Research Article

On Pointwise Approximation of Conjugate Functions by Some Hump Matrix Means of Conjugate Fourier Series

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Received 20 October 2014; Revised 7 January 2015; Accepted 21 January 2015

Academic Editor: Rodolfo H. Torres

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The results generalizing some theorems on $(N, p_n)(E, \gamma)$ summability are shown. The same degrees of pointwise approximation as in earlier papers by weaker assumptions on considered functions and examined summability methods are obtained. From presented pointwise results, the estimation on norm approximation is derived. Some special cases as corollaries are also formulated.

1. Introduction

Let L^p ($1 \leq p < \infty$) (resp., L^∞) be the class of all 2π -periodic real-valued functions integrable in the Lebesgue sense with p th power (essentially bounded) over $Q = [-\pi, \pi]$ with the norm

$$\|f\| := \|f(\cdot)\|_{L^p} = \begin{cases} \left(\int_Q |f(t)|^p dt \right)^{1/p} & \text{when } 1 \leq p < \infty, \\ \text{ess sup}_{t \in Q} |f(t)| & \text{when } p = \infty \end{cases} \quad (1)$$

and consider the conjugate trigonometric Fourier series

$$\tilde{S}f(x) := \sum_{\nu=1}^{\infty} (a_\nu(f) \sin \nu x - b_\nu(f) \cos \nu x) \quad (2)$$

with the partial sums $\tilde{S}_k f$. We know that if $f \in L^1$ then

$$\tilde{f}(x) := -\frac{1}{\pi} \int_0^\pi \psi_x(t) \frac{1}{2} \cot \frac{t}{2} dt = \lim_{\epsilon \rightarrow 0^+} \tilde{f}(x, \epsilon), \quad (3)$$

where

$$\tilde{f}(x, \epsilon) := -\frac{1}{\pi} \int_\epsilon^\pi \psi_x(t) \frac{1}{2} \cot \frac{t}{2} dt, \quad (4)$$

with

$$\psi_x(t) := f(x+t) - f(x-t), \quad (5)$$

exists for almost all x [1, Th.(3.1)IV].

Let $A := (a_{n,k})$ and $B := (b_{n,k})$ be infinite lower triangular matrices of real numbers such that

$$a_{n,k} \geq 0,$$

$$b_{n,k} \geq 0,$$

$$\text{when } k = 0, 1, 2, \dots, n,$$

$$a_{n,k} = 0,$$

$$b_{n,k} = 0,$$

$$\text{when } k > n, \quad (6)$$

$$\sum_{k=0}^n a_{n,k} = 1,$$

$$\sum_{k=0}^n b_{n,k} = 1,$$

$$\text{where } n = 0, 1, 2, \dots,$$

and let, for $m = 0, 1, 2, \dots, n$,

$$\begin{aligned} A_{n,m} &= \sum_{k=0}^m a_{n,k}, \\ \bar{A}_{n,m} &= \sum_{k=m}^n a_{n,k}. \end{aligned} \quad (7)$$

Let the AB-transformation of $\tilde{S}_k f$ be given by

$$\tilde{T}_{n,A,B} f(x) := \sum_{r=0}^n \sum_{k=0}^r a_{n,r} b_{r,k} \tilde{S}_k f(x) \quad (n = 0, 1, 2, \dots). \quad (8)$$

We define two classes of sequences (see [2]).

Sequence $c := (c_k)$ of nonnegative numbers tending to zero is called the *Rest Bounded Variation Sequence*, or briefly $c \in \text{RBVS}$, if it has the property

$$\sum_{k=m}^{\infty} |c_k - c_{k+1}| \leq K(c) c_m, \quad (9)$$

for all positive integer m , where $K(c)$ is a constant depending only on c .

Sequence $c := (c_k)$ of nonnegative numbers will be called the *Head Bounded Variation Sequence*, or briefly $c \in \text{HBVS}$, if it has the property

$$\sum_{k=0}^{m-1} |c_k - c_{k+1}| \leq K(c) c_m, \quad (10)$$

for all positive integer m , or only for all $m \leq n$ if the sequence c has only finite nonzero terms and the last nonzero term is c_n .

Now, we define the other classes of sequences.

Following Leindler [3], sequence $c := (c_r)$ of nonnegative numbers tending to zero is called the *Mean Rest Bounded Variation Sequence*, or briefly $c \in \text{MRBVS}$, if it has the property

$$\sum_{r=m}^{\infty} |c_r - c_{r+1}| \leq K(c) \frac{1}{m+1} \sum_{r=m/2}^m c_r, \quad (11)$$

for all positive integer m .

Analogously as in [4], sequence $c := (c_r)$ of nonnegative numbers will be called the *Mean Head Bounded Variation Sequence*, or briefly $c \in \text{MHBVS}$, if it has the property

$$\sum_{r=0}^{n-m-1} |c_r - c_{r+1}| \leq K(c) \frac{1}{m+1} \sum_{r=n-m}^n c_r, \quad (12)$$

for all positive integers $m < n$, where the sequence c has only finite nonzero terms and the last nonzero term is c_n . It is clear that (see [5])

$$\begin{aligned} \text{RBVS} &\not\subseteq \text{MRBVS}, \\ \text{HBVS} &\not\subseteq \text{MHBVS}. \end{aligned} \quad (13)$$

Consequently, we assume that the sequence $(K(\alpha_n))_{n=0}^{\infty}$ is bounded, that is, that there exists a constant K such that

$$0 \leq K(\alpha_n) \leq K \quad (14)$$

holds for all n , where $K(\alpha_n)$ denote the constants for the sequences $\alpha_n = (a_{n,r})_{r=0}^n$, $n = 0, 1, 2, \dots$ appearing in the inequalities (11) and (12) as $K(c)$.

