Research Article

Sharp Bounds for Toader Mean in terms of Arithmetic and Second Contraharmonic Means

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We present the best possible parameters $\lambda_1, \mu_1 \in \mathbb{R}$ and $\lambda_2, \mu_2 \in (1/2, 1)$ such that double inequalities $\lambda_1 C(a, b) + (1 - \lambda_1)A(a, b) < T(a, b) < \mu_1 C(a, b) + (1 - \mu_1)A(a, b)$, $C[\lambda_2 a + (1 - \lambda_2)b, \lambda_2 b + (1 - \lambda_2)a] < T(a, b) < C[\mu_2 a + (1 - \mu_2)b, \mu_2 b + (1 - \mu_2)a]$ hold for all $a, b > 0$ with $a \neq b$, where $A(a, b) = (a + b)/2$, $C(a, b) = (a^3 + b^3)/(a^2 + b^2)$ and $T(a, b) = 2 \int_0^{\pi/2} \sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta} d\theta / \pi$ are the arithmetic, second contraharmonic, and Toader means of a and b , respectively.

1. Introduction

For $a, b > 0$ the Toader mean $T(a, b)$ [1], second contraharmonic mean $C(a, b)$, and arithmetic mean $A(a, b)$ of a and b are given by

$$T(a, b) = \frac{2}{\pi} \int_0^{\pi/2} \sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta} d\theta$$

$$= \begin{cases} \frac{2a\mathcal{E}(\sqrt{1 - (b/a)^2})}{\pi}, & a > b, \\ \frac{2b\mathcal{E}(\sqrt{1 - (a/b)^2})}{\pi}, & a < b, \\ a, & a = b, \end{cases} \quad (1)$$

$$C(a, b) = \frac{a^3 + b^3}{a^2 + b^2},$$

$$A(a, b) = \frac{a + b}{2},$$

(2)

respectively, where $\mathcal{E}(r) = \int_0^{\pi/2} (1 - r^2 \sin^2 t)^{1/2} dt$ ($r \in (0, 1)$) is the complete elliptic integral of the second kind. The Toader

mean $T(a, b)$ is well known in mathematical literature for many years; it satisfies

$$T(a, b) = R_E(a^2, b^2),$$

$$T(1, r) = \frac{2}{\pi} \mathcal{E}(\sqrt{1 - r^2}) \quad (3)$$

for all $a, b > 0$ and $0 < r < 1$, where

$$R_E(a, b) = \frac{1}{\pi} \int_0^\infty \frac{[a(t + b) + b(t + a)]t}{(t + a)^{3/2}(t + b)^{3/2}} dt \quad (4)$$

stands for the symmetric complete elliptic integral of the second kind (see [2–4]); therefore it cannot be expressed in terms of the elementary transcendental functions.

Recently, the Toader mean $T(a, b)$ has been the subject of intensive research. In particular, many remarkable inequalities for the Toader mean can be found in the literature [5–9].

Let $p \in \mathbb{R}$, $q \in [0, 1]$, and $a, b > 0$. Then the p th power mean $M_p(a, b)$, p th Gini mean $G_p(a, b)$, p th Lehmer

mean $L_p(a, b)$, and q th generalized Seiffert mean $S_q(a, b)$ are defined by

$$\begin{aligned} M_p(a, b) &= \begin{cases} \left(\frac{a^p + b^p}{2} \right)^{1/p}, & p \neq 0, \\ \sqrt{ab}, & p = 0, \end{cases} \\ G_p(a, b) &= \begin{cases} \left(\frac{a^{p-1} + b^{p-1}}{a + b} \right)^{1/(p-2)}, & p \neq 2, \\ (a^a b^b)^{1/(a+b)}, & p = 2, \end{cases} \\ L_p(a, b) &= \frac{a^{p+1} + b^{p+1}}{a^p + b^p}, \\ S_q(a, b) &= \begin{cases} \frac{p(a-b)}{\arctan[2p(a-b)/(a+b)]}, & 0 < p \leq 1, a \neq b, \\ \frac{a+b}{2}, & p = 0, a \neq b, \\ a, & a = b, \end{cases} \end{aligned} \quad (5)$$

respectively. It is well known that $M_p(a, b)$, $G_p(a, b)$, $L_p(a, b)$, and $S_q(a, b)$ are continuous and strictly increasing with respect to $p \in \mathbb{R}$ and $q \in [0, 1]$ for fixed $a, b > 0$ with $a \neq b$, respectively.

