Research Article

Almost λ_r -Statistical and Strongly Almost λ_r -Convergence of Order β of Sequences of Fuzzy Numbers

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The main purpose of this article is to introduce the concepts of almost λ_r -statistical convergence and strongly almost λ_r -convergence of order β of sequences of fuzzy numbers with respect to an Orlicz function. We give some relations between strongly almost λ_r -convergence and almost λ_r -statistical convergence of order β of sequences of fuzzy numbers.

1. Introduction

The concept of fuzzy set was introduced by Zadeh [1]. Matloka [2] introduced sequences of fuzzy numbers and provided that every convergent sequence of fuzzy numbers is bounded. Recently, sequences of fuzzy numbers have been discussed by Altınok et al. [3, 4], Aytar and Pehlivan [5], Çolak et al. [6, 7], Gökhan et al. [8], Nuray [9], and Talo and Başar [10].

The idea of statistical convergence was introduced by Fast [11] and the notion was linked with summability theory by Alotaibi et al. [12], Connor [13], Et et al. [14, 15], Fridy [16], Işık [17], Mohiuddine et al. [18, 19], and Tripathy [20]. Recently, the notion was generalized by Gadjiev and Orhan [21], Çolak [22], and Çolak and Bektaş [23].

In this paper, we study the concepts of and examine some properties of almost λ_r -statistical convergence and strongly almost λ_r -convergence of order β of sequences of fuzzy numbers. The results which we obtained in this study are much more general than those obtained by Başarır et al. [24].

2. Definitions and Preliminaries

A fuzzy set u on $\mathbb{R}$ is called a fuzzy number if it has the following properties:

- (i) u is normal;

- (ii) u is fuzzy convex;

- (iii) u is upper semicontinuous;

- (iv) $\text{supp } u = \text{cl}\{x \in \mathbb{R} : u(x) > 0\}$ is compact, where cl denoted the closure of the enclosed set.

A sequence $X = (X_k)$ of fuzzy numbers is a function X from the set $\mathbb{N}$ of all positive integers into $L(\mathbb{R})$, where $L(\mathbb{R})$ is fuzzy number space. A sequence $X = (X_k)$ is said to be bounded if the set $\{X_k : k \in \mathbb{N}\}$ is bounded. A sequence $X = (X_k)$ is said to be convergent if there exists a positive integer k_0 such that $d(X_k, X_0) < \varepsilon$ for $k > k_0$, for every $\varepsilon > 0$. By $w^{\mathcal{F}}$, $\ell_{\infty}^{\mathcal{F}}$, and $c^{\mathcal{F}}$ we denote the set of all, bounded, and convergent sequences of fuzzy numbers, respectively [2].

An Orlicz function is a function $N : [0, \infty) \rightarrow [0, \infty)$, which is continuous, nondecreasing, and convex with $N(0) = 0$, $N(x) > 0$ for $x > 0$, and $N(x) \rightarrow \infty$ as $x \rightarrow \infty$.

The generalized de la Vallée-Poussin mean is defined by $t_m(x) = (1/\lambda_m) \sum_{k \in I_m} x_k$, where $\lambda = (\lambda_m)$ is a nondecreasing sequence of positive numbers such that $\lambda_{m+1} \leq \lambda_m + 1$, $\lambda_1 = 1$, and $\lambda_m \rightarrow \infty$ as $m \rightarrow \infty$ and $I_m = [m - \lambda_m + 1, m]$.

The space $\hat{c}$ was introduced by Lorentz [25] and Maddox [26] has defined x to be strongly almost convergent to a number L if $\lim_{m \rightarrow \infty} (1/m) \sum_{k=1}^m |x_{k+n} - L| = 0$, uniformly in n .