Now we can give the conditions to be used later on. We assume that for all n and $0 \leq m < n$

$$\sum_{r=m}^{n-1} |a_{n,r} - a_{n,r+1}| \leq K \frac{1}{m+1} \sum_{r=m/2}^m a_{n,r}, \quad (15)$$

$$\sum_{r=0}^{n-m-1} |a_{n,r} - a_{n,r+1}| \leq K \frac{1}{m+1} \sum_{r=n-m}^n a_{n,r}$$

hold if $(a_{n,r})_{r=0}^n$ belongs to MRBVS and MHBVS, for $n = 1, 2, \dots$, respectively.

We also define two hump matrices in the following way: a lower triangular matrix $C = (c_{n,k})$ is called a *maximal hump matrix* if, for each n , there exists integer $k_0 = k_0(n)$, such that $(c_{n,k})_{k=0}^{k_0-1}$ is nondecreasing for $0 \leq k < k_0$ and $(c_{n,k})_{k=k_0}^n$ is nonincreasing for $k_0 \leq k \leq n$, but otherwise we will have a *minimal hump matrix*. The hump matrices were defined and considered in [6, 7].

As a measure of approximation of $\tilde{f}$ by $\tilde{T}_{n,A,B} f$, we use the pointwise modulus of continuity of f in the space L^p defined by the formula

$$\begin{aligned} \tilde{\omega}_x^p f(\delta)_\beta &= \begin{cases} \left\{ \frac{1}{\delta} \int_0^\delta \left| \psi_x(u) \sin^\beta \frac{u}{2} \right|^p du \right\}^{1/p} & \text{when } 1 \leq p < \infty, \\ \text{ess sup}_{0 < u \leq \delta} \left| \psi_x(u) \sin^\beta \frac{u}{2} \right| & \text{when } p = \infty \end{cases} \end{aligned} \quad (16)$$

and the classical one

$$\tilde{\omega}_\beta f(\delta)_{L^p} = \sup_{0 < t \leq \delta} \left\| \sin^\beta \frac{t}{2} \psi_\cdot(t) \right\|_{L^p}. \quad (17)$$

The deviation $\tilde{T}_{n,A} f - \tilde{f} = \tilde{T}_{n,A,B} f - \tilde{f}$, with $b_{r,r} = 1$ and 0 otherwise, was estimated at the point as well as in the norm of L^p by Qureshi [8] and Lal and Nigam [9]. These results were generalized by Qureshi [10]. The next generalization was obtained by Lal [11]. In the case

$$\begin{aligned} a_{n,r} &= \frac{1}{n+1}, \quad \text{when } r = 0, 1, 2, \dots, n, \\ a_{n,r} &= 0, \quad \text{when } r > n, \\ b_{r,k} &= \frac{\binom{r}{k} \gamma^r}{(1+\gamma)^r}, \quad \text{when } k = 0, 1, 2, \dots, r, \\ b_{n,r} &= 0, \quad \text{when } k > r \text{ with } \gamma > 0, \end{aligned} \quad (18)$$

the deviation $\tilde{T}_{n,A,B} f - \tilde{f}$ was estimated by Sonker and Singh [12] as follows.

Theorem 1. Let $f(x)$ be a 2π -periodic, Lebesgue integrable function which belongs to the $\text{Lip}(\alpha, p)$ -class with $p \geq 1$ and $\alpha p \geq 1$. Then the degree of approximation of $\tilde{f}(x)$, the conjugate of $f(x)$ by $(C, 1)$ (E, γ) means of series (2), is given by

$$\left\| \frac{1}{n+1} \sum_{r=0}^n \frac{1}{(1+\gamma)^r} \sum_{k=0}^r \binom{r}{k} \gamma^k \tilde{S}_k f(\cdot) - \tilde{f}(\cdot) \right\|_{L^p} = O\left((n+1)^{-\alpha+1/p}\right), \quad (19)$$

provided

$$\left\{ \int_0^{\pi/(n+1)} \left(\frac{|\psi_x(t)|}{t^\alpha} \right)^p dt \right\}^{1/p} = O\left((n+1)^{-1}\right), \quad (20)$$

$$\left\{ \int_{\pi/(n+1)}^\pi \left(\frac{t^{-\delta} |\psi_x(t)|}{t^\alpha} \right)^p dt \right\}^{1/p} = O\left((n+1)^\delta\right),$$

where δ is an arbitrary positive number with $(\alpha + \delta)q + 1 < 0$ and $p^{-1} + q^{-1} = 1$, $p > 1$.

In this paper we will consider the deviations $\tilde{T}_{n,A,B}f(\cdot) - \tilde{f}(\cdot)$ and $\tilde{T}_{n,A,B}f(\cdot) - \tilde{f}(\cdot, \pi/(n+1))$ in general form. In the theorems we formulate the general conditions for the functions and the modulus of continuity obtaining the same degrees of approximation as above and sometimes essentially better one. Finally, we also give some results on norm approximation with essentially better degrees of approximation. The obtained results generalize the results from [4, 9].

We will write $I_1 \ll I_2$ if there exists positive constant K , sometimes depending on some parameters, such that $I_1 \leq KI_2$.

2. Statement of the Results

Let

$$L^p(\tilde{w}_x)_\beta = \left\{ f \in L^p : \tilde{w}_x^\beta f(\delta)_\beta \leq \tilde{w}_x(\delta) \right\}, \quad (21)$$

where $\tilde{w}_x$ is a positive, with $\tilde{w}_x(0) = 0$, and nondecreasing continuous function.

We can now formulate our main results. At the beginning, we formulate the results on the degrees of pointwise summability of conjugate series.

