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Research Article

A Lyapunov-Type Inequality for a Fractional Differential Equation under a Robin Boundary Condition

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We establish a new Lyapunov-type inequality for a class of fractional differential equations under Robin boundary conditions. The obtained inequality is used to obtain an interval where a linear combination of certain Mittag-Leffler functions has no real zeros.

1. Introduction and Preliminaries

Let the following $\chi : [a,b] \to \mathbb{R}$ be a continuous function. The well-known Lyapunov inequality [1] states that a necessary condition for the boundary value problem

$$\begin{aligned} &u''(t) + \chi(t)\,u(t) = 0, \quad a < t < b \\ &u(a) = u(b) = 0 \end{aligned} \tag{1}$$

to have nontrivial solutions is that

$$\int_a^b |\chi(t)|\,dt > \frac{4}{b-a}. \tag{2}$$

This result found many practical applications in differential and difference equations (oscillation theory, disconjugacy, eigenvalue problems, etc.); see, for instance, [2–7]. On the other hand, many improvements of (2) have been carried out, and similar inequalities have been obtained for other types of differential equations; compare to the Pachpatte monograph [8]. The search for Lyapunov-type inequalities in which the starting differential equation is constructed via fractional differential operators has begun very recently. For example, in [9], a Lyapunov-type inequality was obtained for differential equations depending on the Riemann-Liouville fractional derivative; that is, for the boundary value problem

$$\begin{aligned} &\left({}_aD^{\alpha} u\right)(t) + \chi(t)\,u(t) = 0, \quad a < t < b, 1 < \alpha \le 2, \\ &u(a) = u(b) = 0, \end{aligned} \tag{3}$$

where ${}_aD^{\alpha}$ denotes the Riemann-Liouville fractional derivative of order α. Precisely, the author proved that if (3) has a nontrivial solution, then we have

$$\int_a^b |\chi(t)|\,dt > \Gamma(\alpha)\left(\frac{4}{b-a}\right)^{\alpha-1}. \tag{4}$$

Clearly, if we let $\alpha = 2$ in (4), one obtains Lyapunov's classical inequality (2).

In [10], a Lyapunov-type inequality was obtained for the Caputo fractional boundary value problem

$$\begin{aligned} &\left({}_a^CD^{\alpha} u\right)(t) + \chi(t)\,u(t) = 0, \quad a < t < b, 1 < \alpha \le 2, \\ &u(a) = u(b) = 0, \end{aligned} \tag{5}$$

where ${}_a^CD^{\alpha}$ is the Caputo fractional derivative of order α. It was proved that if (5) has a nontrivial solution, then we have

$$\int_a^b |\chi(t)|\,dt > \frac{\Gamma(\alpha)\,\alpha^{\alpha}}{[(\alpha-1)(b-a)]^{\alpha-1}}. \tag{6}$$

Similarly, if we let $\alpha = 2$ in (6), one obtains Lyapunov's classical inequality (2).

Motivated by the above works, we consider in this paper a Caputo fractional differential equation under Robin boundary conditions. More precisely, we consider the boundary value problem

$$\begin{aligned} &\left({}_a^C D^\alpha u\right)(t) + \chi(t)\, u(t) = 0, \quad a < t < b, 1 < \alpha \le 2, \\ &u(a) - u'(a) = u(b) + u'(b) = 0, \end{aligned} \tag{7}$$

and we get a corresponding Lyapunov-type inequality. This result is then used to obtain a real interval where a linear combination of certain Mittag-Leffler functions has no (real) zeros.

Before presenting the main results, let us start by recalling the concepts of the Riemann-Liouville fractional integral and the Caputo fractional derivative of order $\alpha \ge 0$. For more details, we refer to [11].

Definition 1. Let $\alpha \ge 0$ and let f be a real function defined on $[a, b]$. The Riemann-Liouville fractional integral of order α is defined by $({}_aI^0 f) \equiv f$ and

$$\left({}_aI^\alpha f\right)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)\, ds, \quad \alpha > 0, t \in [a, b]. \tag{8}$$

Definition 2. The Caputo fractional derivative of order $\alpha \ge 0$ is defined by $({}_a^C D^0 f) \equiv f$ and $({}_a^C D^\alpha f)(t) = ({}_aI^{m-\alpha} D^m f)(t)$ for $\alpha > 0$, where m is the smallest integer greater than or equal to α.

The following result is standard within the fractional calculus theory involving the Caputo differential operator (see [7]).

Lemma 3. *One has that $u \in C[a, b]$ is a solution to (7) if and only if*

$$u(t) = c_0 + c_1(t-a) - \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \chi(s)\, u(s)\, ds, \tag{9}$$

where c_0 and c_1 are some real constants.

Now, we are ready to state and prove our main results.