Definition 1. Let $\lambda = (\lambda_m)$ be as above, let N be an Orlicz function, let $r = (r_k)$ be any sequence of strictly positive real numbers, and let $0 < \beta \leq 1$ be given. A sequence $X = (X_k)$ is said to be almost λ_r -statistically convergent of order β to X_0 , with respect to the Orlicz function N , if for every $\varepsilon > 0$

$$\lim_{m \rightarrow \infty} \frac{1}{\lambda_m^\beta} \left| \left\{ k \in I_m : \left[N \left(\frac{d(t_{kn}(X), X_0)}{s} \right) \right]^{r_k} \geq \varepsilon \right\} \right| = 0, \quad (1)$$

uniformly in $n, s > 0$,

where

$$t_{kn}(X) = \frac{X_n + X_{n+1} + \cdots + X_{n+k}}{k+1} = \frac{1}{k+1} \sum_{i=0}^k X_{n+i}. \quad (2)$$

The set of all almost λ_r -statistically convergent sequences of order β with respect to the Orlicz function N will be denoted by $\widehat{S}_\beta^\mathcal{F}(N, r, \lambda)$. In this case we write $X_k \rightarrow X_0(\widehat{S}_\beta^\mathcal{F}(N, r, \lambda))$. In the special case $\lambda_m = m$, for all $m \in \mathbb{N}$ we will write $\widehat{S}_\beta^\mathcal{F}(N, r)$ instead of $\widehat{S}_\beta^\mathcal{F}(N, r, \lambda)$ and in the special case $r_k = 1$ for all $k \in \mathbb{N}$ and $N(x) = x$ we will write $\widehat{S}_\beta^\mathcal{F}(\lambda)$ instead of $\widehat{S}_\beta^\mathcal{F}(N, r, \lambda)$. If we take $\beta = 1$, then we obtain the set of all almost λ_r -statistically convergent sequence $\widehat{S}^\mathcal{F}(N, r, \lambda)$. If $r_k = 1$ for all $k \in \mathbb{N}$ then we will write $\widehat{S}_\beta^\mathcal{F}(N, \lambda)$ instead of $\widehat{S}_\beta^\mathcal{F}(N, r, \lambda)$.

Definition 2. Let $\lambda = (\lambda_m)$ be as above, let N be an Orlicz function, let $r = (r_k)$ be any sequence of strictly positive real numbers, and let $0 < \beta \leq 1$ be given. We define the following sets:

$$\begin{aligned} & \widehat{w}_\beta^\mathcal{F}(N, r, \lambda) \\ &= \left\{ X \in w^\mathcal{F} : \lim_{m \rightarrow \infty} \frac{1}{\lambda_m^\beta} \sum_{k \in I_m} \left[N \left(\frac{d(t_{kn}(X), X_0)}{s} \right) \right]^{r_k} = 0, \right. \\ & \quad \left. \text{uniformly in } n, s > 0 \right\}, \\ & \widehat{w}_\beta^\mathcal{F}(N, r, \lambda)_0 \\ &= \left\{ X \in w^\mathcal{F} : \lim_{m \rightarrow \infty} \frac{1}{\lambda_m^\beta} \sum_{k \in I_m} \left[N \left(\frac{d(t_{kn}(X), \bar{0})}{s} \right) \right]^{r_k} = 0, \right. \\ & \quad \left. \text{uniformly in } n, s > 0 \right\}. \end{aligned}$$

$$\begin{aligned} & \widehat{w}_\beta^\mathcal{F}(N, r, \lambda)_\infty \\ &= \left\{ X \in w^\mathcal{F} : \sup_{m, n} \frac{1}{\lambda_m^\beta} \sum_{k \in I_m} \left[N \left(\frac{d(t_{kn}(X), \bar{0})}{s} \right) \right]^{r_k} < \infty, \right. \\ & \quad \left. s > 0 \right\}. \end{aligned} \quad (3)$$

If $X \in \widehat{w}_\beta^\mathcal{F}(N, r, \lambda)$, we say that X is strongly almost λ_r -convergent of order β with respect to the Orlicz function N . In this case we write $X_k \rightarrow X_0(\widehat{w}_\beta^\mathcal{F}(N, r, \lambda))$.