Theorem 2. Let $f \in L^1$. If matrix A is a maximal or minimal hump matrix with $k_0^{-1} = O(n^{-1})$ such that $(a_{n,k})_{k=0}^n \in \text{MHBVS} \cup \text{MRBVS}$ and

$$\left| \sum_{r=\mu}^v \sum_{k=0}^r b_{r,k} \cos \frac{(2k+1)t}{2} \right| \ll \tau \quad (22)$$

for $0 \leq \mu \leq \nu$, then

$$\left| \tilde{T}_{n,A,B}f(x) - \tilde{f}\left(x, \frac{\pi}{n+1}\right) \right| = O\left((n+1)^\beta \left[\frac{1}{n+1} \sum_{k=0}^n \tilde{w}_x^1 f\left(\frac{\pi}{k+1}\right)_\beta \right]\right), \quad (23)$$

for almost all considered x , where $\beta \in [0, 1]$.

Theorem 3. Let $f \in L^p(\tilde{w}_x)_\beta$ with $1 < p < \infty$, and let $\tilde{w}_x$ satisfy

$$\left\{ \int_0^{\pi/(n+1)} \left(\frac{\tilde{w}_x(t)}{t \sin^\beta t/2} \right)^{p/(p-1)} dt \right\}^{(p-1)/p} = O\left((n+1)^{\beta+1/p} \tilde{w}_x\left(\frac{\pi}{n+1}\right)\right), \quad (24)$$

$$\left\{ \int_0^{\pi/(n+1)} \left(\frac{|\psi_x(t)|}{\tilde{w}_x(t)} \right)^p \sin^{\beta p} \frac{t}{2} dt \right\}^{1/p} = O\left((n+1)^{-1/p}\right) \quad (25)$$

with some $\beta \geq 0$. If the entries of matrix B satisfy condition (22) for $0 \leq \mu \leq \nu$ and if matrix A is a maximal or minimal hump matrix with $k_0^{-1} = O(n^{-1})$ such that $(a_{n,k})_{k=0}^n \in \text{MHBVS} \cup \text{MRBVS}$, then

$$\left| \tilde{T}_{n,A,B}f(x) - \tilde{f}(x) \right| = O\left((n+1)^\beta \left[\frac{1}{n+1} \sum_{k=0}^n \tilde{w}_x\left(\frac{\pi}{k+1}\right) \right]\right), \quad (26)$$

for almost all considered x such that $\tilde{f}(x)$ exists.

Next, we formulate the results on estimates of L^p norm of the deviation considered above. In case of the deviation $\tilde{T}_{n,A,B}f(\cdot) - \tilde{f}(\cdot)$, let

$$L^p(\tilde{\omega})_\beta = \left\{ f \in L^p : \tilde{\omega}_\beta f(\delta)_{L^p} \leq \tilde{\omega}(\delta) \right\}, \quad (27)$$

where $\tilde{\omega}$ is positive, with $\tilde{\omega}(0) = 0$, and almost nondecreasing continuous function.

Theorem 4. Let $f \in L^p$ ($1 \leq p \leq \infty$). If the entries of matrix B satisfy condition (22) for $0 \leq \mu \leq \nu$ and if matrix A is a maximal or minimal hump matrix with $k_0^{-1} = O(n^{-1})$ such that $(a_{n,k})_{k=0}^n \in \text{MHBVS} \cup \text{MRBVS}$, then

$$\left\| \tilde{T}_{n,A,B}f(\cdot) - \tilde{f}\left(\cdot, \frac{\pi}{n+1}\right) \right\|_{L^p} = O\left((n+1)^\beta \left[\frac{1}{n+1} \sum_{k=0}^n \tilde{\omega}_\beta f\left(\frac{\pi}{k+1}\right)_{L^p} \right]\right), \quad (28)$$

where $0 \leq \beta \leq 1$.

Theorem 5. Let $f \in L^p(\tilde{\omega})_\beta$ with $1 < p < \infty$, where $\tilde{\omega}$ instead of $\tilde{w}_x$ satisfies (24) with some $\beta \geq 0$. If the entries of matrix B satisfy condition (22) for $0 \leq \mu \leq \nu$ and if matrix A is

a maximal or minimal hump matrix with $k_0^{-1} = O(n^{-1})$ such that $(a_{n,k})_{k=0}^n \in \text{MHBVS} \cup \text{MRBVS}$, then

$$\begin{aligned} & \|\tilde{T}_{n,A,B}f(\cdot) - \tilde{f}(\cdot)\|_{L^p} \\ &= O\left((n+1)^\beta \left[\frac{1}{n+1} \sum_{k=0}^n \tilde{\omega}\left(\frac{\pi}{k+1}\right) \right]\right). \end{aligned} \quad (29)$$

Finally, we give corollary and remarks as an application of our results.

Taking $a_{n,r} = p_{n-r} / \sum_{v=0}^n p_v$ when $r = 0, 1, 2, \dots, n$ and $a_{n,r} = 0$ when $r > n$ with $p_v > 0$, $p_v \leq p_{v+1}$ and $b_{r,k} = \binom{r}{k} \gamma^k / (1+\gamma)^r$ when $k = 0, 1, 2, \dots, r$ and $b_{r,k} = 0$ when $k > r$ with $\gamma > 0$, Theorem 2 (Theorem 3 analogously) implies the following.

Corollary 6. If $f \in L^1$, then

$$\begin{aligned} & \left| \frac{1}{\sum_{v=0}^n p_v} \sum_{r=0}^n \frac{p_r}{(1+\gamma)^r} \sum_{k=0}^r \binom{r}{k} \gamma^k \tilde{S}_k f(x) \right. \\ & \quad \left. - \tilde{f}\left(x, \frac{\pi}{n+1}\right) \right| \\ &= O\left((n+1)^\beta \left[\frac{1}{n+1} \sum_{k=0}^n \tilde{\omega}_x^1\left(\frac{\pi}{k+1}\right)_\beta \right]\right), \\ & \left| \frac{1}{\sum_{v=0}^n p_v} \sum_{r=0}^n \frac{p_{n-r}}{(1+\gamma)^r} \sum_{k=0}^r \binom{r}{k} \gamma^k \tilde{S}_k f(x) \right. \\ & \quad \left. - \tilde{f}\left(x, \frac{\pi}{n+1}\right) \right| \\ &= O\left((n+1)^\beta \left[\frac{1}{n+1} \sum_{k=0}^n \tilde{\omega}_x^1\left(\frac{\pi}{k+1}\right)_\beta \right]\right), \end{aligned} \quad (30)$$

for almost all considered x , where $0 \leq \beta \leq 1$, $p_v > 0$, $p_v \leq p_{v+1}$, and $\gamma > 0$.

