Research Article

Continuous Dependence of the Solutions of Nonlinear Integral Quadratic Volterra Equation on the Parameter

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We prove results on the existence and continuous dependence of solutions of a nonlinear quadratic integral Volterra equation on a parameter. This dependence is investigated in terms of Hausdorff distance. The considerations are placed in the Banach space and the Fréchet space.

1. Introduction

In this paper we investigate the following nonlinear quadratic integral Volterra equation:

$$x(t) = a(t) + f(t, x(t)) \int_0^t v(t, s, x(s)) ds, \quad t \in J, \quad (1)$$

both on a bounded interval (i.e., $J = [0, T]$) and on an unbounded one ($J = \mathbb{R}_+ = [0, \infty)$), where a , f , and v are given functions. We will study (1) in the Banach space $C([0, T])$ when $J = [0, T]$ and in the Fréchet space $C(\mathbb{R}_+)$ when $J = \mathbb{R}_+$.

The main aim of the paper is to formulate assumptions that guarantee continuous dependence of solutions of (1) on parameter. In our considerations we do not assume the uniqueness of solutions, while dependence of the set of solutions on a parameter will be expressed in terms of Hausdorff distance of the spaces $C([0, T])$ and $C(\mathbb{R}_+)$.

Quadratic integral equations appear in theories of radiative transfer and neutron transport and in kinetic theory of gases (cf. [1–4]). Up to this time, a lot of papers have appeared on those equations [1–9]; however, to the best of our knowledge, there are no papers on continuous dependence of solutions of this kind of equations on parameter.

Existence results for (1) have been obtained with the help of fixed point theorems expressed in terms of measures of noncompactness.

2. Notation and Auxiliary Facts

In this section we collect some definitions and results which will be needed later. Assume that E is a real Banach space with the norm $\|\cdot\|$ and the zero element θ . Denote by $B(x, r)$ the closed ball centered at x and with radius r . The ball $B(\theta, r)$ will be denoted by B_r . If X is a subset of E , then the symbols $\overline{X}$ and $\text{Conv}X$ stand for the closure and convex closure of X , respectively. The family of all nonempty and bounded subsets of E will be denoted by $\mathfrak{M}_E$ while its subfamily consisting of all relatively compact sets is denoted by $\mathfrak{N}_E$. Following [5, 8, 10] we accept the following definition of a measure of noncompactness.

Definition 1. A mapping $\mu : \mathfrak{M}_E \rightarrow \mathbb{R}_+$ is said to be a measure of noncompactness if it satisfies the following conditions.

- (1°) The family $\ker \mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $\ker \mu \subset \mathfrak{N}_E$.
- (2°) $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$.
- (3°) $\mu(\overline{X}) = \mu(\text{Conv}X) = \mu(X)$.
- (4°) $\mu(\lambda X + (1 - \lambda)Y) \leq \lambda\mu(X) + (1 - \lambda)\mu(Y)$ for $\lambda \in [0, 1]$.
- (5°) If $(X_k)_{k \in \mathbb{N}}$ is a sequence of closed sets from $\mathfrak{M}_E$ such that $X_{k+1} \subset X_k$ ($k = 1, 2, \dots$) and if $\lim_{k \rightarrow \infty} \mu(X_k) = 0$, then the intersection set $X_\infty = \bigcap_{k=1}^{\infty} X_k$ is nonempty.

For our purposes we will only need the following fixed point theorem [6, 8–11].

Theorem 2. *Let Ω be nonempty bounded closed convex subset of the space E and let $V : \Omega \rightarrow \Omega$ be continuous such that $\mu(VX) \leq k\mu(X)$ for any nonempty subset X of Ω , where k is a constant, $k \in [0, 1)$. Then V has a fixed point in the set Ω .*

In the sequel we will work in the Banach space $C([0, T])$ consisting of all real functions defined and continuous on $[0, T]$. The space $C([0, T])$ is furnished with the standard norm

$$\|x\| = \sup \{|x(t)| : t \in [0, T]\}. \quad (2)$$

Now we recollect the definition of the measure of noncompactness which will be used further on. This measure was introduced in [8, 10]. Fix a nonempty bounded subset X of $C([0, T])$ and a positive number $T > 0$. For $x \in X$ and $\varepsilon > 0$ let us denote by $\omega(x, \varepsilon)$ the modulus of continuity of the function x on the interval $[0, T]$; that is,

$$\omega^T(x, \varepsilon) = \sup \{|x(t) - x(s)| : t, s \in [0, T], |t - s| \leq \varepsilon\}. \quad (3)$$

Further, let us put

$$\begin{aligned} \omega^T(X, \varepsilon) &= \sup \{\omega^T(x, \varepsilon) : x \in X\}, \\ \omega_0^T(X) &= \lim_{\varepsilon \rightarrow 0+} \omega^T(X, \varepsilon). \end{aligned} \quad (4)$$

It can be shown [9, 10] that the function ω_0^T is a measure of noncompactness in the space $C([0, T])$.

In what follows, we will also work in the space $C(\mathbb{R}_+)$ consisting of all real functions defined and continuous on $\mathbb{R}_+$. The space $C(\mathbb{R}_+)$ equipped with the family of seminorms

$$\|x\|_n = \sup \{|x(t)| : t \in [0, n]\}, \quad n \in \mathbb{N}, \quad (5)$$

becomes a Fréchet space furnished with the distance

$$d_C(x, y) = \sup \left\{ 2^{-n} \frac{\|x - y\|_n}{1 + \|x - y\|_n} : n \in \mathbb{N} \right\}, \quad (6)$$

$$x, y \in C(\mathbb{R}_+),$$

or equivalently

$$\tilde{d}_C(x, y) = \sum_{n=1}^{\infty} 2^{-n} \frac{\|x - y\|_n}{1 + \|x - y\|_n}, \quad x, y \in C(\mathbb{R}_+). \quad (7)$$

A nonempty subset $X \subset C(\mathbb{R}_+)$ is said to be *bounded* if

$$\sup \{\|x\|_n : x \in X\} < \infty \quad (8)$$

for $n \in \mathbb{N}$.