- (i) If we take $r_k = 1$ for all $k \in \mathbb{N}$, then we get $\widehat{w}_\beta^\mathcal{F}(N, r, \lambda) = \widehat{w}_\beta^\mathcal{F}(N, \lambda)$, $\widehat{w}_\beta^\mathcal{F}(N, r, \lambda)_0 = \widehat{w}_\beta^\mathcal{F}(N, \lambda)_0$, and $\widehat{w}_\beta^\mathcal{F}(N, r, \lambda)_\infty = \widehat{w}_\beta^\mathcal{F}(N, \lambda)_\infty$.
- (ii) If we take $N(x) = x$ and $r_k = 1$ for all $k \in \mathbb{N}$, then we get $\widehat{w}_\beta^\mathcal{F}(N, r, \lambda) = \widehat{w}_\beta^\mathcal{F}(\lambda)$, $\widehat{w}_\beta^\mathcal{F}(N, r, \lambda)_0 = \widehat{w}_\beta^\mathcal{F}(\lambda)_0$, and $\widehat{w}_\beta^\mathcal{F}(N, r, \lambda)_\infty = \widehat{w}_\beta^\mathcal{F}(\lambda)_\infty$.
- (iii) If we take $\beta = 1$ and $r_k = 1$ for all $k \in \mathbb{N}$, then we get $\widehat{w}_\beta^\mathcal{F}(N, r, \lambda) = \widehat{w}^\mathcal{F}(N, \lambda)$, $\widehat{w}_\beta^\mathcal{F}(N, r, \lambda)_0 = \widehat{w}^\mathcal{F}(N, \lambda)_0$, and $\widehat{w}_\beta^\mathcal{F}(N, r, \lambda)_\infty = \widehat{w}^\mathcal{F}(N, \lambda)_\infty$.

3. Main Results

In this section we give the main results of this paper.

The proof of each of the following results is straightforward, so we state the following.

Theorem 3. Let $r = (r_k)$ be bounded, let $\lambda = (\lambda_m)$ be as above, and let $0 < \beta \leq 1$; then $\widehat{w}_\beta^\mathcal{F}(N, r, \lambda)_0 \subset \widehat{w}_\beta^\mathcal{F}(N, r, \lambda) \subset \widehat{w}_\beta^\mathcal{F}(N, r, \lambda)_\infty$.

Theorem 4. Let $r = (r_k)$ be bounded, let $\lambda = (\lambda_m)$ be as above, and let $0 < \beta \leq 1$; then $\widehat{w}_\beta^\mathcal{F}(N, r, \lambda)$, $\widehat{w}_\beta^\mathcal{F}(N, r, \lambda)_0$, $\widehat{w}_\beta^\mathcal{F}(N, r, \lambda)_\infty$, and $\widehat{S}_\beta^\mathcal{F}(N, r, \lambda)$ are closed under the operations of addition and scalar multiplication.

Theorem 5. Let $r = (r_k)$ be bounded, let $\lambda = (\lambda_m)$ be as above, let $0 < \beta \leq 1$, and let N_1, N_2 be Orlicz functions; then

- (i) $\widehat{w}_\beta^\mathcal{F}(N_1, r, \lambda)_0 \cap \widehat{w}_\beta^\mathcal{F}(N_2, r, \lambda)_0 \subset \widehat{w}_\beta^\mathcal{F}(N_1 + N_2, r, \lambda)_0$,
- (ii) $\widehat{w}_\beta^\mathcal{F}(N_1, r, \lambda) \cap \widehat{w}_\beta^\mathcal{F}(N_2, r, \lambda) \subset \widehat{w}_\beta^\mathcal{F}(N_1 + N_2, r, \lambda)$,
- (iii) $\widehat{w}_\beta^\mathcal{F}(N_1, r, \lambda)_\infty \cap \widehat{w}_\beta^\mathcal{F}(N_2, r, \lambda)_\infty \subset \widehat{w}_\beta^\mathcal{F}(N_1 + N_2, r, \lambda)_\infty$.

