Research Article

Sandor Type Inequalities for Sugeno Integral with respect to General (α, m, r) -Convex Functions

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The concept for general (α, m, r) -convex functions, as a generalization of convex functions, is introduced. Then, Sandor type inequalities for the Sugeno integral based on this kind of function are established. Our work generalizes the previous results in the literature. Finally, some conclusions and problems for further investigations are given.

1. Introduction

As a tool for modeling nondeterministic problems, fuzzy measures and fuzzy integrals introduced by Sugeno in [1] have been successfully applied to various fields. The fuzzy integrals provide a useful tool in engineering and social choice, where the aggregation of data is required. However, they are restricted for the special operators used in the construction of these integrals. Thus, many researchers have generalized the Sugeno integral by using some other operators to replace the special operator(s) $\vee$ and/or $\wedge$. They obtained Choquet-like integral [2], Shilkret integral [3], $\perp$ -integral [4], generalized fuzzy integral [5], Sugeno-like integral [6], λ -generalized Sugeno integral [7], pseudointegral [8], interval-valued generalized fuzzy integral [9], and set-valued pseudointegral [10]. García and Álvarez [11] presented two families of fuzzy integrals, the so-called seminormed fuzzy integrals and semiconormed fuzzy integrals. Klement et al. [12] provided a concept of universal integrals generalizing both the Choquet integral and the Sugeno integral. Wang and Klir [13] provided a general overview on fuzzy measurement and fuzzy integration.

The integral inequalities are significant mathematical tools both in theory and in application. Different integral inequalities including Chebyshev inequality, Jensen inequality, Hölder inequality, and Minkowski inequality are widely used in various fields of mathematics, such as probability theory, differential equations, decision-making under risk, forecasting of time series, and information sciences.

Convexity is one of the most powerful tools in establishing analytic inequalities. In particular, there are many important applications in the theory of higher transcendental functions. However, for many problems encountered in economics and engineering, convexity is unsuitable. Hence, it is natural to generalize convexity. Hanson [14] gave the notion of invexity as significant generalization of classical convexity. Ben-Israel and Mond [15] studied the preinvex functions, which are a special case of invex functions. Breckner [16] introduced the s -convex functions and Varošanec [17] presented the h -convex functions as a generalization of convex functions. In [18], Miheșan proposed the definition of (α, m) -convex functions. For recent results and generalizations concerning m -convex and (α, m) -convex functions, see [19, 20]. Latif and Shoaib [21] discussed m -preinvex functions and (α, m) -preinvex functions. Gill et al. [22] provided the concept of r -mean convex functions.

On the other hand, some scholars have shown that several integral inequalities hold both in the classical context and in the fuzzy context. Román-Flores et al. investigated several kinds of fuzzy integral inequalities including Chebyshev type inequality [23], Young type inequality [24], Jensen type inequality [25], Hardy type inequality [26], Convolution type inequality [27], Stolarsky type inequality [28], and Markov type inequality [29]. Agahi et al. proved general Chebyshev type inequality [30], Hölder type inequality [31], Berwald type inequality [32], general Minkowski type inequality [33], and general Barnes-Godun-Levin type inequality [34] for the Sugeno integral. Caballero and Sadarangani presented

Cauchy-Schwarz type inequality [35], Chebyshev type inequality [36], Fritz Carlson type inequality [37], and Sandor type inequality [38] for Sugeno integral. Mesiar and Ouyang proposed Chebyshev type inequality [39], Yong type inequality [40], general Chebyshev type inequality [41], and Minkowski type inequality [42] for Sugeno integral.

Caballero and Sadarangani [38] illustrated a Sandor type inequality of fuzzy integrals for convex function. The purpose of this paper is to investigate Sandor type inequalities for Sugeno integral related to the general (α, m, r) -convexity, which generalize the previous results in the literature. Some examples are given to illustrate the results.

After some preliminaries and summarization of some previous known results in Section 2, Section 3 deals with general Sandor inequalities for Sugeno integral based on general (α, m, r) -convex functions and some examples are given to illustrate the results. Section 4 focuses on Sandor inequalities for Sugeno integral based on general (α, m, r) -concave functions. Finally, our results are applied to some special cases.

2. Preliminaries

In this section, we recall some basic definitions and properties of the fuzzy integral and introduce the notion of general (α, m, r) -convex functions. For details, we refer the reader to [1, 13].

Definition 1 (see [18]). Let $I \subseteq \mathbb{R}$ be an interval, $\lambda \in [0, 1]$. A function $f : I \rightarrow \mathbb{R}$ is said to be convex on I if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y) \quad (1)$$

for all $x, y \in I$. If the above inequalities reverse, then we say that the function f is concave on I .

Let X be a nonempty set and let $P(X) = \{A \mid A : X \rightarrow [0, 1]\}$ be the class of all subsets of X .

Definition 2 (see [13]). Let σ -algebra $\wp$ be a nonempty subclass of $P(X)$ with the following properties:

- (1) $X, \emptyset \in \wp$.
- (2) If $A \in \wp$, then $A^c \in \wp$.
- (3) If $\{A_n\} \in \wp$, then $\bigcup_{n=1}^{\infty} A_n \in \wp$.

Let $(X, \wp, \mu)$ be a fuzzy measure space, where X is a nonempty set. Let $\wp$ be a σ -algebra of subsets of X and let $\mu : \wp \rightarrow [0, \infty)$ be a nonnegative, extended real-valued set function. We say that μ is a fuzzy measure if it satisfies the following:

- (1) $\mu(\emptyset) = 0$.
- (2) $E, F \in \wp$ and $E \subset F$ imply $\mu(E) \leq \mu(F)$.
- (3) $\{E_n\} \subset \wp, E_1 \subset E_2 \subset \dots$, imply $\lim_{n \rightarrow \infty} \mu(E_n) = \mu(\bigcup_{n=1}^{\infty} E_n)$.
- (4) $\{E_n\} \subset \wp, E_1 \supset E_2 \supset \dots, \mu(E_1) < \infty$, imply $\lim_{n \rightarrow \infty} \mu(E_n) = \mu(\bigcap_{n=1}^{\infty} E_n)$.

If f is a nonnegative real-valued function defined on X , we denote the set $\{x \in X : f(x) \geq \alpha\} = \{x \in X : f \geq \alpha\}$ by F_α for $\alpha \geq 0$. Note that if $\alpha \leq \beta$, then $F_\beta \subset F_\alpha$.

Let $(X, \wp, \mu)$ be a fuzzy measure space; we denote by M^+ the set of all nonnegative measurable functions with respect to $\wp$.

Definition 3 (Sugeno [1]). Let $(X, \wp, \mu)$ be a fuzzy measure space, $f \in M^+$, and $A \in \wp$; the Sugeno integral (or the fuzzy integral) of f on A , with respect to the fuzzy measure μ , is defined as

$$(S) \int_A f d\mu = \bigvee_{\alpha \geq 0} [\alpha \wedge \mu(A \cap F_\alpha)]; \quad (2)$$

when $A = X$,

$$(S) \int_X f d\mu = \bigvee_{\alpha \geq 0} [\alpha \wedge \mu(F_\alpha)], \quad (3)$$

where $\bigvee$ and $\wedge$ denote the operations sup and inf on $[0, \infty)$, respectively.

