi_2(x_0, s)} \\
 &\quad \cdot \int_s^{\infty} \left(1 + \ln \frac{r}{s} \right) \|f\|_{L^p(\omega, B(x_0, r))} \\
 &\quad \cdot \omega(B(x_0, r))^{-(1/p)} \frac{dr}{r} \\
 &\leq C \|b\|_* \|f\|_{M_{\varphi_1}^p(\omega, \mathbb{R}^n)}. \quad (82)
 \end{aligned}$$

□

Conflict of Interests

The authors declare that they have no conflict of interests.

Authors' Contribution

The paper is a joint work of all the authors who contributed equally to the final version of the paper. All authors read and approved the final paper.

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□

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