Vuorinen [10] conjectured that inequality

$$M_{3/2}(a, b) < T(a, b) \quad (6)$$

holds for all $a, b > 0$ with $a \neq b$. This conjecture was proved by Qiu and Shen [11] and Barnard et al. [12], respectively.

Alzer and Qiu [13] presented a best possible upper power mean bound for the Toader mean as follows:

$$T(a, b) < M_{\log 2 / (\log \pi - \log 2)}(a, b) \quad (7)$$

for all $a, b > 0$ with $a \neq b$.

In [14, 15], the authors found the best possible parameters $\alpha, \beta \in [0, 1]$ and $\lambda, \mu \in \mathbb{R}$ such that double inequalities $S_\alpha(a, b) < T(a, b) < S_\beta(a, b)$ and $G_\lambda(a, b) < T(a, b) < G_\mu(a, b)$ hold for all $a, b > 0$ with $a \neq b$.

Chu and Wang [16] proved that double inequality

$$L_p(a, b) < T(a, b) < L_q(a, b) \quad (8)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $p \leq 0$ and $q \geq 1/4$.

Inequality (8) leads to

$$\begin{aligned} A(a, b) = L_0(a, b) &< T(a, b) < L_{1/4}(a, b) < L_2(a, b) \\ &= C(a, b) \end{aligned} \quad (9)$$

for all $a, b > 0$ with $a \neq b$.

Let $a, b > 0$ with $a \neq b$ be fixed and $f(x) = C[xa + (1-x)b, xb + (1-x)a]$. Then it is not difficult to verify that $f(x)$ is continuous and strictly increasing on $[1/2, 1]$. Note that

$$f\left(\frac{1}{2}\right) = A(a, b) < T(a, b) < C(a, b) = f(1). \quad (10)$$

Motivated by inequalities (9) and (10), it is natural to ask what are the best possible parameters, $\lambda_1, \mu_1 \in \mathbb{R}$ and $\lambda_2, \mu_2 \in (1/2, 1)$, such that double inequalities

$$\begin{aligned} \lambda_1 C(a, b) + (1 - \lambda_1) A(a, b) &< T(a, b) \\ &< \mu_1 C(a, b) + (1 - \mu_1) A(a, b), \\ C[\lambda_2 a + (1 - \lambda_2)b, \lambda_2 b + (1 - \lambda_2)a] &< T(a, b) \\ &< C[\mu_2 a + (1 - \mu_2)b, \mu_2 b + (1 - \mu_2)a] \end{aligned} \quad (11)$$

hold for all $a, b > 0$ with $a \neq b$? The main purpose of this paper is to answer this question.

2. Main Results

In order to prove our main results we need some basic knowledge and two lemmas, which we present in this section.

For $r \in (0, 1)$ the complete elliptic integral $\mathcal{K}(r)$ of the first kind is defined by

$$\mathcal{K}(r) = \int_0^{\pi/2} (1 - r^2 \sin^2 t)^{-1/2} dt. \quad (12)$$

We clearly see that

$$\mathcal{K}(0^+) = \mathcal{E}(0^+) = \frac{\pi}{2}, \quad (13)$$

$$\mathcal{E}(1^-) = 1,$$

and $\mathcal{K}(r)$ and $\mathcal{E}(r)$ satisfy formulas (see [17, Appendix E, p. 474-475])

$$\begin{aligned} \frac{d\mathcal{K}(r)}{dr} &= \frac{\mathcal{E}(r) - (1 - r^2)\mathcal{K}(r)}{r(1 - r^2)}, \\ \frac{d\mathcal{E}(r)}{dr} &= \frac{\mathcal{E}(r) - \mathcal{K}(r)}{r}, \\ \mathcal{E}\left(\frac{2\sqrt{r}}{1+r}\right) &= \frac{2\mathcal{E}(r) - (1 - r^2)\mathcal{K}(r)}{1+r}. \end{aligned} \quad (14)$$