The properties of fuzzy integral are well known and can be found in [13].

Proposition 4. Let $(X, \wp, \mu)$ be a fuzzy measure space, $A, B \in \wp$, and $f, g \in M^+$; then one obtains the following:

- (1) $(S) \int_A f d\mu \leq \mu(A)$.
- (2) $(S) \int_A k d\mu = k \wedge \mu(A)$, k for nonnegative constant.
- (3) $(S) \int_A f d\mu \leq (S) \int_A g d\mu$, for $f \leq g$.
- (4) $\mu(A \cap \{f \geq \alpha\}) \geq \alpha \Rightarrow (S) \int_A f d\mu \geq \alpha$.
- (5) $\mu(A \cap \{f \geq \alpha\}) \leq \alpha \Rightarrow (S) \int_A f d\mu \leq \alpha$.
- (6) $(S) \int_A f d\mu > \alpha \Leftrightarrow$ there exists $\gamma > \alpha$ such that $\mu(A \cap \{f \geq \gamma\}) > \alpha$.
- (7) $(S) \int_A f d\mu < \alpha \Leftrightarrow$ there exists $\gamma < \alpha$ such that $\mu(A \cap \{f \geq \gamma\}) < \alpha$.

Remark 5. Consider the distribution function F associated with f on A ; that is, $F(\alpha) = \mu(A \cap \{f \geq \alpha\})$. Then, due to (4) and (5) of Proposition 4, we have $F(\alpha) = \alpha \Rightarrow (S) \int_A f d\mu = \alpha$. Thus, from a numerical point of view, the fuzzy integral can be calculated solving the equation $F(\alpha) = \alpha$.

Definition 6. Let $I \subseteq \mathbb{R}$ be an interval, $\lambda, \alpha, m \in [0, 1]$, $r \in \mathbb{R}$. Let g be a continuous and monotonous function on $\mathbb{R}$. A function $f : I \rightarrow \mathbb{R}$ is said to be general (α, m, r) -convex on I if

$$f\left([\lambda x^r + m(1 - \lambda)y^r]^{1/r}\right) \leq g^{-1}\left(\left[\lambda^\alpha (g \circ f)^r(x) + m(1 - \lambda^\alpha)(g \circ f)^r(y)\right]^{1/r}\right), \quad r \neq 0, \quad (4)$$

or

$$f\left(x^\lambda y^{m(1-\lambda)}\right) \leq g^{-1}\left((g \circ f)^{\lambda^\alpha}(x)(g \circ f)^{m(1-\lambda^\alpha)}(y)\right), \quad r = 0, \quad (5)$$

for all $x, y \in I$. If the above inequalities reverse, then we say that the function f is a general (α, m, r) -concave function on I .

Remark 7. If, in Definition 6, $g = id$ (i.e., $g(x) = x$ for any $x \in I$), then one obtains the definition of (α, m, r) -convexity.

If, in Definition 6, $\alpha, m = 1$, then one obtains the definition of general r -mean convexity.

If, in Definition 6, $\alpha, m = 1$ and $g = id$, then one obtains the definition of r -mean convexity [43].

If in Definition 6, $r = 1$, then one obtain the definition of general (α, m) -convexity.

If, in Definition 6, $r = 1$ and $g = id$, then one obtains the definition of (α, m) -convexity [18].

If $(\alpha, m, r) \in \{(0, 0, 1), (\alpha, 0, 1), (1, 0, 1), (1, m, 1), (1, 1, 1), (\alpha, 1, 1)\}$ and $g = id$ in Definition 6, one obtains the following classes of functions: increasing, α -starshaped, starshaped, m -convex, convex, and α -convex, respectively.

3. Sandor Type Inequalities for Fuzzy Integrals Based on General (α, m, r) -Convex Functions

The classical Sandor type inequality provides estimates of the mean value of a nonnegative and convex function: $f : [a, b] \rightarrow \mathbb{R}$; then,

$$\frac{1}{b-a} \int_a^b f^2(x) dx \leq \frac{f^2(a) + f(a)f(b) + f^2(b)}{3}. \quad (6)$$

Unfortunately, the following example shows that Sandor type inequality for fuzzy integral based on general (α, m, r) -convex functions is not valid.

Example 8. Consider $X = [0, 1]$ and let μ be the Lebesgue measure on X . If we take the function $f(x) = (1/4)x^2$ and $g(x) = \sqrt{x}$, then $f(x)$ is a general $(1/2, 1/3, 2)$ -convex function. In fact,

$$\begin{aligned} \frac{1}{4}x^2 &= f\left(\left[x^2 \cdot 1^2 + \frac{1}{3}(1-x^2) \cdot 0^2\right]^{1/2}\right) \\ &\leq \left(\left[\frac{1}{4}x + \frac{1}{3}(1-x) \cdot 0\right]^{1/2}\right)^2 = \frac{1}{4}x \end{aligned} \quad (7)$$

for $x \in [0, 1]$. Straightforward calculus shows that

$$(S) \int_0^1 f^2(x) d\mu = 0.0557. \quad (8)$$

On the other hand,

$$\frac{f^2(0) + f(0)f(1) + f^2(1)}{3} = 0.0208. \quad (9)$$

This proves that the Sandor type inequality is not satisfied for Sugeno integral based on general (α, m, r) -convex functions.

In this section, we will show general Sandor type inequalities for the Sugeno integral based on general (α, m, r) -convex functions.

Theorem 9. Let $(\alpha, m) \in (0, 1]^2$, $r \in \mathbb{R}$, and $r \neq 0$, let g be a continuous and monotonous function, let $f : [0, 1] \rightarrow [0, \infty)$ be a general (α, m, r) -convex function, and let μ be the Lebesgue measure on $\mathbb{R}$. Then, one has the following.

Case 1. If $(g \circ f)^r(1) - m(g \circ f)^r(0) > 0$, then

$$(S) \int_0^1 f^2(x) d\mu \leq \min\{\beta, 1\}, \quad (10)$$

where β satisfies the following equation:

$$\begin{aligned} &((g \circ f)^r(1) - m(g \circ f)^r(0))(1 - \beta)^{\alpha r} - g^r(\sqrt{\beta}) \\ &+ m(g \circ f)^r(0) = 0. \end{aligned} \quad (11)$$

Case 2. If $(g \circ f)^r(1) - m(g \circ f)^r(0) = 0$, then

$$(S) \int_0^1 f^2(x) d\mu \leq \min\{f^2(0) \sqrt[r]{m^2}, 1\}. \quad (12)$$

Case 3. If $(g \circ f)^r(1) - m(g \circ f)^r(0) < 0$, then

$$(S) \int_0^1 f^2(x) d\mu \leq \min\{\beta, 1\}, \quad (13)$$

where β satisfies the following equation:

$$\begin{aligned} &((g \circ f)^r(1) - m(g \circ f)^r(0))\beta^{\alpha r} - g^r(\sqrt{\beta}) \\ &+ m(g \circ f)^r(0) = 0. \end{aligned} \quad (14)$$