Remark 7. In special case, if $p_r = p_{n-r} = 1$, then

$$\begin{aligned} & \left| \frac{1}{n+1} \sum_{r=0}^n \frac{1}{(1+\gamma)^r} \sum_{k=0}^r \binom{r}{k} \gamma^k \tilde{S}_k f(x) - \tilde{f}\left(x, \frac{\pi}{n+1}\right) \right| \\ &= O\left((n+1)^\beta \left[\frac{1}{n+1} \sum_{k=0}^n \tilde{\omega}_x^1\left(\frac{\pi}{k+1}\right)_\beta \right]\right), \end{aligned} \quad (31)$$

and if $f \in \text{Lip}(\alpha, p)$, then

$$\begin{aligned} & \left| \frac{1}{n+1} \sum_{r=0}^n \frac{1}{(1+\gamma)^r} \sum_{k=0}^r \binom{r}{k} \gamma^k \tilde{S}_k f(x) - \tilde{f}\left(x, \frac{\pi}{n+1}\right) \right| \\ &= \begin{cases} O\left((n+1)^{\beta-\alpha}\right) & \text{when } 0 < \alpha < 1 \\ O\left((n+1)^{\beta-1} \log(n+1)\right) & \text{when } \alpha = 1, \end{cases} \end{aligned} \quad (32)$$

for almost all considered x , where $0 \leq \beta \leq 1$ and $\gamma > 0$.

Remark 8. Taking $\beta = 0$, we have, by Theorem 3 with $\tilde{\omega}_x(\delta) = \delta^\alpha$, for $0 < \alpha < 1$ and $\alpha p > 1$, the estimate like in [12] with the better order of approximation without any additional assumptions.

Remark 9. Analyzing the proofs of Theorems 2–5, we can deduce that, taking the assumption $(a_{n,k})_{k=0}^n \in \text{RBVS}$ or $(a_{n,k})_{k=0}^n \in \text{HBVS}$ instead of $(a_{n,k})_{k=0}^n \in \text{MRBVS}$ or $(a_{n,k})_{k=0}^n \in \text{MHBVS}$, respectively, we obtain the results like that from [13].

3. Auxiliary Results

We begin this section by some notations following A. Zygmund [1, Section 5 of Chapter II].

It is clear that

$$\begin{aligned} \tilde{S}_k f(x) &= -\frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) \tilde{D}_k(t) dt, \\ \tilde{T}_{n,A,B} f(x) &= -\frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) \sum_{r=0}^n \sum_{k=0}^r a_{n,r} b_{r,k} \tilde{D}_k(t) dt, \end{aligned} \quad (33)$$

where

$$\tilde{D}_k(t) = \sum_{v=0}^k \sin vt = \frac{\cos(t/2) - \cos((2k+1)t/2)}{2 \sin(t/2)}. \quad (34)$$

Hence

$$\begin{aligned} & \tilde{T}_{n,A,B} f(x) - \tilde{f}\left(x, \frac{\pi}{n+1}\right) \\ &= -\frac{1}{\pi} \int_0^{\pi/(n+1)} \psi_x(t) \sum_{r=0}^n \sum_{k=0}^r a_{n,r} b_{r,k} \tilde{D}_k(t) dt \\ & \quad + \frac{1}{\pi} \int_{\pi/(n+1)}^{\pi} \psi_x(t) \sum_{r=0}^n \sum_{k=0}^r a_{n,r} b_{r,k} \tilde{D}_k^\circ(t) dt, \end{aligned} \quad (35)$$

$$\begin{aligned} & \tilde{T}_{n,A,B} f(x) - \tilde{f}(x) \\ &= \frac{1}{\pi} \int_0^{\pi} \psi_x(t) \sum_{r=0}^n \sum_{k=0}^r a_{n,r} b_{r,k} \tilde{D}_k^\circ(t) dt, \end{aligned}$$

where

$$\tilde{D}_k^\circ(t) = \frac{1}{2} \cot \frac{t}{2} - \sum_{v=0}^k \sin vt = \frac{\cos((2k+1)t/2)}{2 \sin(t/2)}. \quad (36)$$

Now, we formulate some estimates for the conjugate Dirichlet kernels.

Lemma 10 (see [1]). If $0 < |t| \leq \pi/2$, then

$$|\tilde{D}_k^\circ(t)| \leq \frac{\pi}{2|t|} \quad (37)$$

and for any real t one has

$$|\tilde{D}_k(t)| \leq \frac{1}{2} k(k+1) |t|, \quad (38)$$

$$|\tilde{D}_k(t)| \leq k+1.$$

Lemma 11. Let $(b_{r,k})_{k=0}^r$ be such that (22) for $0 \leq \mu \leq \nu$ holds. If $(a_{n,k})_{k=0}^n \in \text{MRBVS}$, then

$$\left| \sum_{r=0}^n \sum_{k=0}^r a_{n,r} b_{r,k} \widetilde{D}_k^\circ(t) \right| \ll \tau A_{n,\tau}, \quad (39)$$

and if $(a_{n,k})_{k=0}^n \in \text{MHBVS}$, then

$$\left| \sum_{r=0}^n \sum_{k=0}^r a_{n,r} b_{r,k} \widetilde{D}_k^\circ(t) \right| \ll \tau \bar{A}_{n,n-\tau} \quad (40)$$

with $\tau = [\pi/t]$ and $t \in [\pi/(n+1), \pi]$, for $n = 0, 1, 2, \dots$

Proof. Let

$$K_n(t) := \sum_{r=0}^n \sum_{k=0}^r a_{n,r} b_{r,k} \cos \frac{(2k+1)t}{2}. \quad (41)$$