Lemma 1 (see [17, Theorem 1.25]). *Let $-\infty < a < b < \infty$, $f, g : [a, b] \rightarrow (-\infty, \infty)$ be continuous on $[a, b]$ and differentiable on (a, b) , and $g'(x) \neq 0$ on (a, b) . If $f'(x)/g'(x)$ is increasing (decreasing) on (a, b) , then so are*

$$\begin{aligned} \frac{f(x) - f(a)}{g(x) - g(a)}, \\ \frac{f(x) - f(b)}{g(x) - g(b)}. \end{aligned} \quad (15)$$

If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

Lemma 2 (see [17, Theorem 3.21]). (1) *Function $r \mapsto [\mathcal{E}(r) - (1 - r^2)\mathcal{K}(r)]/r^2$ is strictly increasing from $(0, 1)$ to $(\pi/4, 1)$.*

(2) *Function $r \mapsto (1 - r^2)^\lambda \mathcal{K}(r)$ is strictly decreasing from $(0, 1)$ to $(0, \pi/2)$ if $\lambda \geq 1/4$.*

Theorem 3. *Double inequality*

$$\begin{aligned} \lambda_1 C(a, b) + (1 - \lambda_1) A(a, b) &< T(a, b) \\ &< \mu_1 C(a, b) + (1 - \mu_1) A(a, b) \end{aligned} \quad (16)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\lambda_1 \leq 1/8$ and $\mu_1 \geq 4/\pi - 1 = 0.2732 \dots$

Proof. Since $A(a, b)$, $T(a, b)$, and $C(a, b)$ are symmetric and homogeneous of degree 1, without loss of generality, we assume that $a > b > 0$. Let $r = (a - b)/(a + b) \in (0, 1)$. Then (1) and (2) lead to

$$\begin{aligned} T(a, b) &= \frac{2a}{\pi} \mathcal{E} \left(\frac{2\sqrt{r}}{1+r} \right) \\ &= \frac{2A(a, b)}{\pi} [2\mathcal{E}(r) - (1 - r^2) \mathcal{K}(r)], \end{aligned} \quad (17)$$

$$C(a, b) = A(a, b) \frac{1 + 3r^2}{1 + r^2}. \quad (18)$$

We clearly see that inequality (16) is equivalent to

$$\lambda_1 < \frac{T(a, b) - A(a, b)}{C(a, b) - A(a, b)} < \mu_1. \quad (19)$$

It follows from (17) and (18) that

$$\begin{aligned} \frac{T(a, b) - A(a, b)}{C(a, b) - A(a, b)} &= \frac{1}{\pi} \frac{2\mathcal{E}(r) - (1 - r^2) \mathcal{K}(r) - \pi/2}{r^2/(1 + r^2)}. \end{aligned} \quad (20)$$

Let

$$\begin{aligned} f(r) &= \frac{2\mathcal{E}(r) - (1 - r^2) \mathcal{K}(r) - \pi/2}{r^2/(1 + r^2)}, \\ f_1(r) &= 2\mathcal{E}(r) - (1 - r^2) \mathcal{K}(r) - \frac{\pi}{2}, \\ f_2(r) &= \frac{r^2}{1 + r^2}. \end{aligned} \quad (21)$$

Then simple computations lead to

$$\begin{aligned} f_1(0^+) &= f_2(0^+) = 0, \\ \frac{f'_1(r)}{f'_2(r)} &= \frac{(1 + r^2)^2 \mathcal{E}(r) - (1 - r^2) \mathcal{K}(r)}{2r^2}. \end{aligned} \quad (22)$$

From Lemmas 1 and 2 together with (21) and (22) we know that $f(r)$ is strictly increasing on $(0, 1)$ and

$$\begin{aligned} f(0^+) &= \frac{f'_1(0^+)}{f'_2(0^+)} = \frac{\pi}{8}, \\ f(1^-) &= 4 - \pi. \end{aligned} \quad (23)$$