Proof. As f is a general (α, m, r) -convex function for $x \in [0, 1]$, we have

$$\begin{aligned} f(x) &= f\left(\left[x^r \cdot 1^r + m(1-x^r) \cdot 0^r\right]^{1/r}\right) \\ &\leq g^{-1}\left(\left[x^{\alpha r}(g \circ f)^r(1) \right. \right. \\ &\quad \left. \left. + m(1-x^{\alpha r})(g \circ f)^r(0)\right]^{1/r}\right) = h(x). \end{aligned} \quad (15)$$

By (3) of Proposition 4, we have

$$(S) \int_0^1 f^2(x) d\mu = (S) \int_0^1 f^2 \left([x^r \cdot 1^r + m(1-x^r) \cdot 0^r]^{1/r} \right) d\mu \leq (S) \int_0^1 g^{-2} \left([x^{\alpha r} (g \circ f)^r(1) \right.$$

$$\left. + m(1-x^{\alpha r}) (g \circ f)^r(0) \right]^{1/r} \right) d\mu = (S)$$

$$\cdot \int_0^1 h^2(x) d\mu.$$

(16)

In order to calculate the right hand side of the last inequality, we consider the distribution function F given by

$$\begin{aligned} F(\beta) &= \mu([0, 1] \cap \{h^2 \geq \beta\}) = \mu([0, 1] \cap \{h \geq \sqrt{\beta}\}) \\ &= \mu\left([0, 1] \cap \left\{x \mid ((g \circ f)^r(1) - m(g \circ f)^r(0)) x^{\alpha r} \geq g^r(\sqrt{\beta}) - m(g \circ f)^r(0)\right\}\right) \\ &= \begin{cases} 1 - \left(\frac{g^r(\sqrt{\beta}) - m(g \circ f)^r(0)}{(g \circ f)^r(1) - m(g \circ f)^r(0)}\right)^{1/\alpha r}, & (g \circ f)^r(1) - m(g \circ f)^r(0) > 0, \\ f^2(0) \sqrt[r]{m^2}, & (g \circ f)^r(1) - m(g \circ f)^r(0) = 0, \\ \left(\frac{g^r(\sqrt{\beta}) - m(g \circ f)^r(0)}{(g \circ f)^r(1) - m(g \circ f)^r(0)}\right)^{1/\alpha r}, & (g \circ f)^r(1) - m(g \circ f)^r(0) < 0 \end{cases} \end{aligned} \quad (17)$$

and the solution of the equation $F(\beta) = \beta$. By (1) of Proposition 4 and Remark 5, we get the following.

Case 1. If $(g \circ f)^r(1) - m(g \circ f)^r(0) > 0$, then

$$(S) \int_0^1 f^2(x) d\mu \leq \min\{\beta, 1\}, \quad (18)$$

where β satisfies the following equation:

$$\begin{aligned} &((g \circ f)^r(1) - m(g \circ f)^r(0))(1 - \beta)^{\alpha r} - g^r(\sqrt{\beta}) \\ &+ m(g \circ f)^r(0) = 0. \end{aligned} \quad (19)$$

Case 2. If $(g \circ f)^r(1) - m(g \circ f)^r(0) = 0$, then

$$(S) \int_0^1 f^2(x) d\mu \leq \min\{f^2(0) \sqrt[r]{m^2}, 1\}. \quad (20)$$

Case 3. If $(g \circ f)^r(1) - m(g \circ f)^r(0) < 0$, then

$$(S) \int_0^1 f^2(x) d\mu \leq \min\{\beta, 1\}, \quad (21)$$

where β satisfies the following equation:

$$\begin{aligned} &((g \circ f)^r(1) - m(g \circ f)^r(0))\beta^{\alpha r} - g^r(\sqrt{\beta}) \\ &+ m(g \circ f)^r(0) = 0. \end{aligned} \quad (22)$$

This completes the proof. $\square$

Remark 10. If $\alpha = 0$ in Theorem 9, then $(S) \int_0^1 f^2(x) d\mu \leq \min\{f^2(1), 1\}$.

Example 11. Consider $X = [0, 1]$ and let μ be the Lebesgue measure on X . If we take the function $f(x) = x^2$ and $g(x) = \sqrt{x}$, then $f(x)$ is a general $(1/2, 1/3, 2)$ -convex function. In fact,

$$\begin{aligned} x^2 &= f\left([x^2 \cdot 1^2 + \frac{1}{3}(1-x^2) \cdot 0^2]^{1/2}\right) \\ &\leq \left(\left[x + \frac{1}{3}(1-x) \cdot 0\right]^{1/2}\right)^2 = x. \end{aligned} \quad (23)$$

By Theorem 9, we have

$$0.2755 = (S) \int_0^1 x^4 d\mu \leq \min\{0.3820, 1\} = 0.3820. \quad (24)$$

Now, we will prove the general case of Theorem 9.

Theorem 12. Let $(\alpha, m) \in (0, 1]^2$, $r \in \mathbb{R}$, and $r \neq 0$, let g be a continuous and monotonous function, let $f: [a, b] \rightarrow [0, \infty)$ be a general (α, m, r) -convex function, and let μ be the Lebesgue measure on $\mathbb{R}$. Then, one obtains the following.

Case 1. If $(g \circ f)^r(b) - m(g \circ f)^r(a) > 0$, then

$$(S) \int_a^b f^2(x) d\mu \leq \min\{\beta, b-a\}, \quad (25)$$

where β satisfies the following equation:

$$\begin{aligned} & m(g \circ f)^r(a)(b^r - ma^r)^\alpha - (b^r - ma^r)^\alpha g^r(\sqrt{\beta}) \\ & + ((g \circ f)^r(b) - m(g \circ f)^r(a))((b - \beta)^r - ma^r)^\alpha \quad (26) \\ & = 0. \end{aligned}$$

Case 2. If $(g \circ f)^r(b) - m(g \circ f)^r(a) = 0$, then

$$(S) \int_a^b f^2(x) d\mu \leq \min \{f^2(a) \sqrt[r]{m^2}, b - a\}. \quad (27)$$

Case 3. If $(g \circ f)^r(b) - m(g \circ f)^r(a) < 0$, then

$$(S) \int_a^b f^2(x) d\mu \leq \min \{\beta, b - a\}, \quad (28)$$

where β satisfies the following equation:

$$\begin{aligned} & m(g \circ f)^r(a)(b^r - ma^r)^\alpha - (b^r - ma^r)^\alpha g^r(\sqrt{\beta}) \\ & + ((g \circ f)^r(b) - m(g \circ f)^r(a))((a + \beta)^r - ma^r)^\alpha \quad (29) \\ & = 0. \end{aligned}$$