The relation $(a_{n,k})_{k=0}^n \in \text{MRBVS}$ implies

$$\begin{aligned} a_{n,s} - a_{n,m} &\leq |a_{n,m} - a_{n,s}| \leq \sum_{k=m}^{s-1} |a_{n,k} - a_{n,k+1}| \\ &\ll \sum_{k=r}^{n-1} |a_{n,k} - a_{n,k+1}| \ll \frac{1}{r+1} \sum_{k \geq r/2}^r a_{n,k}, \quad (42) \\ &\quad (0 \leq r \leq m < s \leq n) \end{aligned}$$

whence

$$a_{n,s} \ll a_{n,m} + \frac{1}{r+1} \sum_{k \geq r/2}^r a_{n,k}, \quad (0 \leq r \leq m < s \leq n), \quad (43)$$

and thus, by our assumption,

$$\left| \sum_{l=\tau+1}^r \sum_{k=0}^l b_{l,k} \cos \frac{(2k+1)t}{2} \right| \ll \tau, \quad (44)$$

we obtain

$$\begin{aligned} |K_n(t)| &= \left| \left(\sum_{r=0}^{\tau} + \sum_{r=\tau+1}^n \right) a_{n,r} \sum_{k=0}^r b_{r,k} \cos \frac{(2k+1)t}{2} \right| \\ &\leq \sum_{r=0}^{\tau} a_{n,r} \sum_{k=0}^r b_{r,k} + \sum_{r=\tau+1}^{n-1} |a_{n,r} - a_{n,r+1}| \\ &\quad \cdot \left| \sum_{l=\tau+1}^r \sum_{k=0}^l b_{l,k} \cos \frac{(2k+1)t}{2} \right| \\ &\quad + a_{n,n} \left| \sum_{l=\tau+1}^n \sum_{k=0}^l b_{l,k} \cos \frac{(2k+1)t}{2} \right| \ll A_{n,\tau} \\ &\quad + \left(\frac{1}{\tau+1} \sum_{k \geq \tau/2}^{\tau} a_{n,k} + \frac{1}{\tau+1} \sum_{k \geq \tau/2}^{\tau} a_{n,n} \right) \tau \leq A_{n,\tau} \\ &\quad + \left[\frac{1}{\tau+1} \sum_{k \geq \tau/2}^{\tau} a_{n,k} \right. \\ &\quad \left. + \frac{1}{\tau+1} \sum_{k \geq \tau/2}^{\tau} \left(a_{n,k} + \frac{1}{k+1} \sum_{l \geq k/2}^k a_{n,l} \right) \right] \tau \leq A_{n,\tau} \quad (45) \\ &\quad + \left[\frac{1}{\tau+1} \sum_{k \geq \tau/2}^{\tau} a_{n,k} \right. \\ &\quad \left. + \frac{1}{\tau+1} \sum_{k \geq \tau/2}^{\tau} \left(a_{n,k} + \frac{1}{\tau/2+1} \sum_{l \geq \tau/4}^{\tau} a_{n,l} \right) \right] \tau \leq A_{n,\tau} \\ &\quad + \left[\frac{1}{\tau+1} \sum_{k \geq \tau/2}^{\tau} a_{n,k} \right. \\ &\quad \left. + \frac{1}{\tau+1} \sum_{k \geq \tau/2}^{\tau} \left(a_{n,k} + \frac{1}{\tau/2+1} \sum_{l \geq \tau/4}^{\tau} a_{n,l} \right) \right] \tau \leq A_{n,\tau} \\ &\quad + \left[\frac{1}{\tau+1} \sum_{k \geq \tau/2}^{\tau} a_{n,k} + \frac{1}{\tau+1} \sum_{k \geq \tau/2}^{\tau} a_{n,k} + \frac{1}{\tau+1} \sum_{l=0}^{\tau} a_{n,l} \right] \\ &\quad \cdot \tau \leq A_{n,\tau} + 3 \sum_{l=0}^{\tau} a_{n,l} = 4A_{n,\tau}. \end{aligned}$$

The relation $(a_{n,k})_{k=0}^n \in \text{MHBVS}$ implies

$$\begin{aligned} a_{n,m} - a_{n,s} &\leq |a_{n,m} - a_{n,s}| \leq \sum_{k=m}^{s-1} |a_{n,k} - a_{n,k+1}| \\ &\ll \sum_{k=0}^{r-1} |a_{n,k} - a_{n,k+1}| \ll \frac{1}{n-r+1} \sum_{k=r}^n a_{n,k}, \quad (46) \\ &\quad (0 \leq m < s \leq r \leq n) \end{aligned}$$

whence

$$a_{n,m} \ll a_{n,s} + \frac{1}{n-r+1} \sum_{k=r}^n a_{n,k}, \quad (47)$$

$$(0 \leq m < s \leq r \leq n),$$

and thus, by our assumption,

$$\left| \sum_{l=0}^r \sum_{k=0}^l b_{l,k} \cos \frac{(2k+1)t}{2} \right| \ll \tau, \quad (48)$$