Therefore, Theorem 3 follows from (19)–(21) and (23) together with the monotonicity of $f(r)$. $\square$

Theorem 4. *Let $\lambda_2, \mu_2 \in (1/2, 1)$. Then double inequality*

$$\begin{aligned} C[\lambda_2 a + (1 - \lambda_2) b, \lambda_2 b + (1 - \lambda_2) a] &< T(a, b) \\ &< C[\mu_2 a + (1 - \mu_2) b, \mu_2 b + (1 - \mu_2) a] \end{aligned} \quad (24)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\lambda_2 \leq 1/2 + \sqrt{2}/8 = 0.6767 \dots$ and $\mu_2 \geq 1/2 + \sqrt{(4 - \pi)/(3\pi - 4)}/2 = 0.6988 \dots$

Proof. Let $\lambda_2^* = 1/2 + \sqrt{2}/8$ and $\mu_2^* = 1/2 + \sqrt{(4 - \pi)/(3\pi - 4)}/2$. We first prove that

$$T(a, b) > C[\lambda_2^* a + (1 - \lambda_2^*) b, \lambda_2^* b + (1 - \lambda_2^*) a], \quad (25)$$

$$T(a, b) < C[\mu_2^* a + (1 - \mu_2^*) b, \mu_2^* b + (1 - \mu_2^*) a] \quad (26)$$

for all $a, b > 0$ with $a \neq b$.

Without loss of generality, we assume that $a > b$. Let $r = (a - b)/(a + b) \in (0, 1)$ and $p \in (1/2, 1)$. Then (2) leads to

$$\begin{aligned} C[pa + (1 - p)b, pb + (1 - p)a] \\ = A(a, b) \frac{3(2p - 1)^2 r^2 + 1}{(2p - 1)^2 r^2 + 1}. \end{aligned} \quad (27)$$

It follows from (17) and (27) that

$$\begin{aligned} T(a, b) - C[pa + (1 - p)b, pb + (1 - p)a] \\ = A(a, b) \left[\frac{2}{\pi} (2\mathcal{E}(r) - (1 - r^2) \mathcal{K}(r)) \right. \\ \left. + \frac{2}{(2p - 1)^2 r^2 + 1} - 3 \right]. \end{aligned} \quad (28)$$

Let

$$\begin{aligned} g(r) &= \frac{2}{\pi} (2\mathcal{E}(r) - (1 - r^2) \mathcal{K}(r)) \\ &\quad + \frac{2}{(2p - 1)^2 r^2 + 1} - 3, \end{aligned} \quad (29)$$

$$g_1(r) = \frac{1}{r} g'(r). \quad (30)$$

Then making use of Lemma 2 and simple computations lead to

$$g(0) = 0, \quad (31)$$

$$g(1) = \frac{4}{\pi} + \frac{2}{(2p-1)^2 + 1} - 3, \quad (32)$$

$$g_1(r) = \frac{2}{\pi} \frac{\mathcal{E}(r) - (1-r^2)\mathcal{K}(r)}{r^2} - \frac{4(2p-1)^2}{[(2p-1)^2 r^2 + 1]^2}, \quad (33)$$

$$g_1(0) = \frac{1}{2} - 4(2p-1)^2, \quad (34)$$

$$g_1(1) = \frac{2}{\pi} - \frac{4(2p-1)^2}{[(2p-1)^2 + 1]^2}. \quad (35)$$

We divide the proof into two cases.

Case 1. Consider $p = \lambda_2^* = 1/2 + \sqrt{2}/8$. Then (34) becomes

$$g_1(0) = 0. \quad (36)$$

It follows from Lemma 2(1) and (33) together with (36) that

$$g_1(r) > 0 \quad (37)$$

for all $r \in (0, 1)$.

Therefore, inequality (25) follows easily from (28)–(31) and (37).