Proof. As f is a general (α, m, r) -convex function for $x \in [a, b]$, we have

$$\begin{aligned} f(x) &= f\left(\left[m\left(1 - \frac{x^r - ma^r}{b^r - ma^r}\right)a^r + \frac{x^r - ma^r}{b^r - ma^r}b^r\right]^{1/r}\right) \\ &\leq g^{-1}\left(\left[m\left(1 - \left(\frac{x^r - ma^r}{b^r - ma^r}\right)^\alpha\right)(g \circ f)^r(a) + \left(\frac{x^r - ma^r}{b^r - ma^r}\right)^\alpha (g \circ f)^r(b)\right]^{1/r}\right) = h(x). \end{aligned} \quad (30)$$

By (3) of Proposition 4, we have

$$\begin{aligned} (S) \int_a^b f^2(x) d\mu &= (S) \int_a^b f^2\left(\left[m\left(1 - \frac{x^r - ma^r}{b^r - ma^r}\right)a^r + \frac{x^r - ma^r}{b^r - ma^r}b^r\right]^{1/r}\right) d\mu \leq (S) \\ &\cdot \int_a^b g^{-2}\left(\left[m\left(1 - \left(\frac{x^r - ma^r}{b^r - ma^r}\right)^\alpha\right)(g \circ f)^r(a) + \left(\frac{x^r - ma^r}{b^r - ma^r}\right)^\alpha (g \circ f)^r(b)\right]^{1/r}\right) d\mu = (S) \\ &\cdot \int_a^b h^2(x) d\mu. \end{aligned} \quad (31)$$

In order to calculate the right hand side of the last inequality, we consider the distribution function F given by

$$\begin{aligned} F(\beta) &= \mu\left([a, b] \cap \{h \geq \sqrt{\beta}\}\right) \\ &= \mu\left([a, b] \cap \left\{x \mid m\left(1 - \left(\frac{x^r - ma^r}{b^r - ma^r}\right)^\alpha\right)(g \circ f)^r(a) + \left(\frac{x^r - ma^r}{b^r - ma^r}\right)^\alpha (g \circ f)^r(b) \geq g^r(\sqrt{\beta})\right\}\right) \\ &= \mu\left([a, b] \cap \left\{x \mid \left(\frac{x^r - ma^r}{b^r - ma^r}\right)^\alpha ((g \circ f)^r(b) - m(g \circ f)^r(a)) \geq g^r(\sqrt{\beta}) - m(g \circ f)^r(a)\right\}\right) \\ &= \begin{cases} b - \left(\left(\frac{g^r(\sqrt{\beta}) - m(g \circ f)^r(a)}{(g \circ f)^r(b) - m(g \circ f)^r(a)}\right)^{1/\alpha} (b^r - ma^r) + ma^r\right)^{1/r}, & (g \circ f)^r(b) - m(g \circ f)^r(a) > 0, \\ f^2(a) \sqrt[r]{m^2}, & (g \circ f)^r(b) - m(g \circ f)^r(a) = 0, \\ \left(\left(\frac{g^r(\sqrt{\beta}) - m(g \circ f)^r(a)}{(g \circ f)^r(b) - m(g \circ f)^r(a)}\right)^{1/\alpha} (b^r - ma^r) + ma^r\right)^{1/r} - a, & (g \circ f)^r(b) - m(g \circ f)^r(a) < 0 \end{cases} \end{aligned} \quad (32)$$

and the solution of the equation $F(\beta) = \beta$. By (1) of Proposition 4 and Remark 5, we get the following.

Case 1. If $(g \circ f)^r(b) - m(g \circ f)^r(a) > 0$, then

$$(S) \int_a^b f^2(x) d\mu \leq \min \{\beta, b - a\}, \quad (33)$$

where β satisfies the following equation:

$$m(g \circ f)^r(a)(b^r - ma^r)^\alpha - (b^r - ma^r)^\alpha g^r(\sqrt{\beta}) \\ + ((g \circ f)^r(b) - m(g \circ f)^r(a))((b - \beta)^r - ma^r)^\alpha \quad (34) \\ = 0.$$

Case 2. If $(g \circ f)^r(b) - m(g \circ f)^r(a) = 0$, then

$$(S) \int_a^b f^2(x) d\mu \leq \min \{f^2(a) \sqrt[r]{m^2}, b - a\}. \quad (35)$$

Case 3. If $(g \circ f)^r(b) - m(g \circ f)^r(a) < 0$, then

$$(S) \int_a^b f^2(x) d\mu \leq \min \{\beta, b - a\}, \quad (36)$$

where β satisfies the following equation:

$$m(g \circ f)^r(a)(b^r - ma^r)^\alpha - (b^r - ma^r)^\alpha g^r(\sqrt{\beta}) \\ + ((g \circ f)^r(b) - m(g \circ f)^r(a))((a + \beta)^r - ma^r)^\alpha \quad (37) \\ = 0.$$

This completes the proof. $\square$

Remark 13. If $\alpha = 0$ in Theorem 12, then $(S) \int_a^b f^2(x) d\mu \leq \min\{f^2(b), b - a\}$.

Example 14. Consider $X = [1, 4]$ and let μ be the Lebesgue measure on X . If we take the function $f(x) = x^{3/2}$ and $g(x) = \sqrt[3]{x}$, then $f(x)$ is a general $(1/3, 0, 3)$ -convex function. In fact,

$$x^{3/2} = f\left(\left[\left(\frac{x^3 - 0 \cdot 1^3}{4^3 - 0 \cdot 1^3}\right) \cdot 4^3\right.\right. \\ \left.\left.+ 0\left(1 - \left(\frac{x^3 - 0 \cdot 1^3}{4^3 - 0 \cdot 1^3}\right)\right) \cdot 1^3\right]^{1/3}\right) \\ \leq \left(\left[\left(\frac{x^3 - 0 \cdot 1^3}{4^3 - 0 \cdot 1^3}\right)^{1/3} \cdot 4^{3/2}\right.\right. \\ \left.\left.+ 0\left(1 - \left(\frac{x^3 - 0 \cdot 1^3}{4^3 - 0 \cdot 1^3}\right)^{1/3}\right) \cdot 1^{3/2}\right]^{1/3}\right)^3 = 2x. \quad (38)$$

By Theorem 12, we have

$$2.6212 = (S) \int_1^4 x^3 d\mu \leq \min \{3.1172, 3\} = 3. \quad (39)$$

Now we consider some special cases of general (α, m, r) -convex functions in Theorem 12.

Remark 15. Let $(\alpha, m) \in [0, 1]^2$, $r \in \mathbb{R}$, and $r \neq 0$, $g = id$, let $f : [a, b] \rightarrow [0, \infty)$ be an (α, m, r) -convex function, and let μ be the Lebesgue measure on $\mathbb{R}$. Then, one obtains the following.