Theorem 6. Let $0 < r_k \leq q_k$ and (q_k/r_k) be bounded, let $\lambda = (\lambda_m)$ be as above, and let $0 < \beta \leq 1$; then $\widehat{w}_\beta^\mathcal{F}(N, q, \lambda) \subseteq \widehat{w}_\beta^\mathcal{F}(N, r, \lambda)$.

Theorem 7. Let $r = (r_k)$ be bounded, let $\lambda = (\lambda_m)$ be as above, and let $0 < \beta \leq 1$. If $\liminf_{m \rightarrow \infty} (\lambda_m^\beta/m^\beta) > 0$, then $\widehat{S}_\beta^\mathcal{F}(N, p) \subset \widehat{S}_\beta^\mathcal{F}(N, r, \lambda)$.

Proof. For given $\varepsilon > 0$ we have

$$\begin{aligned} & \left\{ k \leq m : \left[N \left(\frac{d(t_{kn}(X), X_0)}{s} \right) \right]^{r_k} \geq \varepsilon \right\} \\ & \supset \left\{ k \in I_m : \left[N \left(\frac{d(t_{kn}(X), X_0)}{s} \right) \right]^{r_k} \geq \varepsilon \right\} \end{aligned} \quad (4)$$

and so

$$\begin{aligned} & \frac{1}{m^\beta} \left| \left\{ k \leq m : \left[N \left(\frac{d(t_{kn}(X), X_0)}{s} \right) \right]^{r_k} \geq \varepsilon \right\} \right| \\ & \geq \frac{\lambda_m^\beta}{m^\beta} \frac{1}{\lambda_m^\beta} \left| \left\{ k \in I_m : \left[N \left(\frac{d(t_{kn}(X), X_0)}{s} \right) \right]^{r_k} \geq \varepsilon \right\} \right|. \end{aligned} \quad (5)$$

Hence we get $x \in \widehat{S}_\beta^\mathcal{F}(N, r, \lambda)$. $\square$

Theorem 8. Let $r = (r_k)$ be bounded, let $\lambda = (\lambda_m)$ be as above, and let $0 < \beta \leq \gamma \leq 1$; then $\widehat{S}_\beta^\mathcal{F}(N, r, \lambda) \subseteq \widehat{S}_\gamma^\mathcal{F}(N, r, \lambda)$.

Proof. The following inequality yields the proof:

$$\begin{aligned} & \frac{1}{\lambda_m^\gamma} \left| \left\{ k \in I_m : \left[N \left(\frac{d(t_{kn}(X), X_0)}{s} \right) \right]^{r_k} \geq \varepsilon \right\} \right| \\ & \leq \frac{1}{\lambda_m^\beta} \left| \left\{ k \in I_m : \left[N \left(\frac{d(t_{kn}(X), X_0)}{s} \right) \right]^{r_k} \geq \varepsilon \right\} \right|. \end{aligned} \quad (6)$$

$\square$

Theorem 9. Let $r = (r_k)$ be bounded, let $\lambda = (\lambda_m)$ be as above, let $0 < \beta \leq \gamma \leq 1$, and let N be an Orlicz function; then

- (i) $\widehat{w}_\beta^\mathcal{F}(N, r, \lambda) \subseteq \widehat{w}_\gamma^\mathcal{F}(N, r, \lambda)$,
- (ii) $\widehat{w}_\beta^\mathcal{F}(N, r, \lambda)_0 \subseteq \widehat{w}_\gamma^\mathcal{F}(N, r, \lambda)_0$,
- (iii) $\widehat{w}_\beta^\mathcal{F}(N, r, \lambda)_\infty \subseteq \widehat{w}_\gamma^\mathcal{F}(N, r, \lambda)_\infty$.

Proof. It is easy so it is omitted. $\square$

Theorem 10. If $\lim r_k > 0$ and X is strongly almost λ_r -convergent of order β to X_0 , with respect to the Orlicz function N , then X_0 is unique.