Case 2. Consider $p = \mu_2^* = 1/2 + \sqrt{(4-\pi)/(3\pi-4)}/2$. Then (32), (34), and (35) lead to

$$g(1) = 0, \quad (38)$$

$$g_1(0) = -\frac{36-11\pi}{6\pi-8} < 0, \quad (39)$$

$$g_1(1) = \frac{3\pi^2 - 14\pi + 16}{\pi^2} > 0. \quad (40)$$

It follows from Lemma 2(1), (33), (39), and (40) that there exists $r_0 \in (0, 1)$ such that $g_1(r) < 0$ for $r \in (0, r_0)$ and $g_1(r) > 0$ for $r \in (r_0, 1)$. Then (30) leads to the conclusion that $g(r)$ is strictly decreasing on $(0, r_0]$ and strictly increasing on $(r_0, 1]$.

Therefore, inequality (26) follows easily from (28), (29), (31), (38), and the piecewise monotonicity of $g(r)$.

Next, we prove that $\lambda_2 = \lambda_2^* = 1/2 + \sqrt{2}/8$ is the best possible parameter on $(1/2, 1)$ such that inequality

$$T(a, b) > C[\lambda_2 a + (1-\lambda_2)b, \lambda_2 b + (1-\lambda_2)a] \quad (41)$$

holds for all $a, b > 0$ with $a \neq b$.

Indeed, if $\lambda_2^* = 1/2 + \sqrt{2}/8 < p < 1$, then (34) leads to $g_1(0) < 0$ and there exists $\delta_1 \in (0, 1)$ such that

$$g_1(r) < 0 \quad (42)$$

for $r \in (0, \delta_1)$.

Equations (28)–(31) and inequality (42) imply that

$$T(a, b) < C[pa + (1-p)b, pb + (1-p)a] \quad (43)$$

for $(a-b)/(a+b) \in (0, \delta_1)$.

Finally, we prove that $\mu_2 = \mu_2^* = 1/2 + \sqrt{(4-\pi)/(3\pi-4)}/2$ is the best possible parameter on $(1/2, 1)$ such that double inequality

$$T(a, b) < C[\mu_2 a + (1-\mu_2)b, \mu_2 b + (1-\mu_2)a] \quad (44)$$

for all $a, b > 0$ with $a \neq b$.

In fact, if $1/2 < p < \mu_2^* = 1/2 + \sqrt{(4-\pi)/(3\pi-4)}/2$, then (32) leads to $g(1) > 0$ and there exists $\delta_2 \in (0, 1)$ such that

$$g(r) > 0 \quad (45)$$

for $r \in (1-\delta_2, 1)$.

Equations (28) and (29) together with inequality (45) imply that

$$T(a, b) > C[pa + (1-p)b, pb + (1-p)a] \quad (46)$$

for $(a-b)/(a+b) \in (1-\delta_2, 1)$. $\square$

Let $r \in (0, 1)$, $a = 1$, $b = \sqrt{1-r^2}$, $\lambda_1 = 1/8$, $\mu_1 = 4/\pi - 1$, $\lambda_2 = 1/2 + \sqrt{2}/8$, and $\mu_2 = 1/2 + \sqrt{(4-\pi)/(3\pi-4)}/2$. Then Theorems 3 and 4 lead to Corollary 5 as follows.

Corollary 5. *Double inequalities*

$$\begin{aligned} & \frac{\pi \left[1 + (1-r^2)^{3/2} \right]}{16(2-r^2)} + \frac{7\pi \left[1 + (1-r^2)^{1/2} \right]}{32} < \mathcal{E}(r) \\ & < \frac{(4-\pi) \left[1 + (1-r^2)^{3/2} \right]}{2(2-r^2)} + \left(\frac{\pi}{2} - 1 \right) \left[1 + (1-r^2)^{1/2} \right], \\ & \frac{\pi}{2} \frac{32 - 21r^2 + 21(1-r^2)^{1/2} + 11(1-r^2)^{3/2}}{36 - 18r^2 + 28(1-r^2)^{1/2}} < \mathcal{E}(r) < \frac{\pi}{2} \\ & \cdot \frac{3\pi - 4 - 3(\pi-2)r^2 + 3(\pi-2)(1-r^2)^{1/2} + 2(1-r^2)^{3/2}}{\pi(2-r^2) + 4(\pi-2)(1-r^2)^{1/2}} \end{aligned} \quad (47)$$

hold for all $r \in (0, 1)$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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