Further, let $\mathfrak{M}_C$ denote the family of all nonempty and bounded subsets of $C(\mathbb{R}_+)$ and $\mathfrak{N}_C$ the family of all relatively compact subsets of $C(\mathbb{R}_+)$. Obviously $\mathfrak{N}_C \subset \mathfrak{M}_C$.

We accept the following definition of the notion of a sequence of measures of noncompactness [12, 13].

Definition 3. A sequence of functions $\{\mu_n\}_{n \in \mathbb{N}}$, where $\mu_n : \mathfrak{M}_C \rightarrow [0, \infty)$, is said to be a *sequence of measures of noncompactness* in $C(\mathbb{R}_+)$ if it satisfies the following conditions.

- (1°) The family $\ker\{\mu_n\} = \{X \in \mathfrak{M}_C : \mu_n(X) = 0 \text{ for } n \in \mathbb{N}\}$ is nonempty and $\ker\{\mu_n\} \subset \mathfrak{N}_C$.
- (2°) $X \subset Y \Rightarrow \mu_n(X) \leq \mu_n(Y)$ for $n \in \mathbb{N}$.
- (3°) $\mu_n(\overline{X}) = \mu_n(\text{Conv}X) = \mu_n(X)$ for $n \in \mathbb{N}$.
- (4°) If $(X_k)_{k \in \mathbb{N}}$ is a sequence of closed sets from $\mathfrak{M}_C$ such that $X_{k+1} \subset X_k$ ($k = 1, 2, \dots$) and if $\lim_{k \rightarrow \infty} \mu_n(X_k) = 0$ for each $n \in \mathbb{N}$, then the intersection set $X_\infty = \bigcap_{k=1}^{\infty} X_k$ is nonempty.

We have the following two facts (see [12, 13]).

Theorem 4. *The family of mappings $\{\omega_0^n\}_{n \in \mathbb{N}}$, where $\omega_0^n : \mathfrak{M}_C \rightarrow [0, \infty)$, satisfies the conditions (1°)–(4°) from Definition 3 and, moreover, $\ker\{\omega_0^n\} = \mathfrak{N}_C$.*

Theorem 5. *Let Ω be a nonempty, bounded, closed, and convex subset of the space $C(\mathbb{R}_+)$ and let $V : \Omega \rightarrow \Omega$ be continuous mapping. Suppose that there exists a sequence of numbers $(k_n) \subset [0, 1)$ such that*

$$\omega_0^n(VX) \leq k_n \omega_0^n(X) \quad (9)$$

for nonempty $X \subset \Omega$ and $n \in \mathbb{N}$. Then V has at least one fixed point in the set Ω .

Let (Z, d) be an arbitrary metric space. For any two nonempty and bounded subsets of Z we define their Hausdorff distance $D_Z(X, Y)$ by formula

$$D_Z(X, Y) := \inf \{r \geq 0 : X \subset B(Y, r), Y \subset B(X, r)\}, \quad (10)$$

where

$$B(X, r) := \bigcup_{x \in X} \{z \in Z : d(x, z) \leq r\}. \quad (11)$$

In the next chapters, we will consider Hausdorff distance $D_{C([0, T])}$ in the family $\mathfrak{N}_{C([0, T])}$ of nonempty and relatively compact subsets of the Banach space $C[0, T]$ and $D_{C(\mathbb{R}_+)}$ in the family $\mathfrak{N}_{C(\mathbb{R}_+)}$ of nonempty and relatively compact subsets of the Fréchet space $C(\mathbb{R}_+)$ (with distance d_C).

Let (Λ, d) be arbitrary metric space, and let us consider a mapping $F : \Lambda \rightarrow \mathfrak{N}_{C(\mathbb{R}_+)}$. Since we will consider the continuity of such mappings (with respect to the distance $D_{C(\mathbb{R}_+)}$ in $\mathfrak{N}_{C(\mathbb{R}_+)}$), we need the following lemma.

Lemma 6. A mapping $F : \Lambda \rightarrow \mathfrak{N}_{C(\mathbb{R}_+)}$ is continuous at a point $\lambda_0 \in \Lambda$ if and only if

$$\forall \varepsilon > 0 \forall n \in \mathbb{N} \exists \delta > 0 \forall \lambda \in \Lambda, d(\lambda_0, \lambda) < \delta \forall x \in F(\lambda_0) \exists y \in F(\lambda) \|x - y\|_n \leq \varepsilon, \quad (12)$$

$$\forall \varepsilon > 0 \forall n \in \mathbb{N} \exists \delta > 0 \forall \lambda \in \Lambda, d(\lambda_0, \lambda) < \delta \forall y \in F(\lambda) \exists x \in F(\lambda_0) \|x - y\|_n \leq \varepsilon. \quad (13)$$

Proof. From (6) we derive that for $\eta > 0$ we have

$$d_C(x, y) \leq \eta \iff \|x - y\|_k \leq \frac{\eta}{2^{-k} - \eta}, \quad (14)$$

for $k = 1, \dots, n(\eta)$,

where

$$n(\eta) := \begin{cases} \max \left\{ k \in \mathbb{N} : k < \log_2 \frac{1}{\eta} \right\}, & \text{for } \eta \leq \frac{1}{2}, \\ 0, & \text{for } \eta > \frac{1}{2}. \end{cases} \quad (15)$$

The continuity of the mapping $F \rightarrow \mathfrak{N}_{C(\mathbb{R}_+)}$ at a point $\lambda_0 \in \Lambda$ means

$$\forall \varepsilon > 0 \exists \delta > 0 \forall \lambda \in \Lambda, d(\lambda_0, \lambda) < \delta \forall x \in F(\lambda_0) \exists y \in F(\lambda) d_C(x, y) \leq \varepsilon, \quad (16)$$

$$\forall \varepsilon > 0 \exists \delta > 0 \forall \lambda \in \Lambda, d(\lambda_0, \lambda) < \delta \forall y \in F(\lambda) \exists x \in F(\lambda_0) d_C(x, y) \leq \varepsilon. \quad (17)$$

We will show that conditions (12) and (13) are equivalent to conditions (16) and (17).