Case 1. If $f^r(b) - mf^r(a) > 0$, then

$$(S) \int_a^b f^2(x) d\mu \leq \min \{\beta, b - a\}, \quad (40)$$

where β satisfies the following equation:

$$(f^r(b) - mf^r(a))((b - \beta)^r - ma^r)^\alpha \\ - (b^r - ma^r)^\alpha \sqrt[r]{\beta^r} + mf^r(a)(b^r - ma^r)^\alpha = 0. \quad (41)$$

Case 2. If $f^r(b) - mf^r(a) = 0$, then

$$(S) \int_a^b f^2(x) d\mu \leq \min \{f^2(a) \sqrt[r]{m^2}, b - a\}. \quad (42)$$

Case 3. If $f^r(b) - mf^r(a) < 0$, then

$$(S) \int_a^b f^2(x) d\mu \leq \min \{\beta, b - a\}, \quad (43)$$

where β satisfies the following equation:

$$(f^r(b) - mf^r(a))((a + \beta)^r - ma^r)^\alpha \\ - (b^r - ma^r)^\alpha \sqrt[r]{\beta^r} + mf^r(a)(b^r - ma^r)^\alpha = 0. \quad (44)$$

Remark 16. Let $\alpha = m = 1$, $r \in \mathbb{R}$, and $r \neq 0$, let g be a continuous and monotonous function, let $f : [a, b] \rightarrow [0, \infty)$ be a general r -mean convex function, and let μ be the Lebesgue measure on $\mathbb{R}$. Then, one obtains the following.

Case 1. If $(g \circ f)^r(b) - (g \circ f)^r(a) > 0$, then

$$(S) \int_a^b f^2(x) d\mu \leq \min \{\beta, b - a\}, \quad (45)$$

where β satisfies the following equation:

$$(g \circ f)^r(a)(b^r - a^r) - (b^r - a^r) g^r(\sqrt{\beta}) \\ + ((g \circ f)^r(b) - (g \circ f)^r(a))((b - \beta)^r - a^r) \quad (46) \\ = 0.$$

Case 2. If $(g \circ f)^r(b) - (g \circ f)^r(a) = 0$, then

$$(S) \int_a^b f^2(x) d\mu \leq \min \{f^2(a) \sqrt[r]{m^2}, b - a\}. \quad (47)$$

Case 3. If $(g \circ f)^r(b) - (g \circ f)^r(a) < 0$, then

$$(S) \int_a^b f^2(x) d\mu \leq \min \{\beta, b-a\}, \quad (48)$$

where β satisfies the following equation:

$$\begin{aligned} & (g \circ f)^r(a) (b^r - a^r) - (b^r - a^r) g^r(\sqrt{\beta}) \\ & + ((g \circ f)^r(b) - (g \circ f)^r(a)) ((a + \beta)^r - a^r) = 0. \end{aligned} \quad (49)$$

Remark 17. Let $\alpha = m = 1$, $r \in \mathbb{R}$, and $r \neq 0$, $g = id$, let $f: [a, b] \rightarrow [0, \infty)$ be an r -mean convex function, and μ be the Lebesgue measure on $\mathbb{R}$. Then, one obtains the following.

Case 1. If $f^r(b) - f^r(a) > 0$, then

$$(S) \int_a^b f^2(x) d\mu \leq \min \{\beta, b-a\}, \quad (50)$$

where β satisfies the following equation:

$$\begin{aligned} & f^r(a) (b^r - a^r) - (b^r - a^r) \sqrt{\beta^r} \\ & + (f^r(b) - f^r(a)) ((b - \beta)^r - a^r) = 0. \end{aligned} \quad (51)$$

Case 2. If $f^r(b) - f^r(a) = 0$, then

$$(S) \int_a^b f^2(x) d\mu \leq \min \{f^2(a) \sqrt[m]{m^2}, b-a\}. \quad (52)$$

Case 3. If $f^r(b) - f^r(a) < 0$, then

$$(S) \int_a^b f^2(x) d\mu \leq \min \{\beta, b-a\}, \quad (53)$$

where β satisfies the following equation:

$$\begin{aligned} & f^r(a) (b^r - a^r) - (b^r - a^r) \sqrt{\beta^r} \\ & + (f^r(b) - f^r(a)) ((a + \beta)^r - a^r) = 0. \end{aligned} \quad (54)$$

Remark 18. Let $(\alpha, m) \in [0, 1]^2$ and $r = 1$, let g be a continuous and monotonous function, let $f: [a, b] \rightarrow [0, \infty)$ be a general (α, m) -convex function, and let μ be the Lebesgue measure on $\mathbb{R}$. Then, one obtains the following.

Case 1. If $(g \circ f)(b) - m(g \circ f)(a) > 0$, then

$$(S) \int_a^b f^2(x) d\mu \leq \min \{\beta, b-a\}, \quad (55)$$

where β satisfies the following equation:

$$\begin{aligned} & m(g \circ f)(a) (b - ma)^\alpha - (b - ma)^\alpha g(\sqrt{\beta}) \\ & + ((g \circ f)(b) - m(g \circ f)(a)) ((b - \beta) - ma)^\alpha = 0. \end{aligned} \quad (56)$$

Case 2. If $(g \circ f)(b) - m(g \circ f)(a) = 0$, then

$$(S) \int_a^b f^2(x) d\mu \leq \min \{f^2(a) m^2, b-a\}. \quad (57)$$

Case 3. If $(g \circ f)(b) - m(g \circ f)(a) < 0$, then

$$(S) \int_a^b f^2(x) d\mu \leq \min \{\beta, b-a\}, \quad (58)$$

where β satisfies the following equation:

$$\begin{aligned} & m(g \circ f)(a) (b - ma)^\alpha - (b - ma)^\alpha g(\sqrt{\beta}) \\ & + ((g \circ f)(b) - m(g \circ f)(a)) ((a + \beta) - ma)^\alpha = 0. \end{aligned} \quad (59)$$

Remark 19. Let $g = id$ and $r = 1$ in Theorem 12; then we obtain the general Sandor inequalities for fuzzy integral based on (α, m) -convex functions.

Remark 20. Let $g = id$ and $\alpha = m = r = 1$ in Theorem 12; then we obtain the Sandor inequalities for fuzzy integral of convex functions [38].

4. Sandor Type Inequalities for Fuzzy Integral Based on General (α, m, r) -Concave Functions

In this section, we provide Sandor type inequalities for fuzzy integral based on general (α, m, r) -concave functions.

Theorem 21. Let $(\alpha, m) \in (0, 1]^2$, $r \in \mathbb{R}$, and $r \neq 0$, let g be a continuous and monotonous function, let $f: [0, 1] \rightarrow [0, \infty)$ be a general (α, m, r) -concave function, and let μ be the Lebesgue measure on $\mathbb{R}$. Then, one obtains the following.

