Research Article

Two Positive Solutions of Third-Order BVP with Integral Boundary Condition and Sign-Changing Green's Function

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We are concerned with the following third-order boundary value problem with integral boundary condition: $u'''(t) = f(t, u(t))$, $t \in [0, 1]$, $u'(0) = u(1) = 0$, $u''(\eta) + \int_{\alpha}^{\beta} u(t)dt = 0$, where $1/2 < \alpha \leq \beta \leq 1$, $\alpha + \beta \leq 4/3$, and $\eta \in (1/2, \alpha]$. Although the corresponding Green's function is sign-changing, we still obtain the existence of at least two positive and decreasing solutions under some suitable conditions on f by using the two-fixed-point theorem due to Avery and Henderson. An example is also included to illustrate the main results obtained.

1. Introduction

Third-order differential equations arise from a variety of different areas of applied mathematics and physics, for example, in the deflection of a curved beam having a constant or varying cross-section, a three-layer beam, electromagnetic waves or gravity driven flows, and so on [1].

Recently, the existence of single or multiple positive solutions for some third-order three-point boundary value problems (BVPs for short) and for some third-order BVPs with integral boundary conditions has received much attention from many authors; see [2–15] and the references therein. It is necessary to point out that all the abovementioned papers are achieved when the corresponding Green's functions are nonnegative, which is a very important condition.

In 2008, Palamides and Smyrlis [16] studied the existence of at least one positive solution to the singular third-order three-point BVP with an indefinitely signed Green's function:

$$\begin{aligned} u'''(t) &= a(t)f(t, u(t)), \quad t \in (0, 1), \\ u(0) &= u(1) = u''(\eta) = 0, \end{aligned} \quad (1)$$

where $\eta \in (17/24, 1)$. Their technique was a combination of the Guo-Krasnoselskii fixed point theorem and properties of the corresponding vector field.

Inspired greatly by [16], Sun and Zhao discussed the third-order three-point BVP with sign-changing Green's function:

$$\begin{aligned} u'''(t) &= f(t, u(t)), \quad t \in [0, 1], \\ u'(0) &= u(1) = u''(\eta) = 0. \end{aligned} \quad (2)$$

They proved the existence of at least one or three positive solutions when $\eta \in (1/2, 1)$ by using the Guo-Krasnoselskii and Leggett-Williams fixed point theorems in [17, 18] and obtained the existence of a positive solution when $\eta \in [2 - \sqrt{2}, 1)$ by using iterative technique in [19]. For similar results, one can refer to [20, 21].

It is worth mentioning that there are other types of works on sign-changing Green's functions which prove the existence of sign-changing solutions, positive in some cases; see Infante and Webb's papers [22–24].

In this paper, we consider the following third-order BVP involving integral boundary condition and sign-changing Green's function:

$$\begin{aligned} u'''(t) &= f(t, u(t)), \quad t \in [0, 1], \\ u'(0) &= u(1) = 0, \quad u''(\eta) + \int_{\alpha}^{\beta} u(t)dt = 0. \end{aligned} \quad (3)$$

Throughout this paper, we always assume that $1/2 < \alpha \leq \beta \leq 1$, $\alpha + \beta \leq 4/3$, and $\eta \in (1/2, \alpha]$. By imposing suitable conditions on f , we obtain the existence of at least two positive and decreasing solutions for the BVP (3). The main tool used is the well-known Avery-Henderson two-fixed-point theorem.

To end this section, we state some fundamental definitions and the two-fixed-point theorem due to Avery and Henderson [25].

Let P be a cone in a real Banach space E .

Definition 1. A functional $\psi : P \rightarrow R$ is said to be increasing on P provided $\psi(x) \leq \psi(y)$ for all $x, y \in P$ with $x \leq y$, where $x \leq y$ if and only if $y - x \in P$.