First we suppose that (12) and (13) are fulfilled. Let us fix $\varepsilon > 0$. The condition (12) guarantees that there exist numbers $\delta_k > 0$, $k = 1, \dots, n(\varepsilon)$, such that

$$\forall \lambda \in \Lambda, d(\lambda_0, \lambda) < \delta_k \forall x \in F(\lambda_0) \exists y \in F(\lambda) \|x - y\|_k \leq \frac{\varepsilon}{2^{-k} - \varepsilon} \quad (18)$$

for $k = 1, \dots, n(\varepsilon)$.

Let us put $\delta := \min\{\delta_k : k = 1, \dots, n(\varepsilon)\}$. Applying (14) we get

$$\forall \lambda \in \Lambda, d(\lambda_0, \lambda) < \delta \forall x \in F(\lambda_0) \exists y \in F(\lambda) d_C(x, y) \leq \varepsilon. \quad (19)$$

In a similar way we show that (17) is also satisfied.

Now we assume that (16) and (17) are fulfilled. We will prove conditions (12) and (13). Let us fix $\varepsilon > 0$ and $n \in \mathbb{N}$. There exists a number $\eta > 0$ such that $\eta < 2^{-n}$ and $\eta / (2^{-n} - \eta) = \varepsilon$. Condition (16) implies that there is a number $\delta > 0$ such that

$$\forall \lambda \in \Lambda, d(\lambda_0, \lambda) < \delta \forall x \in F(\lambda_0) \exists y \in F(\lambda) d_C(x, y) \leq \eta. \quad (20)$$

Hence, using (14) we obtain that

$$\|x - y\|_n \leq \frac{\eta}{2^{-n} - \eta} = \varepsilon \quad (21)$$

and therefore (16) is confirmed. In a similar way we show that condition (17) is also satisfied. $\square$

3. Existence Result

In this section we give an existence result for the following nonlinear integral Volterra equation:

$$x(t) = a(t) + f(t, x(t)) \int_0^t v(t, s, x(s)) ds, \quad t \in [0, T]. \quad (22)$$

In the last years there have been published a few dozen papers on nonlinear quadratic integral equations. From among papers cited here, majority of them was concerned with different kinds of equations from (22); see [5–7, 13]. The authors of papers [8, 9, 11, 13] examined equations similar to (22); however, their considerations were conducted in the Banach space $BC(\mathbb{R}_+)$ and therefore their assumptions were too restrictive. In the paper [12] we investigated similar equations to (22) with different kinds of assumptions than these given below.

This theorem will be a starting point of our further investigations on the continuous dependence of solutions on parameter.

Observe that the above equation includes several classes of functional, integral, and functional integral equations considered in the literature [1–4].

Equation (22) will be considered under the following assumptions.

(H₁) The function $a : [0, T] \rightarrow \mathbb{R}$ is continuous on $[0, T]$.

(H₂) The function $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous and there exists a constant k_f such that $|f(t, x) - f(t, y)| \leq k_f |x - y|$ for any $t \in [0, T]$ and for all $x, y \in \mathbb{R}$.

(H₃) The function $v : [0, T] \times [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous. Moreover, there exists a function $g : [0, T] \times [0, T] \rightarrow \mathbb{R}_+$ being continuous on $[0, T] \times [0, T]$ such that

$$|v(t, s, x)| \leq g(t, s) \quad (23)$$

for all $t, s \in [0, T]$, $x \in \mathbb{R}$.

(H₄) Consider

$$k_f \sup_{t \in [0, T]} \int_0^t g(t, s) ds < 1. \quad (24)$$

Then we can formulate our existence result.

Theorem 7. Under assumptions (H₁)–(H₄), (22) has at least one solution in the space $C([0, T])$.

Proof. Consider the operator $V : C([0, T]) \rightarrow C([0, T])$ defined by the formula

$$(Vx)(t) := a(t) + f(t, x(t)) \int_0^t v(t, s, x(s)) ds, \quad t \in [0, T]. \quad (25)$$

Using assumptions (H_1) – (H_3) and reasoning similarly as in [5–11], we can show that the operator V is well defined and is continuous on $C([0, T])$. Next let us put

$$r := \left(1 - k_f \sup_{t \in [0, T]} \int_0^t g(t, s) ds\right)^{-1} \times \left(\sup_{t \in [0, T]} |a(t)| + \sup_{t \in [0, T]} |f(t, 0)| \sup_{t \in [0, T]} \int_0^t g(t, s) ds\right). \quad (26)$$

Notice that for $x \in B(\theta, r)$ we have in virtue of (H_1) – (H_3)

$$|(Vx)(t)| \leq \sup_{t \in [0, T]} |a(t)| + \left(\sup_{t \in [0, T]} |f(t, 0)| + k_f r\right) \sup_{t \in [0, T]} \int_0^t g(t, s) ds = r \quad (27)$$

for $t \in [0, T]$ and therefore the operator V transforms the ball $B(\theta, r)$ into itself.