we get

$$\begin{aligned} |K_n(t)| &= \left| \left(\sum_{r=n-\tau}^n + \sum_{r=0}^{n-\tau} \right) a_{n,r} \sum_{k=0}^r b_{r,k} \cos \frac{(2k+1)t}{2} \right| \\ &= \bar{A}_{n,n-\tau} + \sum_{r=0}^{n-\tau-1} |a_{n,r} - a_{n,n-\tau}| \\ &\quad \cdot \left| \sum_{l=0}^r \sum_{k=0}^l b_{l,k} \cos \frac{(2k+1)t}{2} \right| \\ &\quad + a_{n,n-\tau-1} \left| \sum_{l=0}^{n-\tau} \sum_{k=0}^l b_{l,k} \cos \frac{(2k+1)t}{2} \right| \ll \bar{A}_{n,n-\tau} \\ &\quad + \left[\frac{1}{\tau+1} \sum_{k=n-\tau}^n a_{n,k} + a_{n,n-\tau-1} \right] \tau \ll \bar{A}_{n,n-\tau} \\ &\quad + \left[\frac{1}{\tau+1} \sum_{k=n-\tau}^n a_{n,k} + \frac{1}{\tau+1} \sum_{k=n-\tau}^{n-\tau/2} a_{n,n-\tau-1} \right] \tau \\ &\ll \bar{A}_{n,n-\tau} + \left[\frac{1}{\tau+1} \sum_{k=n-\tau}^n a_{n,k} \right. \\ &\quad \left. + \frac{1}{\tau+1} \sum_{k=n-\tau}^{n-\tau/2} \left(a_{n,k} + \frac{1}{\tau+1} \sum_{\nu=n-\tau/2}^n a_{n,\nu} \right) \right] \tau \\ &\leq \bar{A}_{n,n-\tau} + \sum_{k=n-\tau}^n a_{n,k} + \sum_{k=n-\tau}^n \frac{1}{\tau+1} \sum_{\nu=n-\tau}^n a_{n,\nu} \\ &\ll \bar{A}_{n,n-\tau}. \end{aligned} \quad (49)$$

Now, our proof is complete. $\square$

Lemma 12. If matrix A is a maximal or minimal hump matrix with $k_0^{-1} = O(n^{-1})$, then

$$\frac{n+1}{m+1} \sum_{r=0}^m a_{n,r} = O(1) \quad (50)$$

or

$$\frac{n+1}{m+1} \sum_{r=0}^m a_{n,n-r} = O(1) \quad (51)$$

for $m = 0, 1, 2, \dots$, respectively.

Proof. Since the above formulas are similar, we prove the first one only. If $0 < m \leq k_0$, then

$$\begin{aligned} &\frac{1}{m} \sum_{r=0}^{m-1} a_{n,r} - \frac{1}{m+1} \sum_{r=0}^m a_{n,r} \\ &= \frac{1}{m(m+1)} \sum_{r=0}^m a_{n,r} - \frac{a_{n,m}}{m} \\ &= \frac{1}{m(m+1)} \sum_{r=0}^m (a_{n,r} - a_{n,m}) \leq 0, \end{aligned} \quad (52)$$

whence $(1/(m+1)) \sum_{r=0}^m a_{n,r}$ nondecreases with respect to m and therefore

$$\begin{aligned} \frac{1}{m+1} \sum_{r=0}^m a_{n,r} &\leq \frac{1}{k_0+1} \sum_{r=0}^{k_0} a_{n,r} \leq \frac{1}{k_0+1} \sum_{r=0}^n a_{n,r} \\ &= \frac{1}{k_0+1} = O(n^{-1}). \end{aligned} \quad (53)$$

But if $m \geq k_0$, then

$$\begin{aligned} \frac{1}{m+1} \sum_{r=0}^m a_{n,r} &\leq \frac{1}{k_0+1} \sum_{r=0}^m a_{n,r} \leq \frac{1}{k_0+1} \sum_{r=0}^n a_{n,r} \\ &= \frac{1}{k_0+1} = O(n^{-1}) \end{aligned} \quad (54)$$

and our proof is complete. $\square$

Lemma 13. If $f \in L^p$ ($1 \leq p \leq \infty$) and $\beta \geq 0$, then

$$\bar{w}_x^p f\left(\frac{\pi}{n+1}\right)_\beta \leq \left\{ \frac{2}{n+1} \sum_{r=0}^n \left[\bar{w}_x^p f\left(\frac{\pi}{r+1}\right)_\beta \right]^p \right\}^{1/p} \quad (55)$$

holds for every natural n and all real x .

Proof. The proof follows by the easy account

$$\begin{aligned} &\left[\bar{w}_x^p f\left(\frac{\pi}{n+1}\right)_\beta \right]^p \\ &= \frac{n+1}{\pi} \int_0^{\pi/(n+1)} |\psi_x(u)|^p du \sum_{r=0}^n \frac{2(r+1)}{(n+1)(n+2)} \\ &\leq \frac{2}{n+1} \sum_{r=0}^n \frac{(r+1)(n+1)}{\pi(n+2)} \int_0^{\pi/(n+1)} |\psi_x(u)|^p du \\ &\leq \frac{2}{n+1} \sum_{r=0}^n \frac{r+1}{\pi} \int_0^{\pi/(r+1)} |\psi_x(u)|^p du \\ &= 2 \frac{1}{n+1} \sum_{r=0}^n \left[\bar{w}_x^p f\left(\frac{\pi}{r+1}\right)_\beta \right]^p. \end{aligned} \quad (56)$$

Now, our proof is complete. $\square$

4. Proofs of the Results

4.1. *Proof of Theorem 2.* We start with the obvious relations

$$\begin{aligned} \tilde{T}_{n,A,B}f(x) - \tilde{f}\left(x, \frac{\pi}{n+1}\right) &= -\frac{1}{\pi} \int_0^{\pi/(n+1)} \psi_x(t) \sum_{r=0}^n \sum_{k=0}^r a_{n,r} b_{r,k} \widetilde{D}_k(t) dt \\ &\quad + \frac{1}{\pi} \int_{\pi/(n+1)}^{\pi} \psi_x(t) \sum_{r=0}^n \sum_{k=0}^r a_{n,r} b_{r,k} \widetilde{D}_k^\circ(t) dt = \widetilde{I}_1 + \widetilde{I}_2^\circ, \\ \left| \tilde{T}_{n,A}f(x) - \tilde{f}\left(x, \frac{\pi}{n+1}\right) \right| &\leq |\widetilde{I}_1| + |\widetilde{I}_2^\circ|. \end{aligned} \quad (57)$$