Definition 2. Let $\gamma : P \rightarrow [0, +\infty)$ be continuous. For each $d > 0$, one defines the set

$$P(\gamma, d) = \{u \in P : \gamma(u) < d\}. \quad (4)$$

Theorem 3. Let ψ and γ be increasing, nonnegative, and continuous functionals on P , and let ω be a nonnegative continuous functional on P with $\omega(0) = 0$ such that, for some $c > 0$ and $M > 0$,

$$\gamma(u) \leq \omega(u) \leq \psi(u), \quad \|u\| \leq M\gamma(u) \quad (5)$$

for all $u \in \overline{P(\gamma, c)}$. Suppose there exist a completely continuous operator $A : \overline{P(\gamma, c)} \rightarrow P$ and $0 < a < b < c$ such that

$$\omega(\lambda u) \leq \lambda \omega(u) \quad \text{for } 0 \leq \lambda \leq 1, \quad u \in \partial P(\omega, b), \quad (6)$$

and

- (1) $\gamma(Au) > c$ for all $u \in \partial P(\gamma, c)$;
- (2) $\omega(Au) < b$ for all $u \in \partial P(\omega, b)$;
- (3) $P(\psi, a) \neq \emptyset$, and $\psi(Au) > a$ for all $u \in \partial P(\psi, a)$.

Then A has at least two fixed points u_1 and u_2 in $\overline{P(\gamma, c)}$ such that

$$\begin{aligned} a &< \psi(u_1) \quad \text{with } \omega(u_1) < b, \\ b &< \omega(u_2) \quad \text{with } \gamma(u_2) < c. \end{aligned} \quad (7)$$

2. Main Results

For any $y \in C[0, 1]$, we consider the BVP

$$\begin{aligned} u'''(t) &= y(t), \quad t \in [0, 1], \\ u'(0) &= u(1) = 0, \quad u''(\eta) + \int_{\alpha}^{\beta} u(t) dt = 0. \end{aligned} \quad (8)$$

After a direct computation, one may obtain the expression of Green's function $G(t, s)$ of the BVP (8) as follows:

$$\begin{aligned} G(t, s) &= -\frac{3(1-t^2)}{\rho} \int_{\alpha}^{\beta} K(\tau, s) d\tau - K(t, s) \\ &\quad + \begin{cases} 0, & s \geq \eta, \\ \frac{3(1-t^2)}{\rho}, & s < \eta, \end{cases} \end{aligned} \quad (9)$$

where

$$\begin{aligned} \rho &= 6 - 3(\beta - \alpha) + (\beta^3 - \alpha^3), \\ K(t, s) &= \frac{1}{2} \begin{cases} (1-t)(1+t-2s), & 0 \leq s \leq t \leq 1, \\ (1-s)^2, & 0 \leq t \leq s \leq 1. \end{cases} \end{aligned} \quad (10)$$

Remark 4. Consider the following: $\rho > 0$.

Remark 5. Consider the following: $0 \leq K(t, s) \leq (1/2)(1-s)^2$ for $(t, s) \in [0, 1] \times [0, 1]$.

Remark 6. $G(t, s)$ has the following properties:

$$G(t, s) \geq 0 \quad \text{for } 0 \leq s < \eta, \quad G(t, s) \leq 0 \quad \text{for } \eta \leq s \leq 1. \quad (11)$$

Moreover, for $s \geq \eta$,

$$\begin{aligned} \max \{G(t, s) : t \in [0, 1]\} &= G(1, s) = 0, \\ \min \{G(t, s) : t \in [0, 1]\} &= G(0, s) = -\frac{3}{\rho} \int_{\alpha}^{\beta} K(\tau, s) d\tau - \frac{1}{2}(1-s)^2 \end{aligned} \quad (12)$$

and, for $s < \eta$,

$$\begin{aligned} \max \{G(t, s) : t \in [0, 1]\} &= G(0, s) = \frac{3}{\rho} - \frac{3}{\rho} \int_{\alpha}^{\beta} K(\tau, s) d\tau - \frac{1}{2}(1-s)^2, \\ \min \{G(t, s) : t \in [0, 1]\} &= G(1, s) = 0. \end{aligned} \quad (13)$$

In the remainder of this paper, we always assume that $f : [0, 1] \times [0, +\infty) \rightarrow [0, +\infty)$ is continuous and satisfies the following two conditions:

- (C1) for each $u \in [0, +\infty)$, the mapping $t \mapsto f(t, u)$ is decreasing;
- (C2) for each $t \in [0, 1]$, the mapping $u \mapsto f(t, u)$ is increasing.