2. Main Results

2.1. A Lyapunov-Type Inequality. At first, we consider the following notations:

$$\begin{aligned} &\gamma = b - a + 2, \\ &h_1(t, s) = 1 + t - a - \frac{\gamma (t-s)^{\alpha-1}}{(b-s)^{\alpha-1} + (\alpha-1)(b-s)^{\alpha-2}}, \\ &h_2(t, s) = 1 + t - a. \end{aligned} \tag{10}$$

Now, let us write problem (7) in its equivalent integral form.

Lemma 4. *One has that $u \in C[a, b]$ is a solution to (7) if and only if u satisfies the integral equation*

$$u(t) = \int_a^b G(t, s)\, \chi(s)\, u(s)\, ds, \tag{11}$$

where

$$G(t, s) = \frac{(b-s)^{\alpha-2}(b-s+\alpha-1)}{\gamma\Gamma(\alpha)} H(t, s), \tag{12}$$

$$H(t, s) = \begin{cases} h_1(t, s), & a \le s \le t \le b, \\ h_2(t, s), & a \le t \le s \le b. \end{cases} \tag{13}$$

Proof. From Lemma 3, we have

$$u(t) = c_0 + c_1(t-a) - \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \chi(s)\, u(s)\, ds, \tag{14}$$

where c_0 and c_1 are some real constants. Then,

$$u'(t) = c_1 - \frac{1}{\Gamma(\alpha)} \int_a^t (\alpha-1)(t-s)^{\alpha-2} \chi(s)\, u(s)\, ds. \tag{15}$$

Since $u(a) - u'(a) = 0$, we obtain

$$c_0 = c_1. \tag{16}$$

From the boundary condition $u(b) + u'(b) = 0$, we get

$$c_1 = \frac{1}{\gamma\Gamma(\alpha)} \int_a^b \left((\alpha-1)(b-s)^{\alpha-2} + (b-s)^{\alpha-1}\right) \chi(s)\, u(s)\, ds. \tag{17}$$

Using (14), (16), and (17), we obtain the desired result. □

Lemma 5. *For all $(t, s) \in [a, b] \times (a, b)$, one has*

$$|H(t, s)| \le \max\left\{1 + b - a, \frac{2-\alpha}{\alpha-1}(b-a) - 1\right\}. \tag{18}$$

Proof. It is easy to see that, for $a \le t \le s \le b$, we have

$$0 \le h_2(t, s) \le 1 + b - a. \tag{19}$$

On the other hand, for $a \le s < t \le b$, we have

$$\frac{\partial h_1}{\partial t}(t, s) = 1 - \frac{(\alpha-1)\gamma(t-s)^{\alpha-2}}{(b-s)^{\alpha-1} + (\alpha-1)(b-s)^{\alpha-2}}. \tag{20}$$

Hence,

$$\lim_{t \to s^+} \frac{\partial h_1}{\partial t}(t, s) \in [-\infty, 0). \tag{21}$$

Now, for fixed s in (a, b), we want to study the variation of the function $t \mapsto h_1(t, s)$ for t in $[s, b]$. First, we have

$$\left.\frac{\partial h_1}{\partial t}(t, s)\right|_{t=b} = 1 - \frac{(\alpha-1)\gamma}{\alpha-1+b-s}. \tag{22}$$

Let

$$a^* = b + (1-\gamma)(\alpha-1). \tag{23}$$

We distinguish two eventual cases according to the value of a^*. □

Case 1. If $a^* \le a$, in this case, we have

$$\frac{\partial h_1}{\partial t}(t,s)\Big|_{t=b} \le 0, \quad s \in (a,b). \tag{24}$$

From (20), (21), and (24), we deduce

$$\frac{\partial h_1}{\partial t}(t,s) \le 0, \quad s < t. \tag{25}$$

This yields

$$\begin{aligned} h_1(b,s) &= 1+b-a-\frac{\gamma(b-s)}{b-s+\alpha-1} \\ &\le h_1(t,s) \le h_1(s,s) \le 1+b-a. \end{aligned} \tag{26}$$

Observe that, in this case, we have

$$\frac{2-\alpha}{\alpha-1} \le \frac{1}{b-a}. \tag{27}$$

Using the above inequality, we obtain

$$\begin{aligned} &(1+b-a)(b-s)+(\alpha-1)(1+b-a)-\gamma(b-s) \\ &\quad = s-b+(\alpha-1)(1+b-a) \\ &\quad \ge (b-a)(\alpha-2)+\alpha-1 \\ &\quad \ge 0, \end{aligned} \tag{28}$$

which implies that

$$h_1(b,s) \ge 0. \tag{29}$$

From (26) and (29), we deduce

$$0 \le h_1(t,s) \le 1+b-a. \tag{30}$$

Case 2. If $a < a^* \le b$, in this case, we have two possibilities.