Now, let us take a nonempty subset X of the ball $B(\theta, r)$ and fix $x \in X$. Choose $t_1, t_2 \in [0, T]$ such that $|t_2 - t_1| < \varepsilon$. Without loss of generality we can assume that $t_2 > t_1$. Then we obtain

$$\begin{aligned} |(Vx)(t_2) - (Vx)(t_1)| &\leq |a(t_2) - a(t_1)| \\ &\quad + (|f(t_2, x(t_2)) - f(t_1, x(t_2))| \\ &\quad + |f(t_1, x(t_2)) - f(t_1, x(t_1))|) \\ &\quad \times \int_0^{t_2} |v(t_2, s, x(s))| ds + |f(t_1, x(t_1))| \\ &\quad \times \left(\int_{t_1}^{t_2} |v(t_2, s, x(s))| ds \right. \\ &\quad \left. + \int_0^{t_1} |v(t_2, s, x(s)) - v(t_1, s, x(s))| ds\right) \\ &\leq \xi^T(a, \varepsilon) + (\xi^T(f, \varepsilon) + k_f \omega^T(x, \varepsilon)) \sup_{t \in [0, T]} \int_0^t g(t, s) ds \\ &\quad + \bar{F}(\varepsilon \bar{V} + T \xi^T(v, \varepsilon)), \end{aligned} \quad (28)$$

where we denoted

$$\begin{aligned} \xi^T(a, \varepsilon) &:= \sup \{|a(t_2) - a(t_1)| : t_2, t_1 \in [0, T], |t_2 - t_1| < \varepsilon\}, \\ \xi^T(f, \varepsilon) &:= \sup \{|f(t_2, x) - f(t_1, x)| : t_2, t_1 \in [0, T], \\ &\quad |t_2 - t_1| < \varepsilon, |x| \leq r\}, \end{aligned}$$

$$\xi^T(v, \varepsilon) := \sup \{|v(t_2, s, x) - v(t_1, s, x)| : t_2, t_1, s \in [0, T], |t_2 - t_1| < \varepsilon, |x| \leq r\},$$

$$\bar{F} := \sup \{|f(t, x)| : t \in [0, T], |x| \leq r\},$$

$$\bar{V} := \sup \{|v(t, s, x)| : t, s \in [0, T], |x| \leq r\}.$$

(29)

From the above estimate we derive the following one:

$$\begin{aligned} \omega^T(VX, \varepsilon) &\leq \xi^T(a, \varepsilon) + \xi^T(f, \varepsilon) \sup_{t \in [0, T]} \int_0^t g(t, s) ds \\ &\quad + k_f \sup_{t \in [0, T]} \int_0^t g(t, s) ds \cdot \omega^T(X, \varepsilon) \\ &\quad + \bar{F}(\varepsilon \bar{V} + T \xi^T(v, \varepsilon)). \end{aligned} \quad (30)$$

Observe that $\xi^T(a, \varepsilon) \rightarrow 0$, $\xi^T(f, \varepsilon) \rightarrow 0$, and $\xi^T(v, \varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$, which are simple consequences of the uniform continuity of the functions a , f , and v on the sets $[0, T]$, $\{(t, x) : t \in [0, T], |x| \leq r\}$, and $\{(t, s, x) : t, s \in [0, T], |x| \leq r\}$, respectively. Hence we get

$$\omega_0^T(VX) \leq k_f \sup_{t \in [0, T]} \int_0^t g(t, s) ds \cdot \omega_0^T(X). \quad (31)$$

Since $k_f \sup_{t \in [0, T]} \int_0^t g(t, s) ds < 1$ in view of assumption (H_4) , from the above estimate and Theorem 2 we deduce that the operator V has a fixed point x in the set $B(\theta, r)$. Obviously x is a solution of the functional integral equation (22). $\square$

Our next result is concerned with (22) on the real half-axis $\mathbb{R}_+$. Let us consider equation

$$x(t) = a(t) + f(t, x(t)) \int_0^t v(t, s, x(s)) ds, \quad t \in \mathbb{R}_+. \quad (32)$$

Equation (32) will be considered under the following assumptions.

- (H'_1) The function $a : \mathbb{R}_+ \rightarrow \mathbb{R}$ is continuous on $\mathbb{R}_+$.
- (H'_2) The function $f : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous and there exists a constant k_f such that $|f(t, x) - f(t, y)| \leq k_f |x - y|$ for any $t \in \mathbb{R}_+$ and for all $x, y \in \mathbb{R}$.
- (H'_3) The function $v : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous. Moreover, there exists a function $g : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ being continuous on $\mathbb{R}_+ \times \mathbb{R}_+$ such that

$$|v(t, s, x)| \leq g(t, s) \quad (33)$$

for all $t, s \in \mathbb{R}_+, x \in \mathbb{R}$.

(H'_4) Consider

$$k_f \sup_{t \in [0, T]} \int_0^t g(t, s) ds < 1, \quad T \in \mathbb{R}_+. \quad (34)$$

Then we can formulate the next existence result.

Theorem 8. Under assumptions $(H'_1)-(H'_4)$, (32) has at least one solution in the space $C(\mathbb{R}_+)$.

Proof. In contrast to papers [5, 6, 8, 9, 11], we place our considerations in the Fréchet space $C(\mathbb{R}_+)$, instead of the Banach space $BC(\mathbb{R}_+)$.