Case 1. If $(g \circ f)^r(1) - m(g \circ f)^r(0) > 0$, then

$$(S) \int_0^1 f^2(x) d\mu \geq \min \{\beta, 1\}, \quad (60)$$

where β satisfies the following equation:

$$\begin{aligned} & ((g \circ f)^r(1) - m(g \circ f)^r(0))(1 - \beta)^{\alpha r} - g^r(\sqrt{\beta}) \\ & + m(g \circ f)^r(0) = 0. \end{aligned} \quad (61)$$

Case 2. If $(g \circ f)^r(1) - m(g \circ f)^r(0) = 0$, then

$$(S) \int_0^1 f^2(x) d\mu \geq \min \{f^2(0) \sqrt[r]{m^2}, 1\}. \quad (62)$$

Case 3. If $(g \circ f)^r(1) - m(g \circ f)^r(0) < 0$, then

$$(S) \int_0^1 f^2(x) d\mu \geq \min \{\beta, 1\}, \quad (63)$$

where β satisfies the following equation:

$$\begin{aligned} & ((g \circ f)^r(1) - m(g \circ f)^r(0))\beta^{\alpha r} - g^r(\sqrt{\beta}) \\ & + m(g \circ f)^r(0) = 0. \end{aligned} \quad (64)$$

Proof. As f is a general (α, m, r) -concave function for $x \in [0, 1]$, we have

$$\begin{aligned} f(x) &= f([x^r \cdot 1^r + m(1 - x^r) \cdot 0^r]^{1/r}) \\ &\geq g^{-1}([x^{\alpha r} (g \circ f)^r(1) \\ &+ m(1 - x^{\alpha r}) (g \circ f)^r(0)]^{1/r}) = h(x). \end{aligned} \quad (65)$$

By (3) of Proposition 4, we have

$$\begin{aligned} (S) \int_0^1 f^2(x) d\mu &= (S) \int_0^1 f^2([x^r \cdot 1^r + m(1 - x^r) \\ &\cdot 0^r]^{1/r}) d\mu \geq (S) \int_0^1 g^{-2}([x^{\alpha r} (g \circ f)^r(1) \\ &+ m(1 - x^{\alpha r}) (g \circ f)^r(0)]^{1/r}) d\mu = (S) \\ &\cdot \int_0^1 h^2(x) d\mu. \end{aligned} \quad (66)$$

In order to calculate the right hand side of the last inequality, we consider the distribution function F given by

$$\begin{aligned} F(\beta) &= \mu([0, 1] \cap \{h^2 \geq \beta\}) = \mu([0, 1] \cap \{h \geq \sqrt{\beta}\}) \\ &= \mu([0, 1] \cap \{x \mid ((g \circ f)^r(1) - m(g \circ f)^r(0))x^{\alpha r} \geq g^r(\sqrt{\beta}) - m(g \circ f)^r(0)\}) \\ &= \begin{cases} 1 - \left(\frac{g^r(\sqrt{\beta}) - m(g \circ f)^r(0)}{(g \circ f)^r(1) - m(g \circ f)^r(0)} \right)^{1/\alpha r}, & (g \circ f)^r(1) - m(g \circ f)^r(0) > 0, \\ f(0) \sqrt[r]{m}, & (g \circ f)^r(1) - m(g \circ f)^r(0) = 0, \\ \left(\frac{g^r(\sqrt{\beta}) - m(g \circ f)^r(0)}{(g \circ f)^r(1) - m(g \circ f)^r(0)} \right)^{1/\alpha r}, & (g \circ f)^r(1) - m(g \circ f)^r(0) < 0 \end{cases} \end{aligned} \quad (67)$$

and the solution of the equation $F(\beta) = \beta$. By (1) of Proposition 4 and Remark 5, we get the following.

Case 1. If $(g \circ f)^r(1) - m(g \circ f)^r(0) > 0$, then

$$(S) \int_0^1 f^2(x) d\mu \geq \min \{\beta, 1\}, \quad (68)$$

where β satisfies the following equation:

$$\begin{aligned} & ((g \circ f)^r(1) - m(g \circ f)^r(0))(1 - \beta)^{\alpha r} - g^r(\sqrt{\beta}) \\ & + m(g \circ f)^r(0) = 0. \end{aligned} \quad (69)$$

Case 2. If $(g \circ f)^r(1) - m(g \circ f)^r(0) = 0$, then

$$(S) \int_0^1 f^2(x) d\mu \geq \min \{f^2(0) \sqrt[r]{m^2}, 1\}. \quad (70)$$

Case 3. If $(g \circ f)^r(1) - m(g \circ f)^r(0) < 0$, then

$$(S) \int_0^1 f^2(x) d\mu \geq \min \{\beta, 1\}, \quad (71)$$

where β satisfies the following equation:

$$\begin{aligned} & ((g \circ f)^r(1) - m(g \circ f)^r(0))\beta^{\alpha r} - g^r(\sqrt{\beta}) \\ & + m(g \circ f)^r(0) = 0. \end{aligned} \quad (72)$$

This completes the proof. $\square$

Remark 22. If $\alpha = 0$ in Theorem 21, then (S) $\int_0^1 f^2(x) d\mu \geq \min\{f^2(1), 1\}$.

Example 23. Consider $X = [0, 1]$ and let μ be the Lebesgue measure on X . If we take the function $f(x) = \sqrt[3]{x}$ and $g(x) = x^2$, then $f(x)$ is a general $(1/2, 1/3, 2)$ -concave function. In fact,

$$\begin{aligned} \sqrt[3]{x} &= f\left(\left[x^2 \cdot 1^2 + \frac{1}{3}(1-x^2) \cdot 0^2\right]^{1/2}\right) \\ &\geq \left(\left[x + \frac{1}{3}(1-x) \cdot 0\right]^2\right)^{1/2} = x. \end{aligned} \quad (73)$$

By Theorem 21, we have

$$0.4302 = (S) \int_0^1 \sqrt[3]{x^2} d\mu \geq \min\{0.3820, 1\} = 0.3820. \quad (74)$$

Now, we will prove the general case of Theorem 21.

Theorem 24. Let $(\alpha, m) \in (0, 1]^2$, $r \in \mathbb{R}$, and $r \neq 0$, let g be a continuous and monotonous function, let $f : [a, b] \rightarrow [0, \infty)$ be a general (α, m, r) -concave function, and let μ be the Lebesgue measure on $\mathbb{R}$. Then, one obtains the following.

Case 1. If $(g \circ f)^r(b) - m(g \circ f)^r(a) > 0$, then

$$(S) \int_a^b f^2(x) d\mu \geq \min\{\beta, b-a\}, \quad (75)$$

where β satisfies the following equation:

$$\begin{aligned} &m(g \circ f)^r(a)(b^r - ma^r)^\alpha - (b^r - ma^r)^\alpha g^r(\sqrt{\beta}) \\ &+ ((g \circ f)^r(b) - m(g \circ f)^r(a))((b - \beta)^r - ma^r)^\alpha \end{aligned} \quad (76)$$

$= 0.$

Case 2. If $(g \circ f)^r(b) - m(g \circ f)^r(a) = 0$, then

$$(S) \int_a^b f^2(x) d\mu \geq \min\{f^2(a)\sqrt[m^2]{m^2}, b-a\}. \quad (77)$$

Case 3. If $(g \circ f)^r(b) - m(g \circ f)^r(a) < 0$, then

$$(S) \int_a^b f^2(x) d\mu \geq \min\{\beta, b-a\}, \quad (78)$$

where β satisfies the following equation:

$$\begin{aligned} &m(g \circ f)^r(a)(b^r - ma^r)^\alpha - (b^r - ma^r)^\alpha g^r(\sqrt{\beta}) \\ &+ ((g \circ f)^r(b) - m(g \circ f)^r(a))((a + \beta)^r - ma^r)^\alpha \end{aligned} \quad (79)$$