Proof. Suppose that $X_k \rightarrow X_0(\widehat{w}_\beta^\mathcal{F}(N, r, \lambda))$, $X_k \rightarrow Y_0(\widehat{w}_\beta^\mathcal{F}(N, r, \lambda))$, and $\lim r_k = \ell > 0$; then there exist s_1 and s_2 such that

$$\begin{aligned} & \lim_m \frac{1}{\lambda_m^\beta} \sum_{k \in I_m} \left[N \left(\frac{d(t_{kn}(X), X_0)}{s_1} \right) \right]^{r_k} = 0, \\ & \lim_m \frac{1}{\lambda_m^\beta} \sum_{k \in I_m} \left[N \left(\frac{d(t_{kn}(X), Y_0)}{s_2} \right) \right]^{p_k} = 0, \quad \text{uniformly in } n. \end{aligned} \quad (7)$$

Let $s = \max(2s_1, 2s_2)$. Then we have

$$\begin{aligned} & \lim_m \frac{1}{\lambda_m^\beta} \sum_{k \in I_m} \left[N \left(\frac{d(X_0, Y_0)}{s} \right) \right]^{r_k} \\ & \leq \frac{G}{\lambda_m^\beta} \sum_{k \in I_m} \left[N \left(\frac{d(t_{kn}(X), X_0)}{s_1} \right) \right]^{r_k} \\ & \quad + \frac{G}{\lambda_m^\beta} \sum_{k \in I_m} \left[N \left(\frac{d(t_{kn}(X), Y_0)}{s_2} \right) \right]^{r_k} \rightarrow 0, \end{aligned} \quad (8)$$

$(m \rightarrow \infty),$

where $\sup_k r_k = H$ and $G = \max(1, 2^{H-1})$. Thus

$$\lim_m \frac{1}{\lambda_m^\beta} \sum_{k \in I_m} \left[N \left(\frac{d(X_0, Y_0)}{s} \right) \right]^{r_k} = 0. \quad (9)$$

Also, since clearly

$$\lim_k \left[N \left(\frac{d(X_0, Y_0)}{s} \right) \right]^{r_k} = \left[N \left(\frac{d(X_0, Y_0)}{s} \right) \right]^\ell, \quad (10)$$

therefore $[N(d(X_0, Y_0)/s)]^\ell = 0$. Hence $X_0 = Y_0$. $\square$

Theorem 11. Let $\lambda = (\lambda_m)$ be as above; then $\widehat{w}^\mathcal{F}(N, \lambda)_\infty = \widehat{m}^\mathcal{F}(N)_\infty$ and $\widehat{w}^\mathcal{F}(\lambda)_\infty = \ell_\infty^\mathcal{F}$, where

$$\begin{aligned} \widehat{m}^\mathcal{F}(N)_\infty &= \left\{ X \in w^\mathcal{F} : \sup_{k,n} \left[N \left(\frac{d(t_{kn}(X), \bar{0})}{s} \right) \right] < \infty, \right. \\ & \quad \left. \text{for some } s > 0 \right\}. \end{aligned} \quad (11)$$

Proof. Let $X \in \widehat{w}^\mathcal{F}(N, \lambda)_\infty$. Then we can find a constant $K_1 > 0$ such that

$$\begin{aligned} & \frac{1}{\lambda_1} \left[N \left(\frac{d(t_{kn}(X), \bar{0})}{s} \right) \right] \\ & \leq \frac{1}{\lambda_m} \sum_{k \in I_m} \left[N \left(\frac{d(t_{kn}(X), \bar{0})}{s} \right) \right] \\ & \leq K_1 \quad \text{for each } k, n. \end{aligned} \quad (12)$$

Hence $X \in \widehat{m}^\mathcal{F}(N)_\infty$.

Conversely, if $X \in \widehat{m}^\mathcal{F}(N)_\infty$ we can find a constant $K_2 > 0$ such that $N(d(t_{kn}(X), \bar{0})/s) \leq K_2$ for each k, n and so

$$\frac{1}{\lambda_m} \sum_{k \in I_m} N \left(\frac{d(t_{kn}(X), \bar{0})}{s} \right) \leq \frac{K_2}{\lambda_m} \sum_{k \in I_m} 1 \leq K_2 \quad \text{for each } k, n. \quad (13)$$

Thus $X \in \widehat{w}^{\mathcal{F}}(N, \lambda)_{\infty}$. Similar proof can be made for the other.