Let us define the operator $V : C(\mathbb{R}_+) \rightarrow C(\mathbb{R}_+)$ defined by the formula

$$(Vx)(t) := a(t) + f(t, x(t)) \int_0^t v(t, s, x(s)) ds, \quad t \in \mathbb{R}_+. \quad (35)$$

Using assumptions $(H'_1)-(H'_3)$ and standard arguments, we can show that the operator V is well defined and is continuous on $C(\mathbb{R}_+)$. Next let us put

$$r(t) := \left(1 - k_f \int_0^t g(t, s) ds\right)^{-1} \times \left(|a(t)| + |f(t, 0)| \int_0^t g(t, s) ds\right), \quad t \in \mathbb{R}_+, \quad (36)$$

$$\Omega := \{x \in C(\mathbb{R}_+) : |x(t)| \leq r(t) \text{ for } t \in \mathbb{R}_+\}. \quad (37)$$

Notice that for $x \in \Omega$ we have in virtue of $(H'_1)-(H'_3)$ and (36)

$$|(Vx)(t)| \leq |a(t)| + (|f(t, 0)| + k_f r(t)) \int_0^t g(t, s) ds = r(t) \quad (38)$$

and therefore the operator V transforms Ω into itself.

Applying (31) with $T = n$, $n \in \mathbb{N}$, we get for $X \subset \Omega$

$$\omega_0^n(VX) \leq k_f \sup_{t \in [0, n]} \int_0^t g(t, s) ds \cdot \omega_0^n(X). \quad (39)$$

Since $k_f \sup_{t \in [0, n]} \int_0^t g(t, s) ds < 1$ in view of assumption (H'_4) , from the above estimate and Theorem 5, we deduce that the operator V has a fixed point x in the set Ω . Obviously x is a solution of the functional integral equation (22). $\square$

4. Continuous Dependence of Solutions on Parameter and the Examples

In this section we will investigate (1) depending on parameter λ ; that is, we will consider equation of the type

$$x(t) = a(\lambda, t) + f(\lambda, t, x(t)) \int_0^t v(\lambda, t, s, x(s)) ds, \quad t \in J, \quad (40)$$

where λ is an element of metric space (Λ, d) , $J = [0, T]$ or $J = \mathbb{R}_+$.

For fixed $\lambda \in \Lambda$ we will denote the operator $V_\lambda : C(J) \rightarrow C(J)$ specified by formula

$$(V_\lambda x)(t) := a(\lambda, t) + f(\lambda, t, x(t)) \int_0^t v(\lambda, t, s, x(s)) ds, \quad x \in C(J), \quad t \in J, \quad (41)$$

and put

$$\text{Fix}(\lambda) := \{x \in C(J) : V_\lambda x = x\}. \quad (42)$$

Obviously $\text{Fix}(\lambda)$ is the set of all solutions of (40) and it is compact in view of condition (31), so $\text{Fix}(\lambda) \in \mathfrak{N}_{C(J)}$.

The aim of this paper is to provide the conditions concerning the functions involved in (40), which imply that the sets $\text{Fix}(\lambda)$ are nonempty and change continuously in the space $C(J)$ with respect to parameter λ (in view of Hausdorff metric $D_{C(J)}$).

First we consider case of the bounded set $J = [0, T]$. Let us take the following assumptions.

(A₁) $a : \Lambda \times [0, T] \rightarrow \mathbb{R}$ is a function such that the function $t \mapsto a(\lambda, t)$ is continuous on $[0, T]$ for arbitrarily fixed $\lambda \in \Lambda$ and moreover there exists a nondecreasing function $p_a : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, continuous at 0 and $p_a(0) = 0$, such that

$$|a(\lambda_1, t) - a(\lambda_2, t)| \leq p_a(d(\lambda_1, \lambda_2)) \quad (43)$$

for $\lambda_1, \lambda_2 \in \Lambda, t \in [0, T]$.

(A₂) $f : \Lambda \times [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is a function such that the function $(t, x) \mapsto f(\lambda, t, x)$ is continuous and bounded on the set $[0, T] \times \mathbb{R}$ for arbitrarily fixed $\lambda \in \Lambda$ and there are a constant k_f and a nondecreasing function $p_f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, continuous at 0 and $p_f(0) = 0$, such that

$$|f(\lambda, t, x) - f(\lambda, t, y)| \leq k_f |x - y| \quad (44)$$

for any $\lambda \in \Lambda, t \in [0, T], x, y \in \mathbb{R}$, and

$$|f(\lambda_1, t, x) - f(\lambda_2, t, x)| \leq p_f(d(\lambda_1, \lambda_2)) \quad (45)$$

for all $\lambda_1, \lambda_2 \in \Lambda, t \in [0, T], x \in \mathbb{R}$.

(A₃) $v : \Lambda \times [0, T] \times [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is a function such that the function $(t, s, x) \mapsto v(\lambda, t, s, x)$ is continuous on the set $[0, T] \times [0, T] \times \mathbb{R}$ for arbitrarily fixed $\lambda \in \Lambda$ and there are a continuous function $g : [0, T] \times [0, T] \rightarrow \mathbb{R}_+$ and a nondecreasing function $p_v : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, continuous at 0 and $p_v(0) = 0$, such that

$$|v(\lambda, t, s, x)| \leq g(t, s) \quad (46)$$

for all $\lambda \in \Lambda, t, s \in [0, T], x \in \mathbb{R}$, and

$$|v(\lambda_1, t, s, x) - v(\lambda_2, t, s, x)| \leq p_v(d(\lambda_1, \lambda_2)) \quad (47)$$

for all $\lambda_1, \lambda_2 \in \Lambda, t, s \in [0, T], x \in \mathbb{R}$.

(A₄) Consider

$$k_f \sup_{t \in [0, T]} \int_0^t g(t, s) ds < 1. \quad (48)$$

The following example shows that conditions (A₁)–(A₄) are not enough for continuous dependence of the solutions on parameter λ .