By Lemmas 10 and 13, we have

$$\begin{aligned} |\widetilde{I}_1| &\ll (n+1)^2 \int_0^{\pi/(n+1)} t |\psi_x(t)| dt \\ &\leq (n+1)^2 \int_0^{\pi/(n+1)} t |\psi_x(t)| \sin^{\frac{t}{2}} \sin^{-\beta} \frac{t}{2} dt \\ &\ll (n+1)^2 \int_0^{\pi/(n+1)} |\psi_x(t)| \sin^{\frac{t}{2}} dt \left(\frac{\pi}{n+1}\right)^{1-\beta} \\ &\ll (n+1)^\beta \bar{w}_x^1 f\left(\frac{\pi}{n+1}\right)_\beta \\ &\ll (n+1)^{\beta-1} \sum_{k=0}^n \bar{w}_x^1 f\left(\frac{\pi}{k+1}\right)_\beta, \end{aligned} \quad (58)$$

for $\beta \in [0, 1]$.

Using Lemmas 11 and 12 we obtain

$$\begin{aligned} |\widetilde{I}_2^\circ| &\ll \int_{\pi/(n+1)}^{\pi} \frac{|\psi_x(t)|}{t} \sum_{k=0}^{\tau} a_{n,k} dt \\ &\ll \int_{\pi/(n+1)}^{\pi} \frac{|\psi_x(t)|}{t} \frac{\tau}{n+1} dt \end{aligned} \quad (59)$$

or

$$\begin{aligned} |\widetilde{I}_2^\circ| &\ll \int_{\pi/(n+1)}^{\pi} \frac{|\psi_x(t)|}{t} \sum_{k=n-\tau}^n a_{n,k} dt \\ &= \int_{\pi/(n+1)}^{\pi} \frac{|\psi_x(t)|}{t} \sum_{k=0}^{\tau} a_{n,n-k} dt \\ &\ll \int_{\pi/(n+1)}^{\pi} \frac{|\psi_x(t)|}{t} \frac{\tau}{n+1} dt, \end{aligned} \quad (60)$$

whence

$$\begin{aligned} |\widetilde{I}_2^\circ| &\ll \frac{1}{n+1} \int_{\pi/(n+1)}^{\pi} t^{-2} \left[|\psi_x(t)| \sin^{\frac{t}{2}} \right] \frac{1}{\sin^{\beta}(t/2)} dt \\ &\ll (n+1)^\beta \frac{1}{n+1} \int_{\pi/(n+1)}^{\pi} t^{-2} \left[|\psi_x(t)| \sin^{\frac{t}{2}} \right] dt \\ &= (n+1)^\beta \frac{1}{n+1} \\ &\quad \cdot \int_{\pi/(n+1)}^{\pi} t^{-2} \frac{d}{dt} \left[\int_0^t |\psi_x(u)| \sin^{\frac{u}{2}} du \right] dt = (n+1)^{\beta-1} \left\{ \left[t^{-2} \int_0^t |\psi_x(u)| \sin^{\frac{u}{2}} du \right]_{t=\pi/(n+1)}^{\pi} \right. \\ &\quad \left. + 2 \int_{\pi/(n+1)}^{\pi} t^{-3} \left[\int_0^t |\psi_x(u)| \sin^{\frac{u}{2}} du \right] dt \right\} = (n+1)^{\beta-1} \\ &\quad \cdot \left\{ \left[\pi^{-1} \bar{w}_x^1 f(\pi)_\beta - \left(\frac{\pi}{n+1}\right)^{-1} \bar{w}_x^1 f\left(\frac{\pi}{n+1}\right)_\beta \right] \right. \\ &\quad \left. + 2 \int_{\pi/(n+1)}^{\pi} t^{-2} \bar{w}_x^1 f(t)_\beta dt \right\} \ll (n+1)^{\beta-1} \\ &\quad \cdot \left\{ \sum_{k=0}^n \bar{w}_x^1 f\left(\frac{\pi}{k+1}\right)_\beta \right. \\ &\quad \left. + \int_1^{n+1} \bar{w}_x^1 f\left(\frac{\pi}{t}\right)_\beta dt \right\} \ll (n+1)^{\beta-1} \\ &\quad \cdot \sum_{k=0}^n \bar{w}_x^1 f\left(\frac{\pi}{k+1}\right)_\beta. \end{aligned} \quad (61)$$

Collecting these estimates, we obtain the desired result.

4.2. *Proof of Theorem 3.* We start with the obvious relations

$$\begin{aligned} \tilde{T}_{n,A}f(x) - \tilde{f}(x) &= \frac{1}{\pi} \int_0^{\pi/(n+1)} \psi_x(t) \sum_{r=0}^n \sum_{k=0}^r a_{n,r} b_{r,k} \widetilde{D}_k(t) dt \\ &\quad + \frac{1}{\pi} \int_{\pi/(n+1)}^{\pi} \psi_x(t) \sum_{r=0}^n \sum_{k=0}^r a_{n,r} b_{r,k} \widetilde{D}_k^\circ(t) dt \\ &= \widetilde{I}_1^\circ + \widetilde{I}_2^\circ, \\ \left| \tilde{T}_{n,A}f(x) - \tilde{f}(x) \right| &\leq |\widetilde{I}_1^\circ| + |\widetilde{I}_2^\circ|. \end{aligned} \quad (62)$$