(i) If $a^* \le s < b$, in this case, we have

$$\frac{\partial h_1}{\partial t}(t,s)\Big|_{t=b} \le 0. \tag{31}$$

Therefore, we conclude that

$$h_1(b,s) \le h_1(t,s) \le h_1(s,s) \le 1+b-a. \tag{32}$$

On the other hand, we have

$$\begin{aligned} &(1+b-a)(b-s+\alpha-1)-\gamma(b-s) \\ &\quad = (1+b-a-\gamma)(b-s)+(\alpha-1)(1+b-a) \\ &\quad = s-b+(\alpha-1)(1+b-a) \\ &\quad \ge (2-\gamma+b-a)(\alpha-1) \ge 0, \end{aligned} \tag{33}$$

which implies that

$$h_1(b,s) \ge 0. \tag{34}$$

From (32) and (34), we deduce

$$0 \le h_1(t,s) \le 1+b-a. \tag{35}$$

(ii) If $a < s < a^*$, in this case, we have

$$\frac{\partial h_1}{\partial t}(t,s)\Big|_{t=b} > 0. \tag{36}$$

Hence, there would exist $t^* \in (s,b)$ such that

$$\frac{\partial h_1}{\partial t}(t,s)\Big|_{t=t^*} = 0. \tag{37}$$

As mentioned above, it is easy to verify that $h_1(s,s) \ge 0$ and $h_1(b,s) \le 0$. This yields

$$h_1(t^*,s) \le h_1(b,s) \le h_1(s,s) \le \frac{r}{p}+b-a. \tag{38}$$

Then,

$$|h_1(t,s)| \le \max\{-h_1(t^*,s), 1+b-a\}. \tag{39}$$

Observe that $(\partial h_1/\partial t)(t,s)|_{t=t^*} = 0$ is equivalent to

$$(\alpha-1)\gamma(t^*-s)^{\alpha-2} = (b-s)^{\alpha-1}+(\alpha-1)(b-s)^{\alpha-2}. \tag{40}$$

Therefore, we get

$$\begin{aligned} h_1(t^*,s) &= \frac{\alpha-2}{\alpha-1}t^*+\frac{s}{\alpha-1}+1-a \\ &\ge \frac{\alpha-2}{\alpha-1}b+\frac{a}{\alpha-1}+1-a \\ &= (b-a)\frac{\alpha-2}{\alpha-1}+1. \end{aligned} \tag{41}$$

Finally, using the above inequality and (39), we obtain

$$|h_1(t,s)| \le \max\left\{1+b-a, \frac{2-\alpha}{\alpha-1}(b-a)-1\right\}, \tag{42}$$

which makes end to the proof.

Our first main result is as follows.

Theorem 6. *If (7) admits a nontrivial continuous solution, then*

$$\begin{aligned} &\int_a^b (b-s)^{\alpha-2}(b-s+\alpha-1)|\chi(s)|\,ds \\ &\quad \ge \frac{\gamma\Gamma(\alpha)}{\max\{b-a+1, ((2-\alpha)/(\alpha-1))(b-a)-1\}}. \end{aligned} \tag{43}$$

Proof. Let $B = C[a,b]$ be the Banach space endowed with norm

$$\|x\|_\infty = \max_{a\le t\le b} |x(t)|, \quad x \in B. \tag{44}$$

From Lemma 4, for all $t \in [a,b]$, we have

$$u(t) = \frac{1}{\gamma\Gamma(\alpha)} \times \int_a^b (b-s)^{\alpha-2}(b-s+\alpha-1) H(t,s)\chi(s)u(s)\,ds. \tag{45}$$

Now, an application of Lemma 3 yields

$$\|u\|_\infty \le \|u\|_\infty \frac{\max\{b-a+1, ((2-\alpha)/(\alpha-1))(b-a)-1\}}{\gamma\Gamma(\alpha)} \times \int_a^b (b-s)^{\alpha-2}(b-s+\alpha-1)\left|\chi(s)\right| ds, \tag{46}$$

from which the above theorem follows. □

2.2. Nonexistence Result of Real Zeros for a Linear Combination of Certain Mittag-Leffler Functions. Let $\alpha, \beta > 0$ be fixed. The complex function

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+\beta)}, \quad \alpha > 0, \beta > 0, z \in \mathbb{C} \tag{47}$$

is analytic in the whole complex plane; it will be referred to [12–14] as the Mittag-Leffler function with parameters (α, β).