Example 9. Let us consider the equation

$$x(t) = \int_0^t \sqrt{\lambda + \min\{|x(s)|, 2 + 2\sqrt{2}\}} ds, \quad t \in [0, 2], \quad (49)$$

where $\lambda \in \Lambda := [0, 1]$. The space Λ is equipped with the standard Euclidean metric. Obviously it is a particular case of (40), where

$$\begin{aligned} a(\lambda, t) &\equiv 0, & f(\lambda, t, x) &\equiv 1, & k_f &\equiv 0, \\ v(\lambda, t, s, x) &= \sqrt{\lambda + \min\{|x|, 2 + 2\sqrt{2}\}}, \\ g(t, s) &= \sqrt{3 + 2\sqrt{2}}, & p_a(\tau) &= p_f(\tau) = \tau, \\ p_v(\tau) &= \sqrt{\tau}. \end{aligned} \quad (50)$$

Notice that if x is a solution of (49), then

$$|x(t)| \leq \int_0^2 \sqrt{1 + 2 + 2\sqrt{2}} ds \leq 2\sqrt{3 + 2\sqrt{2}} = 2 + 2\sqrt{2} \quad (51)$$

which shows that (49) may be represented by simpler form

$$x(t) = \int_0^t \sqrt{\lambda + |x(s)|} ds, \quad t \in [0, 2]. \quad (52)$$

Using standard calculations we obtain that $\text{Fix}(0) = \{x_1, x_2\}$, where $x_1(t) \equiv 0$ and $x_2(t) = t^2/4$, but for $\lambda > 0$ we have $\text{Fix}(\lambda) = \{x\}$, where $x(t) = t(t/4 + \sqrt{\lambda})$. It is easy to check whether the sets $\text{Fix}(\lambda)$ do not change continuously in point $\lambda = 0$ (in view of Hausdorff metric). It means the conditions $(A_1)-(A_4)$, which guarantee the existence of the solutions together with Theorem 7, are not sufficient for continuous dependence of the solutions on the parameter λ . Thus, we decide to take an additional assumption.

(A_5) There is an integrable function $k_v : [0, T] \rightarrow \mathbb{R}_+$ such that

$$|v(\lambda, t, s, x) - v(\lambda, t, s, y)| \leq k_v(s) |x - y| \quad (53)$$

for $\lambda \in \Lambda, t, s \in [0, T], x, y \in \mathbb{R}$.

Theorem 10. Under assumptions $(A_1)-(A_5)$, (40) has at least one solution in the space $C([0, T])$ for all parameters $\lambda \in \Lambda$ and moreover the set of all solutions $\text{Fix}(\lambda)$ depends continuously (with respect to Hausdorff metric $D_{C([0, T])}$) on the parameter λ .

Proof. Theorem 7 implies that (40) has at least one solution for all λ .

Let us fix arbitrarily $\lambda \in \Lambda$ and $\varepsilon > 0$. Since we want to prove that $\text{Fix} : \Lambda \rightarrow \mathfrak{N}_{C([0, T])}$ is continuous in point λ with respect to Hausdorff metric $D_{C([0, T])}$, we should show that there exists $\delta > 0$ such that

$$\forall_{x \in \text{Fix}(\lambda)} \forall_{\lambda_1 \in B(\lambda, \delta)} \exists_{x_1 \in \text{Fix}(\lambda_1)} \|x - x_1\| < \varepsilon, \quad (54)$$

$$\forall_{\lambda_1 \in B(\lambda, \delta)} \forall_{x_1 \in \text{Fix}(\lambda_1)} \exists_{x \in \text{Fix}(\lambda)} \|x - x_1\| < \varepsilon. \quad (55)$$

Let us fix $x \in \text{Fix}(\lambda)$ and choose arbitrarily $\lambda_1 \in \Lambda, d(\lambda, \lambda_1) < 1$. Then, for any $y \in C([0, T])$ we obtain the following estimation:

$$\begin{aligned} &|x(t) - (V_{\lambda_1} y)(t)| \\ &= |(V_{\lambda} x)(t) - (V_{\lambda_1} y)(t)| \leq |(V_{\lambda} x)(t) - (V_{\lambda} y)(t)| \\ &\quad + |(V_{\lambda} y)(t) - (V_{\lambda_1} y)(t)| \\ &\leq \left| f(\lambda, t, x(t)) \int_0^t v(\lambda, t, s, x(s)) ds \right. \\ &\quad \left. - f(\lambda, t, y(t)) \int_0^t v(\lambda, t, s, y(s)) ds \right| \\ &\quad + |a(\lambda, t) - a(\lambda_1, t)| \\ &\quad + \left| f(\lambda, t, y(t)) \int_0^t v(\lambda, t, s, y(s)) ds \right. \\ &\quad \left. - f(\lambda_1, t, y(t)) \int_0^t v(\lambda_1, t, s, y(s)) ds \right| \\ &\leq |f(\lambda, t, x(t)) - f(\lambda, t, y(t))| \int_0^t v(\lambda, t, s, x(s)) ds \\ &\quad + |f(\lambda, t, y(t))| \int_0^t |v(\lambda, t, s, x(s)) - v(\lambda, t, s, y(s))| ds \\ &\quad + p_a(d(\lambda, \lambda_1)) + |f(\lambda, t, y(t)) - f(\lambda_1, t, y(t))| \\ &\quad \times \int_0^t |v(\lambda, t, s, y(s))| ds + |f(\lambda_1, t, y(t))| \\ &\quad \times \int_0^t |v(\lambda, t, s, y(s)) - v(\lambda_1, t, s, y(s))| ds \\ &\leq k_f |x(t) - y(t)| \int_0^t g(t, s) ds \\ &\quad + \tilde{F} \int_0^t k_v(s) |x(s) - y(s)| ds + p_a(d(\lambda, \lambda_1)) \\ &\quad + p_f(d(\lambda, \lambda_1)) \int_0^t g(t, s) ds + \tilde{F} \int_0^t p_v(d(\lambda, \lambda_1)) ds \\ &\leq |x(t) - y(t)| k_f \sup_{t \in [0, T]} \int_0^t g(t, s) ds \\ &\quad + \tilde{F} \int_0^t k_v(s) |x(s) - y(s)| ds + p_a(d(\lambda, \lambda_1)) \\ &\quad + \frac{p_f(d(\lambda, \lambda_1))}{k_f} + \tilde{F} T p_v(d(\lambda, \lambda_1)), \end{aligned} \quad (56)$$