By the Hölder inequality ($1/p + 1/q = 1$), Lemma 10, (25), and (24),

$$\begin{aligned} |\widetilde{I}_1| &\ll \int_0^{\pi/(n+1)} \frac{|\psi_x(t)|}{t} dt \\ &\leq \left\{ \int_0^{\pi/(n+1)} \left[\frac{|\psi_x(t)|}{\widetilde{w}_x(t)} \sin^\beta \frac{t}{2} \right]^p dt \right\}^{1/p} \\ &\quad \cdot \left\{ \int_0^{\pi/(n+1)} \left[\frac{\widetilde{w}_x(t)}{t \sin^\beta(t/2)} \right]^q dt \right\}^{1/q} \ll (n+1)^\beta \quad (63) \\ &\quad \cdot \widetilde{w}_x\left(\frac{\pi}{n+1}\right) = (n+1)^{\beta-1} \widetilde{w}_x\left(\frac{\pi}{n+1}\right) \sum_{k=0}^n 1 \\ &\ll (n+1)^{\beta-1} \sum_{k=0}^n \widetilde{w}_x\left(\frac{\pi}{k+1}\right). \end{aligned}$$

We can estimate the term $|\widetilde{I}_2^\circ|$ by the same way like in the proof of Theorem 2:

$$\begin{aligned} |\widetilde{I}_2^\circ| &\ll \int_{\pi/(n+1)}^\pi \frac{|\psi_x(t)|}{t} \sum_{k=n-\tau}^n a_{n,k} dt \\ &= \int_{\pi/(n+1)}^\pi \frac{|\psi_x(t)|}{t} \sum_{k=0}^\tau a_{n,n-k} dt \\ &\ll \int_{\pi/(n+1)}^\pi \frac{|\psi_x(t)|}{t} \frac{\tau}{n+1} dt \quad (64) \\ &\ll (n+1)^{\beta-1} \sum_{k=0}^n \widetilde{w}_x^1 f\left(\frac{\pi}{k+1}\right)_\beta \\ &\ll (n+1)^{\beta-1} \sum_{k=0}^n \widetilde{w}_x\left(\frac{\pi}{k+1}\right). \end{aligned}$$

Collecting these estimates, we obtain the desired result.

4.3. Proofs of Theorems 4-5. The proofs are similar to these above and follow from the evident inequality

$$\|\widetilde{w}^p f(\delta)_\beta\|_{L^p} \ll \widetilde{w}_\beta f(\delta)_{L^p} \quad (65)$$

and additionally in case of Theorem 5 from the estimate

$$\begin{aligned} &\left\| \left\{ \int_0^{\pi/(n+1)} \left(\frac{|\psi_x(t)|}{\widetilde{w}(t)} \right)^p \sin^{\beta p} \frac{t}{2} dt \right\}^{1/p} \right\|_{L^p} \\ &\leq \left\{ \int_0^{\pi/(n+1)} \left(\frac{\|\psi_x \sin^\beta \frac{t}{2}\|_{L^p}}{\widetilde{w}(t)} \right) dt \right\}^{1/p} \quad (66) \\ &\leq \left\{ \int_0^{\pi/(n+1)} \left(\frac{\widetilde{w}_\beta f(t)_{L^p}}{\widetilde{w}(t)} \right)^p dt \right\}^{1/p} \\ &\leq \left\{ \int_0^{\pi/(n+1)} dt \right\}^{1/p} = O_x((n+1)^{-1/p}) \end{aligned}$$

for $f \in L^p(\widetilde{w})_\beta$.

4.4. Proof of Corollary 6. First of all we note that our matrix A is the hump matrix with $k_0 = n$.

Next, we have to verify the assumptions of Lemma 11. For the first one, we note that any nondecreasing sequence belongs to the class MRBVS and any nonincreasing sequence belongs to the class MHBVS. The second one follows from the following calculations:

$$\begin{aligned} &\left| \sum_{r=\mu}^v (1+\gamma)^{-r} \sum_{k=0}^r \binom{r}{k} \gamma^k \cos \frac{(2k+1)t}{2} \right| \\ &= \left| \operatorname{Re} \sum_{r=\mu}^v (1+\gamma)^{-r} \sum_{k=0}^r \binom{r}{k} \gamma^k \exp \frac{(2k+1)it}{2} \right| \\ &= \left| \operatorname{Re} \sum_{r=\mu}^v (1+\gamma)^{-r} \exp \frac{it}{2} \sum_{k=0}^r \binom{r}{k} \gamma^k \exp ikt \right| \\ &= \left| \operatorname{Re} \sum_{r=\mu}^v (1+\gamma)^{-r} \exp \frac{it}{2} (1+\gamma \exp it)^r \right| \\ &= \left| \operatorname{Re} \left[\exp \frac{it}{2} \sum_{r=\mu}^v \left(\frac{1+\gamma \exp it}{1+\gamma} \right)^r \right] \right| = \left| \operatorname{Re} \left[\exp \frac{it}{2} \right. \right. \\ &\quad \cdot \left. \left(\frac{1+\gamma \exp it}{1+\gamma} \right)^\mu \right. \\ &\quad \cdot \left. \left. \frac{1 - ((1+\gamma \exp it)/(1+\gamma))^{\nu-\mu+1}}{1 - (1+\gamma \exp it)/(1+\gamma)} \right] \right| \quad (67) \\ &\leq \left| \frac{1 - ((1+\gamma \exp it)/(1+\gamma))^{\nu-\mu+1}}{1 - (1+\gamma \exp it)/(1+\gamma)} \right| \\ &\leq \frac{2}{|1 - (1+\gamma \exp it)/(1+\gamma)|} \\ &= \frac{2(1+\gamma)}{|1+\gamma - 1 - \gamma \exp it|} = \frac{2(1+\gamma)}{\gamma |1 - \exp it|} \\ &= \frac{2(1+\gamma)}{\gamma \sqrt{(1 - \cos t)^2 + \sin^2 t}} = \frac{2(1+\gamma)}{\gamma \sqrt{2 - 2 \cos t}} \\ &= \frac{2(1+\gamma)}{\gamma \sqrt{2} \sqrt{2 \sin^2(t/2)}} = \frac{1+\gamma}{\gamma |\sin(t/2)|} \leq \frac{1+\gamma}{\gamma} \frac{\pi}{t} \\ &\ll \tau. \end{aligned}$$

Thus proof is complete.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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