$= 0.$

Proof. As f is a general (α, m, r) -concave function for $x \in [a, b]$, we have

$$\begin{aligned} f(x) &= f\left(\left[m\left(1 - \frac{x^r - ma^r}{b^r - ma^r}\right)a^r\right.\right. \\ &\quad \left.\left.+ \frac{x^r - ma^r}{b^r - ma^r}b^r\right]^{1/r}\right) \\ &\geq g^{-1}\left(\left[m\left(1 - \left(\frac{x^r - ma^r}{b^r - ma^r}\right)^\alpha\right)(g \circ f)^r(a)\right.\right. \\ &\quad \left.\left.+ \left(\frac{x^r - ma^r}{b^r - ma^r}\right)^\alpha(g \circ f)^r(b)\right]^{1/r}\right) = h(x). \end{aligned} \quad (80)$$

By (3) of Proposition 4, we have

$$\begin{aligned} (S) \int_a^b f^2(x) d\mu &= (S) \int_a^b f^2\left(\left[m\left(1 - \frac{x^r - ma^r}{b^r - ma^r}\right)a^r\right.\right. \\ &\quad \left.\left.+ \frac{x^r - ma^r}{b^r - ma^r}b^r\right]^{1/r}\right) d\mu \geq (S) \\ &\cdot \int_a^b g^{-2}\left(\left[m\left(1 - \left(\frac{x^r - ma^r}{b^r - ma^r}\right)^\alpha\right)(g \circ f)^r(a)\right.\right. \\ &\quad \left.\left.+ \left(\frac{x^r - ma^r}{b^r - ma^r}\right)^\alpha(g \circ f)^r(b)\right]^{1/r}\right) d\mu = (S) \\ &\cdot \int_a^b h^2(x) d\mu. \end{aligned} \quad (81)$$

In order to calculate the right hand side of the last inequality, we consider the distribution function F given by

$$\begin{aligned} F(\beta) &= \mu\left([a, b] \cap \{h \geq \sqrt{\beta}\}\right) \\ &= \mu\left([a, b] \cap \left\{x \mid m\left(1 - \left(\frac{x^r - ma^r}{b^r - ma^r}\right)^\alpha\right)(g \circ f)^r(a) + \left(\frac{x^r - ma^r}{b^r - ma^r}\right)^\alpha(g \circ f)^r(b) \geq g^r(\sqrt{\beta})\right\}\right) \\ &= \mu\left([a, b] \cap \left\{x \mid \left(\frac{x^r - ma^r}{b^r - ma^r}\right)^\alpha((g \circ f)^r(b) - m(g \circ f)^r(a)) \geq g^r(\sqrt{\beta}) - m(g \circ f)^r(a)\right\}\right) \end{aligned}$$

$$= \begin{cases} b - \left(\left(\frac{g^r(\sqrt{\beta}) - m(g \circ f)^r(a)}{(g \circ f)^r(b) - m(g \circ f)^r(a)} \right)^{1/\alpha} (b^r - ma^r) + ma^r \right)^{1/r}, & (g \circ f)^r(b) - m(g \circ f)^r(a) > 0, \\ f^2(a) \sqrt[r]{m^2}, & (g \circ f)^r(b) - m(g \circ f)^r(a) = 0, \\ \left(\left(\frac{g^r(\sqrt{\beta}) - m(g \circ f)^r(a)}{(g \circ f)^r(b) - m(g \circ f)^r(a)} \right)^{1/\alpha} (b^r - ma^r) + ma^r \right)^{1/r} - a, & (g \circ f)^r(b) - m(g \circ f)^r(a) < 0 \end{cases} \quad (82)$$

and the solution of the equation $F(\beta) = \beta$. By (1) of Proposition 4 and Remark 5, we get the following.

Case 1. If $(g \circ f)^r(b) - m(g \circ f)^r(a) > 0$, then

$$(S) \int_a^b f^2(x) d\mu \geq \min\{\beta, b - a\}, \quad (83)$$

where β satisfies the following equation:

$$\begin{aligned} & m(g \circ f)^r(a) (b^r - ma^r)^\alpha - (b^r - ma^r)^\alpha g^r(\sqrt{\beta}) \\ & + ((g \circ f)^r(b) - m(g \circ f)^r(a)) ((b - \beta)^r - ma^r)^\alpha = 0. \end{aligned} \quad (84)$$

Case 2. If $(g \circ f)^r(b) - m(g \circ f)^r(a) = 0$, then

$$(S) \int_a^b f^2(x) d\mu \geq \min\{f^2(a) \sqrt[r]{m^2}, b - a\}. \quad (85)$$

Case 3. If $(g \circ f)^r(b) - m(g \circ f)^r(a) < 0$, then

$$(S) \int_a^b f^2(x) d\mu \geq \min\{\beta, b - a\}, \quad (86)$$

where β satisfies the following equation:

$$\begin{aligned} & m(g \circ f)^r(a) (b^r - ma^r)^\alpha - (b^r - ma^r)^\alpha g^r(\sqrt{\beta}) \\ & + ((g \circ f)^r(b) - m(g \circ f)^r(a)) ((a + \beta)^r - ma^r)^\alpha = 0. \end{aligned} \quad (87)$$

This completes the proof. $\square$

Remark 25. If $\alpha = 0$ in Theorem 24, then $(S) \int_a^b f^2(x) d\mu \geq \min\{f^2(b), b - a\}$.

Example 26. Consider $X = [1, 4]$ and let μ be the Lebesgue measure on X . If we take the function $f(x) = \sqrt{x}$ and

$g(x) = \sqrt[3]{x}$, then $f(x)$ is a general $(1/3, 0, 3)$ -concave function. In fact,

$$\begin{aligned} \sqrt{x} &= f\left(\left[\left(\frac{x^3 - 0 \cdot 1^3}{4^3 - 0 \cdot 1^3}\right) \cdot 4^3\right. \right. \\ &\quad \left. \left. + 0\left(1 - \left(\frac{x^3 - 0 \cdot 1^3}{4^3 - 0 \cdot 1^3}\right)\right) \cdot 1^3\right]^{1/3}\right) \\ &\geq \left(\left[\left(\frac{x^3 - 0 \cdot 1^3}{4^3 - 0 \cdot 1^3}\right)^{1/3} \cdot 4^{1/2}\right. \right. \\ &\quad \left. \left. + 0\left(1 - \left(\frac{x^3 - 0 \cdot 1^3}{4^3 - 0 \cdot 1^3}\right)^{1/3}\right) \cdot 1^{3/2}\right]^{1/3}\right)^3 = \frac{x}{2}. \end{aligned} \quad (88)$$

By Theorem 24, we have

$$2 = (S) \int_1^4 x d\mu \geq \min\{1.5279, 3\} = 1.5279. \quad (89)$$

Now we consider some special cases of general (α, m, r) -concave functions in Theorem 24.

Remark 27. Let $(\alpha, m) \in [0, 1]^2$, $r \in \mathbb{R}$, and $r \neq 0$, $g = id$, let $f: [a, b] \rightarrow [0, \infty)$ be an (α, m, r) -concave function, and let μ be the Lebesgue measure on $\mathbb{R}$. Then, one obtains the following.