where

$$\tilde{F} := p_f(1) + \sup\{|f(\lambda, t, x)| : t \in [0, T], x \in \mathbb{R}\} < \infty. \quad (57)$$

Summarizing, for fixed $\lambda \in \Lambda$ and $x \in \text{Fix}(\lambda)$ and for arbitrary $\lambda_1 \in B(\lambda, 1)$, $y \in C([0, T])$, we derive

$$\begin{aligned} & |x(t) - (V_{\lambda_1} y)(t)| \\ & \leq q |x(t) - y(t)| + \tilde{F} \int_0^t k_v(s) |x(s) - y(s)| ds + b(\lambda_1), \end{aligned} \quad (58)$$

where we denoted

$$\begin{aligned} q &:= k_f \sup_{t \in [0, T]} \int_0^t g(t, s) ds, \\ b(\lambda_1) &:= p_a(d(\lambda, \lambda_1)) + \frac{p_f(d(\lambda, \lambda_1))}{k_f} + \tilde{F} T p_v(d(\lambda, \lambda_1)). \end{aligned} \quad (59)$$

Consider the following equation:

$$\psi_{\lambda_1}(t) = \frac{\tilde{F}}{1-q} \int_0^t k_v(s) \psi_{\lambda_1}(s) ds + \frac{b(\lambda_1)}{1-q}, \quad t \in [0, T], \quad (60)$$

where $\psi_{\lambda_1} : [0, T] \rightarrow \mathbb{R}_+$ is unknown function. Standard calculations show that solution ψ_{λ_1} of the above equation is given by formula

$$\psi_{\lambda_1}(t) = \frac{b(\lambda_1)}{1-q} \exp\left(\frac{\tilde{F}}{1-q} \int_0^t k_v(s) ds\right), \quad t \in [0, T]. \quad (61)$$

Keeping in mind $\lim_{\lambda_1 \rightarrow \lambda} b(\lambda_1) = 0$ we infer that there exists sufficiently small number $\delta > 0$, $\delta < 1$ satisfying inequality

$$\frac{b(\lambda_1)}{1-q} \exp\left(\frac{\tilde{F}}{1-q} \int_0^T k_v(s) ds\right) < \varepsilon, \quad \text{for } \lambda_1 \in B(\lambda, \delta). \quad (62)$$

Now, let us fix $\lambda_1 \in B(\lambda, \delta)$ and define the set

$$\Omega := \{y \in C([0, T]) : |x(t) - y(t)| \leq \psi_{\lambda_1}(t), t \in [0, T]\}. \quad (63)$$

We will prove

$$V_{\lambda_1}(\Omega) \subset \Omega. \quad (64)$$

Indeed, for $y \in \Omega$ and $t \in [0, T]$, in view of (58) and (60) we obtain

$$\begin{aligned} & |x(t) - (V_{\lambda_1} y)(t)| \\ & \leq q |x(t) - y(t)| + \tilde{F} \int_0^t k_v(s) |x(s) - y(s)| ds + b(\lambda_1) \\ & \leq q \psi_{\lambda_1}(t) + \tilde{F} \int_0^t k_v(s) \psi_{\lambda_1}(s) ds + b(\lambda_1) = \psi_{\lambda_1}(t) \end{aligned} \quad (65)$$

and it confirms (64).

If x_1 is a fixed point in V_{λ_1} then, in view of (62), we obtain

$$\|x - x_1\| = \sup_{t \in [0, T]} |x(t) - x_1(t)| \leq \sup_{t \in [0, T]} \psi_{\lambda_1}(t) < \varepsilon \quad (66)$$

which confirms (54). The proof of estimate (55) is analogous. $\square$

Finally we will consider (40) on the real half-axis $\mathbb{R}_+$. First we express the necessary assumptions.

(A'₁) $a : \Lambda \times \mathbb{R}_+ \rightarrow \mathbb{R}$ is a function such that the function $t \rightarrow a(\lambda, t)$ is continuous on $\mathbb{R}_+$ for arbitrarily fixed $\lambda \in \Lambda$ and moreover there exists a nondecreasing function $p_a : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, continuous at 0 and $p_a(0) = 0$, such that

$$|a(\lambda_1, t) - a(\lambda_2, t)| \leq p_a(d(\lambda_1, \lambda_2)) \quad (67)$$

for $\lambda_1, \lambda_2 \in \Lambda, t \in \mathbb{R}_+$.

(A'₂) $f : \Lambda \times \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ is a function such that the function $(t, x) \rightarrow f(\lambda, t, x)$ is continuous and bounded on the set $\mathbb{R}_+ \times \mathbb{R}$ for arbitrarily fixed $\lambda \in \Lambda$ and there are a constant k_f and a nondecreasing function $p_f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, continuous at 0 and $p_f(0) = 0$, such that

$$|f(\lambda, t, x) - f(\lambda, t, y)| \leq k_f |x - y| \quad (68)$$

for any $\lambda \in \Lambda, t \in \mathbb{R}_+, x, y \in \mathbb{R}$, and

$$|f(\lambda_1, t, x) - f(\lambda_2, t, x)| \leq p_f(d(\lambda_1, \lambda_2)) \quad (69)$$

for all $\lambda_1, \lambda_2 \in \Lambda, t \in \mathbb{R}_+, x \in \mathbb{R}$.