In Theorems 12 and 13, we will assume that the sequence $r = (r_k)$ is bounded and $0 < h = \inf_k r_k \leq r_k \leq \sup_k p_k = H < \infty$. From now on, $\sum_1$ will be denoting the sum over $k \in I_m$ with $d(t_{kn}(X), X_0) \geq \varepsilon$ and $\sum_2$ will be denoting the sum over $k \in I_m$ with $d(t_{kn}(X), X_0) < \varepsilon$. $\square$

Theorem 12. Let $\lambda = (\lambda_m)$ be as above and let $\beta, \gamma \in (0, 1]$ such that $0 < \beta \leq \gamma \leq 1$; then $\widehat{w}^{\mathcal{F}}_{\beta}(N, r, \lambda) \subset \widehat{S}^{\mathcal{F}}_{\gamma}(\lambda)$.

Proof. One has $X \in \widehat{w}^{\mathcal{F}}_{\beta}(N, r, \lambda)$; then we have

$$\begin{aligned} & \frac{1}{\lambda_m^{\beta}} \sum_{k \in I_m} \left[N \left(\frac{d(t_{kn}(X), X_0)}{s} \right) \right]^{r_k} \\ & \geq \frac{1}{\lambda_m^{\gamma}} \sum_1 \left[N \left(\frac{d(t_{kn}(X), X_0)}{s} \right) \right]^{r_k} \\ & \geq \frac{1}{\lambda_m^{\gamma}} \sum_1 \min([N(\sigma)]^h, [N(\sigma)]^H), \quad \sigma = \frac{\varepsilon}{s} \\ & \geq \frac{1}{\lambda_m^{\gamma}} |\{k \in I_m : d(t_{kn}(X), X_0) \geq \varepsilon\}| \\ & \quad \cdot \min([N(\sigma)]^h, [N(\sigma)]^H). \end{aligned} \quad (14)$$

Hence $X \in \widehat{S}^{\mathcal{F}}_{\gamma}(\lambda)$. $\square$

Theorem 13. Let $\lambda = (\lambda_m)$ be as above and let $\beta, \gamma \in (0, 1]$ such that $0 < \beta \leq \gamma \leq 1$, $X \in \ell_{\infty}^{\mathcal{F}}$, and $\lim_{m \rightarrow \infty} (\lambda_m / \lambda_m^{\beta}) = 1$; then $\widehat{S}^{\mathcal{F}}_{\beta}(\lambda) \subset \widehat{w}^{\mathcal{F}}_{\gamma}(N, r, \lambda)$.

Proof. Suppose that $X \in \ell_{\infty}^{\mathcal{F}}$ and $X_k \rightarrow X_0(\widehat{S}^{\mathcal{F}}_{\gamma}(\lambda))$. Since X is bounded, there exists a constant $B > 0$ such that $d(t_{kn}(X), X_0) \leq B$. Let $\varepsilon > 0$, and then we have

$$\begin{aligned} & \frac{1}{\lambda_m^{\gamma}} \sum_{k \in I_m} \left[N \left(\frac{d(t_{kn}(X), X_0)}{s} \right) \right]^{r_k} \\ & \leq \frac{1}{\lambda_m^{\beta}} \sum_1 \left[N \left(\frac{d(t_{kn}(X), X_0)}{s} \right) \right]^{r_k} \\ & \quad + \frac{1}{\lambda_m^{\beta}} \sum_2 \left[N \left(\frac{d(t_{kn}(X), X_0)}{s} \right) \right]^{r_k} \\ & \leq \frac{1}{\lambda_m^{\beta}} \sum_1 \max \left\{ \left[N \left(\frac{B}{s} \right) \right]^h, \left[N \left(\frac{B}{s} \right) \right]^H \right\} \\ & \quad + \frac{1}{\lambda_m^{\beta}} \sum_2 \left[N \left(\frac{\varepsilon}{s} \right) \right]^{r_k} \end{aligned}$$