Case 1. If $f^r(b) - mf^r(a) > 0$, then

$$(S) \int_a^b f^2(x) d\mu \geq \min\{\beta, b - a\}, \quad (90)$$

where β satisfies the following equation:

$$\begin{aligned} & (f^r(b) - mf^r(a)) ((b - \beta)^r - ma^r)^\alpha \\ & - (b^r - ma^r)^\alpha \sqrt[r]{\beta^r} + mf^r(a) (b^r - ma^r)^\alpha = 0. \end{aligned} \quad (91)$$

Case 2. If $f^r(b) - mf^r(a) = 0$, then

$$(S) \int_a^b f^2(x) d\mu \geq \min\{f^2(a) \sqrt[r]{m^2}, b - a\}. \quad (92)$$

Case 3. If $f^r(b) - mf^r(a) < 0$, then

$$(S) \int_a^b f^2(x) d\mu \geq \min\{\beta, b-a\}, \quad (93)$$

where β satisfies the following equation:

$$\begin{aligned} & (f^r(b) - mf^r(a))((a + \beta)^r - ma^r)^\alpha \\ & - (b^r - ma^r)^\alpha \sqrt[r]{\beta^r} + mf^r(a)(b^r - ma^r)^\alpha = 0. \end{aligned} \quad (94)$$

Remark 28. Let $\alpha = m = 1$, $r \in \mathbb{R}$, and $r \neq 0$, let g be a continuous and monotonous function, let $f : [a, b] \rightarrow [0, \infty)$ be a general r -mean concave function, and let μ be the Lebesgue measure on $\mathbb{R}$. Then, one obtains the following.

Case 1. If $(g \circ f)^r(b) - (g \circ f)^r(a) > 0$, then

$$(S) \int_a^b f^2(x) d\mu \geq \min\{\beta, b-a\}, \quad (95)$$

where β satisfies the following equation:

$$\begin{aligned} & (g \circ f)^r(a)(b^r - a^r) - (b^r - a^r)g^r(\sqrt[r]{\beta}) \\ & + ((g \circ f)^r(b) - (g \circ f)^r(a))((b - \beta)^r - a^r) \\ & = 0. \end{aligned} \quad (96)$$

Case 2. If $(g \circ f)^r(b) - (g \circ f)^r(a) = 0$, then

$$(S) \int_a^b f^2(x) d\mu \geq \min\{f^2(a)\sqrt[r]{m^2}, b-a\}. \quad (97)$$

Case 3. If $(g \circ f)^r(b) - (g \circ f)^r(a) < 0$, then

$$(S) \int_a^b f^2(x) d\mu \geq \min\{\beta, b-a\}, \quad (98)$$

where β satisfies the following equation:

$$\begin{aligned} & (g \circ f)^r(a)(b^r - a^r) - (b^r - a^r)g^r(\sqrt[r]{\beta}) \\ & + ((g \circ f)^r(b) - (g \circ f)^r(a))((a + \beta)^r - a^r) \\ & = 0. \end{aligned} \quad (99)$$

Remark 29. Let $\alpha = m = 1$, $r \in \mathbb{R}$, and $r \neq 0$, $g = id$, let $f : [a, b] \rightarrow [0, \infty)$ be an r -mean concave function, and let μ be the Lebesgue measure on $\mathbb{R}$. Then, one obtains the following.

Case 1. If $f^r(b) - f^r(a) > 0$, then

$$(S) \int_a^b f^2(x) d\mu \geq \min\{\beta, b-a\}, \quad (100)$$

where β satisfies the following equation:

$$\begin{aligned} & f^r(a)(b^r - a^r) - (b^r - a^r)\sqrt[r]{\beta^r} \\ & + (f^r(b) - f^r(a))((b - \beta)^r - a^r) = 0. \end{aligned} \quad (101)$$

Case 2. If $f^r(b) - f^r(a) = 0$, then

$$(S) \int_a^b f^2(x) d\mu \geq \min\{f^2(a)\sqrt[r]{m^2}, b-a\}. \quad (102)$$

Case 3. If $f^r(b) - f^r(a) < 0$, then

$$(S) \int_a^b f^2(x) d\mu \geq \min\{\beta, b-a\}, \quad (103)$$

where β satisfies the following equation:

$$\begin{aligned} & f^r(a)(b^r - a^r) - (b^r - a^r)\sqrt[r]{\beta^r} \\ & + (f^r(b) - f^r(a))((a + \beta)^r - a^r) = 0. \end{aligned} \quad (104)$$

Remark 30. Let $(\alpha, m) \in [0, 1]^2$ and $r = 1$, let g be a continuous and monotonous function, let $f : [a, b] \rightarrow [0, \infty)$ be a general (α, m) -concave function, and let μ be the Lebesgue measure on $\mathbb{R}$. Then, one obtains the following.

Case 1. If $(g \circ f)(b) - m(g \circ f)(a) > 0$, then

$$(S) \int_a^b f^2(x) d\mu \geq \min\{\beta, b-a\}, \quad (105)$$

where β satisfies the following equation:

$$\begin{aligned} & m(g \circ f)(a)(b - ma)^\alpha - (b - ma)^\alpha g(\sqrt[r]{\beta}) \\ & + ((g \circ f)(b) - m(g \circ f)(a))((b - \beta) - ma)^\alpha \\ & = 0. \end{aligned} \quad (106)$$

Case 2. If $(g \circ f)(b) - m(g \circ f)(a) = 0$, then

$$(S) \int_a^b f^2(x) d\mu \geq \min\{f^2(a)m^2, b-a\}. \quad (107)$$

Case 3. If $(g \circ f)(b) - m(g \circ f)(a) < 0$, then

$$(S) \int_a^b f^2(x) d\mu \geq \min\{\beta, b-a\}, \quad (108)$$

where β satisfies the following equation:

$$\begin{aligned} & m(g \circ f)(a)(b - ma)^\alpha - (b - ma)^\alpha g(\sqrt[r]{\beta}) \\ & + ((g \circ f)(b) - m(g \circ f)(a))((a + \beta) - ma)^\alpha \\ & = 0. \end{aligned} \quad (109)$$

Remark 31. Let $g = id$ and $r = 1$ in Theorem 24; then we obtain the general Sandor type inequalities for fuzzy integral based on (α, m) -concave functions.

Remark 32. Let $g = id$ and $\alpha = m = r = 1$ in Theorem 24; then we obtain the Sandor type inequalities for fuzzy integral of concave functions [38].

5. Conclusion

In this paper, we have proved the Sandor type inequalities for the Sugeno integral based on general (α, m, r) -convex functions and general (α, m, r) -concave functions. As open problems for future research, it would be interesting to investigate Sandor type inequalities for the Sugeno integral based on general (α, m, r) -convex functions with respect to arbitrary fuzzy measure and explore Sandor type inequality to other generalizations of fuzzy integrals. We will investigate these problems in the future.

Conflict of Interests

The authors declare that they have no conflict of interests.

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