(A'₃) $v : \Lambda \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ is a function such that the function $(t, s, x) \rightarrow v(\lambda, t, s, x)$ is continuous on the set $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}$ for arbitrarily fixed $\lambda \in \Lambda$ and there are a continuous function $g : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and a nondecreasing function $p_v : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, continuous at 0 and $p_v(0) = 0$, such that

$$|v(\lambda, t, s, x)| \leq g(t, s) \quad (70)$$

for all $\lambda \in \Lambda, t, s \in \mathbb{R}_+, x \in \mathbb{R}$ and

$$|v(\lambda_1, t, s, x) - v(\lambda_2, t, s, x)| \leq p_v(d(\lambda_1, \lambda_2)) \quad (71)$$

for all $\lambda_1, \lambda_2 \in \Lambda, t, s \in \mathbb{R}_+, x \in \mathbb{R}$.

(A'₄) Consider

$$k_f \sup_{t \in [0, T]} \int_0^t g(t, s) ds < 1, \quad T \in \mathbb{R}_+. \quad (72)$$

(A₅') There is an integrable function $k_v : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that

$$|v(\lambda, t, s, x) - v(\lambda, t, s, y)| \leq k_v(s) |x - y| \quad (73)$$

for $\lambda \in \Lambda, t, s \in \mathbb{R}_+, x, y \in \mathbb{R}$.

Under the above assumptions we can formulate a theorem analogous to the previous one.

Theorem 11. Under assumptions (A₁')–(A₅'), (40) has at least one solution in the space $C(\mathbb{R}_+)$ for all parameters $\lambda \in \Lambda$ and, moreover, the set of all solutions $\text{Fix}(\lambda)$ depends continuously (with respect to Hausdorff metric $D_{C(\mathbb{R}_+)}$) on the parameter λ .

Proof. The proof of this theorem uses Lemma 6 and it can be done in the same way as the proof of Theorem 10. We omit details. $\square$

Remark 12. Notice that, in hypotheses (A₁)–(A₃) in Theorem 10, we have assumed uniform continuity of the functions a, f , and v with respect to variable λ . Therefore, there appears a natural question.

Question. Is Theorem 10 true in the case when we assume only continuity of the functions a, f , and v instead of uniform continuity?

Finally, we provide an example of an integral equation of the form (40) for which the assumptions of Theorem 10 are satisfied.

Example 13. Consider the following functional integral equation:

$$x(t) = \lambda t^2 + \lambda \sin\left(\frac{tx(t)}{3}\right) \int_0^t \sqrt[3]{se^{-t} + \frac{\lambda(1+|x(s)|)}{2+|x(s)|}} ds, \quad t \in [0, 1], \quad (74)$$

where $\lambda \in \Lambda := [1, 2]$. The space Λ is equipped with the standard Euclidean metric. Observe that (74) is a special case of (40), where $a(\lambda, t) = \lambda t^2$, $f(\lambda, t, x) = \lambda \sin(tx/3)$, and $v(\lambda, t, s, x) = \sqrt[3]{se^{-t} + \lambda(1+|x|)/(2+|x|)}$. Moreover

$$|a(\lambda_1, t) - a(\lambda_2, t)| \leq p_a(|\lambda_1 - \lambda_2|),$$

$$\text{where } p_a(\tau) = \tau,$$

$$|f(\lambda_1, t, x) - f(\lambda_2, t, x)| \leq p_f(|\lambda_1 - \lambda_2|),$$

$$\text{where } p_f(\tau) = \tau,$$

$$|v(\lambda_1, t, s, x) - v(\lambda_2, t, s, x)| \leq p_v(|\lambda_1 - \lambda_2|),$$

$$\text{where } p_v(\tau) = \frac{\sqrt[3]{2}\tau}{3},$$

$$|f(\lambda, t, x) - f(\lambda, t, y)| \leq \frac{2}{3} |x - y|, \quad \text{i.e. } k_f = \frac{2}{3},$$

$$|v(\lambda, t, s, x)| = \sqrt[3]{se^{-t} + \frac{\lambda(1+|x|)}{2+|x|}} \leq \sqrt[3]{3},$$

$$\text{i.e., } g(t, s) = \sqrt[3]{3},$$

$$\sup_{t \in [0, 1]} \int_0^t g(t, s) ds = \sqrt[3]{3} < \frac{1}{k_f} = \frac{3}{2},$$

$$\begin{aligned} & |v(\lambda, t, s, x) - v(\lambda, t, s, y)| \\ & \leq \frac{|\lambda(1+|x|)/(2+|x|) - \lambda(1+|y|)/(2+|y|)|}{\sqrt[3]{s/e^2} + \sqrt[3]{s/e} \sqrt[3]{s/e} + \sqrt[3]{s/e^2}} \\ & \leq \frac{\sqrt[3]{e^2}}{6} s^{-2/3} |x - y|; \end{aligned} \quad (75)$$

that is, the function $k_v(s) = (\sqrt[3]{e^2}/6) s^{-2/3}$ for $s \in (0, 1]$ is integrable.

Joining all above facts we infer that assumptions (A₁)–(A₅) of Theorem 10 are satisfied. Finally, applying Theorem 10 we conclude that the set of all solutions $\text{Fix}(\lambda)$ of (74) depends continuously on the parameter λ .

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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