$$\begin{aligned} & \leq \max \{ [N(C)]^h, [N(C)]^H \} \\ & \quad \cdot \frac{1}{\lambda_m^{\beta}} |\{k \in I_m : d(t_{kn}(X), X_0) \geq \varepsilon\}| \\ & \quad + \frac{\lambda_m}{\lambda_m^{\beta}} \max \{ [N(\delta)]^h, [N(\delta)]^H \}, \quad \frac{B}{s} = C, \quad \frac{\varepsilon}{s} = \delta. \end{aligned} \quad (15)$$

Hence $X \in \widehat{w}^{\mathcal{F}}_{\gamma}(N, r, \lambda)$.

Let $\lambda = (\lambda_m)$ and $\mu = (\mu_m)$ be two sequences such that $I_m \subset J_m(\lambda_m \leq \mu_m)$ for all $m \in \mathbb{N}$, where $I_m = [m - \lambda_m + 1, m]$ and $J_m = [m - \mu_m + 1, m]$. Suppose also that the parameters β and γ are fixed real numbers such that $0 < \beta \leq \gamma \leq 1$. We will now give the following inclusion relations. $\square$

Theorem 14. Let $\lambda = (\lambda_m)$ and $\mu = (\mu_m)$ be defined as above, let N be an Orlicz function, and let $\beta, \gamma \in (0, 1]$ such that $0 < \beta \leq \gamma \leq 1$. Then the following assertions hold true.

(i) If

$$\liminf_{m \rightarrow \infty} \frac{\lambda_m^{\beta}}{\mu_m^{\gamma}} > 0 \quad (16)$$

then $\widehat{S}^{\mathcal{F}}_{\gamma}(N, \mu) \subseteq \widehat{S}^{\mathcal{F}}_{\beta}(N, \lambda)$.

(ii) If

$$\lim_{n \rightarrow \infty} \frac{\mu_m}{\lambda_m^{\gamma}} = 1 \quad (17)$$

then $\widehat{S}^{\mathcal{F}}_{\beta}(N, \lambda) \subseteq \widehat{S}^{\mathcal{F}}_{\gamma}(N, \mu)$.

Proof. (i) It is easy so it is omitted.

(ii) Suppose that $X \in \widehat{S}^{\mathcal{F}}_{\beta}(N, \lambda)$ and (17) is satisfied. Since $I_m \subset J_m$, we may write

$$\begin{aligned} & \frac{1}{\mu_m^{\gamma}} \left| \left\{ k \in J_m : \left[N \left(\frac{d(t_{kn}(X), X_0)}{s} \right) \right] \geq \varepsilon \right\} \right| \\ & = \frac{1}{\mu_m^{\gamma}} \left| \left\{ m - \mu_m + 1 < k \right. \right. \\ & \quad \left. \left. \leq m - \lambda_m : \left[N \left(\frac{d(t_{kn}(X), X_0)}{s} \right) \right] \geq \varepsilon \right\} \right| \\ & \quad + \frac{1}{\mu_m^{\gamma}} \left| \left\{ k \in I_m : \left[N \left(\frac{d(t_{kn}(X), X_0)}{s} \right) \right] \geq \varepsilon \right\} \right| \\ & \leq \left(\frac{\mu_m - \lambda_m^{\gamma}}{\lambda_m^{\gamma}} \right) \\ & \quad + \frac{1}{\lambda_m^{\beta}} \left| \left\{ k \in I_m : \left[N \left(\frac{d(t_{kn}(X), X_0)}{s} \right) \right] \geq \varepsilon \right\} \right| \\ & \leq \left(\frac{\mu_m}{\lambda_m^{\gamma}} - 1 \right) \\ & \quad + \frac{1}{\lambda_m^{\beta}} \left| \left\{ k \in I_m : \left[N \left(\frac{d(t_{kn}(X), X_0)}{s} \right) \right] \geq \varepsilon \right\} \right| \end{aligned} \quad (18)$$

for all $m \in \mathbb{N}$. Hence $X \in \widehat{S}^{\mathcal{F}}_{\gamma}(N, \mu)$. $\square$

Theorem 15. Let $\lambda = (\lambda_m)$ and $\mu = (\mu_m)$ be defined as above, let N be an Orlicz function, and let $\beta, \gamma \in (0, 1]$ such that $0 < \beta \leq \gamma \leq 1$. Then the following assertions hold true.