At this stage, using the above Lyapunov-type inequality, we give an interval where a linear combination of Mittag-Leffler functions has no real zeros. More precisely, we prove

Theorem 7. *Let $1 < \alpha \le 2$. Then $E_{\alpha,2}(z) + E_{\alpha,1}(z) + zE_{\alpha,\alpha}(z)$ has no real zeros for*

$$z \in \left(\frac{-3\alpha\Gamma(\alpha)}{(1+\alpha)\max\{2, ((2-\alpha)/(\alpha-1))-1\}}, 0\right]. \tag{48}$$

Proof. Let $(a,b) = (0,1)$, and consider the fractional Sturm-Liouville eigenvalue problem

$$\begin{aligned} \left({}_0^C D^\alpha u\right)(t) + \lambda u(t) &= 0, \quad 0 < t < 1, \\ u(a) - u'(a) &= u(b) + u'(b) = 0. \end{aligned} \tag{49}$$

The real values of λ, for which there exists a non-trivial solution to (49), are called eigenvalues of (49); and the corresponding solutions are called eigenfunctions.

As established in [15], the eigenvalues of (49) must be positive; moreover, the positive number λ is an eigenvalue of (49) if and only if

$$E_{\alpha,2}(-\lambda) + E_{\alpha,1}(-\lambda) - \lambda E_{\alpha,\alpha}(-\lambda) = 0. \tag{50}$$

Thanks to Theorem 6, if a positive real eigenvalue λ of (49) exists, then

$$\lambda \int_0^1 (1-s)^{\alpha-2}(\alpha-s)\,ds \ge \frac{3\Gamma(\alpha)}{\max\{2, ((2-\alpha)/(\alpha-1))-1\}}. \tag{51}$$

Hence,

$$\lambda\left(\frac{\alpha+1}{\alpha}\right) \ge \frac{3\Gamma(\alpha)}{\max\{2, ((2-\alpha)/(\alpha-1))-1\}}, \tag{52}$$

which concludes the proof. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Authors' Contribution

All the authors contributed equally and significantly to writing this paper. All authors read and approved the final paper.

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References

[1] A. M. Liapunov, "Problème général de la stabilité du mouvement," *Annals of Mathematics Studies*, vol. 2, pp. 203–407, 1907.

[2] D. Cakmak, "Lyapunov-type integral inequalities for certain higher order differential equations," *Applied Mathematics and Computation*, vol. 216, no. 2, pp. 368–373, 2010.

[3] S. Clark and D. Hinton, "A Liapunov inequality for linear Hamiltonian systems," *Mathematical Inequalities & Applications*, vol. 1, no. 2, pp. 201–209, 1998.

[4] N. Parhi and S. Panigrahi, "Liapunov-type inequality for higher order differential equations," *Mathematica Slovaca*, vol. 52, no. 1, pp. 31–46, 2002.

[5] X. Yang, "On Liapunov-type inequality for certain higher-order differential equations," *Applied Mathematics and Computation*, vol. 134, no. 2-3, pp. 307–317, 2003.

[6] X. Yang and K. Lo, "Lyapunov-type inequality for a class of even-order differential equations," *Applied Mathematics and Computation*, vol. 215, no. 11, pp. 3884–3890, 2010.

[7] Q.-M. Zhang and X. H. Tang, "Lyapunov inequalities and stability for discrete linear Hamiltonian systems," *Applied Mathematics and Computation*, vol. 218, no. 2, pp. 574–582, 2011.

[8] B. G. Pachpatte, *Mathematical Inequalities*, North Holland Mathematical Library, Elsevier, 2005.

[9] R. A. C. Ferreira, "A Lyapunov-type inequality for a fractional boundary value problem," *Fractional Calculus and Applied Analysis*, vol. 16, no. 4, pp. 978–984, 2013.

[10] R. A. Ferreira, "On a Lyapunov-type inequality and the zeros of a certain Mittag-LEFfler function," *Journal of Mathematical Analysis and Applications*, vol. 412, no. 2, pp. 1058–1063, 2014.

[11] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, vol. 204 of *North-Holland Mathematics Studies*, Elsevier Science, Amsterdam, The Netherlands, 2006.

[12] F. Mainardi, "Fractional relaxation-oscillation and fractional diffusion-wave phenomena," *Chaos, Solitons and Fractals*, vol. 7, no. 9, pp. 1461–1477, 1996.

[13] F. Mainardi and R. Gorenflo, "On Mittag-Leffler-type functions in fractional evolution processes," *Journal of Computational and Applied Mathematics*, vol. 118, no. 1-2, pp. 283–299, 2000.

[14] I. Podlubny, *Fractional Differential Equations*, vol. 198 of *Mathematics in Science and Engineering*, Academic Press, San Diego, Calif, USA, 1999.

[15] J.-S. Duan, Z. Wang, Y.-L. Liu, and X. Qiu, "Eigenvalue problems for fractional ordinary differential equations," *Chaos, Solitons & Fractals*, vol. 46, pp. 46–53, 2013.