- (i) If (16) holds, then $\widehat{w}_\gamma^\mathcal{F}(N, \mu) \subset \widehat{w}_\beta^\mathcal{F}(N, \lambda)$,
- (ii) Let condition (17) be satisfied and $X \in \ell_\infty^\mathcal{F}$; then $\widehat{w}_\beta^\mathcal{F}(N, \lambda) \subset \widehat{w}_\gamma^\mathcal{F}(N, \mu)$.

Proof. It is easy so it is omitted. $\square$

Theorem 16. Let $\lambda = (\lambda_m)$ and $\mu = (\mu_m)$ be defined as above, let N be an Orlicz function, and let $\beta, \gamma \in (0, 1]$ such that $0 < \beta \leq \gamma \leq 1$. Then

- (i) if (16) holds, then $\widehat{w}_\gamma^\mathcal{F}(N, \mu) \subset \widehat{S}_\beta^\mathcal{F}(N, \lambda)$;
- (ii) let condition (17) be satisfied and $X \in \ell_\infty^\mathcal{F}$; then $\widehat{S}_\beta^\mathcal{F}(N, \lambda) \subset \widehat{w}_\gamma^\mathcal{F}(N, \mu)$.

Proof. (i) It is easy so it is omitted.

(ii) Suppose that $X \in \widehat{S}_\beta^\mathcal{F}(N, \lambda)$ and let $X = (X_k)$ be bounded; then there exists a constant $B > 0$ such that $d(t_{kn}(X), X_0) \leq B$. Then we may write

$$\begin{aligned}
 & \frac{1}{\mu_m^\gamma} \sum_{k \in I_m} \left[N \left(\frac{d(t_{kn}(X), X_0)}{s} \right) \right] \\
 &= \frac{1}{\mu_m^\gamma} \sum_{k \in J_m - I_m} \left[N \left(\frac{d(t_{kn}(X), X_0)}{s} \right) \right] \\
 & \quad + \frac{1}{\mu_m^\gamma} \sum_{k \in I_m} \left[N \left(\frac{d(t_{kn}(X), X_0)}{s} \right) \right] \\
 &\leq \left(\frac{\mu_m - \lambda_m^\gamma}{\mu_m^\gamma} \right) B + \frac{1}{\mu_m^\gamma} \sum_{k \in I_m} \left[N \left(\frac{d(t_{kn}(X), X_0)}{s} \right) \right] \\
 &= \left(\frac{\mu_m}{\mu_m^\gamma} - 1 \right) B + \frac{1}{\mu_m^\gamma} \sum_1 \left[N \left(\frac{d(t_{kn}(X), X_0)}{s} \right) \right] \\
 & \quad + \frac{1}{\mu_m^\gamma} \sum_2 \left[N \left(\frac{d(t_{kn}(X), X_0)}{s} \right) \right] \\
 &\leq \left(\frac{\mu_m}{\lambda_m^\beta} - 1 \right) B \\
 & \quad + \frac{B}{\lambda_m^\beta} \left| \left\{ k \in I_m : \left[N \left(\frac{d(t_{kn}(X), X_0)}{s} \right) \right] \geq \varepsilon \right\} \right| \\
 & \quad + \frac{\mu_m}{\lambda_m^\gamma} \varepsilon
 \end{aligned} \tag{19}$$

for all $n \in \mathbb{N}$. Hence we obtain $X \in \widehat{w}_\gamma^\mathcal{F}(N, \mu)$. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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