Let real Banach space $C[0, 1]$ be equipped with the norm $\|u\| = \max_{t \in [0, 1]} |u(t)|$ and

$$\begin{aligned} P &= \left\{ u \in C[0, 1] : u(t) \text{ is nonnegative} \right. \\ &\quad \text{and decreasing on } [0, 1], \\ &\quad \left. \text{and } \min_{t \in [1-\theta, \theta]} u(t) \geq \theta^* \|u\| \right\}, \end{aligned} \quad (14)$$

where $\theta \in (1/2, \eta)$ and $\theta^* = (\eta - \theta)/\eta$. Then P is a cone in $C[0, 1]$. Now, we define an operator A as follows:

$$(Au)(t) = \int_0^1 G(t, s) f(s, u(s)) ds, \quad u \in P, \quad t \in [0, 1]. \quad (15)$$

Obviously, if u is a fixed point of A in P , then u is a nonnegative and decreasing solution of the BVP (3).

Lemma 7. $A : P \rightarrow P$ is completely continuous.

Proof. Let $u \in P$. Then, for $t \in [0, \eta]$, we have

$$\begin{aligned}
 (Au)(t) &= \int_0^t \left[-\frac{3(1-t^2)}{\rho} \int_\alpha^\beta K(\tau, s) d\tau \right. \\
 &\quad \left. -\frac{1}{2}(1-t)(1+t-2s) + \frac{3(1-t^2)}{\rho} \right] \\
 &\quad \times f(s, u(s)) ds \\
 &\quad + \int_t^\eta \left[-\frac{3(1-t^2)}{\rho} \int_\alpha^\beta K(\tau, s) d\tau \right. \\
 &\quad \left. -\frac{1}{2}(1-s)^2 + \frac{3(1-t^2)}{\rho} \right] \\
 &\quad \times f(s, u(s)) ds \\
 &\quad + \int_\eta^1 \left[-\frac{3(1-t^2)}{\rho} \int_\alpha^\beta K(\tau, s) d\tau - \frac{1}{2}(1-s)^2 \right] \\
 &\quad \times f(s, u(s)) ds.
 \end{aligned} \tag{16}$$

In view of $u \in P$, $1/2 < \eta \leq \alpha$, $\beta + \alpha \leq 4/3$, Remark 5, and (C1) and (C2), we get

$$\begin{aligned}
 (Au)'(t) &= \int_0^t \left[\frac{6t}{\rho} \int_\alpha^\beta K(\tau, s) d\tau + t - s - \frac{6t}{\rho} \right] f(s, u(s)) ds \\
 &\quad + \frac{6t}{\rho} \int_t^\eta \left[\int_\alpha^\beta K(\tau, s) d\tau - 1 \right] f(s, u(s)) ds \\
 &\quad + \frac{6t}{\rho} \int_\eta^1 \int_\alpha^\beta K(\tau, s) d\tau f(s, u(s)) ds \\
 &\leq \int_0^t \left[\frac{3t}{\rho} \int_\alpha^\beta (1-\tau)(1+\tau-2s) d\tau + t - s - \frac{6t}{\rho} \right] \\
 &\quad \times f(s, u(s)) ds \\
 &\quad + \frac{3t}{\rho} \int_t^\eta \left[\int_\alpha^\beta (1-\tau)(1+\tau-2s) d\tau - 2 \right] f(s, u(s)) ds \\
 &\quad + \frac{3t(\beta-\alpha)}{\rho} \int_\eta^1 (1-s)^2 f(s, u(s)) ds \\
 &= \frac{1}{\rho} \left\{ -[3t(\beta-\alpha)(2-\beta-\alpha) + \rho] \right.
 \end{aligned}$$

$$\begin{aligned}
 &\quad \times \int_0^t s f(s, u(s)) ds \\
 &\quad - t \int_t^\eta [\rho + 3s(\beta-\alpha)(2-\beta-\alpha)] f(s, u(s)) ds \\
 &\quad \left. + 3t(\beta-\alpha) \int_\eta^1 (1-s)^2 f(s, u(s)) ds \right\} \\
 &\leq \frac{f(\eta, u(\eta))}{\rho} \\
 &\quad \times \left\{ -[3t(\beta-\alpha)(2-\beta-\alpha) + \rho] \right. \\
 &\quad \times \int_0^t s ds - t \int_t^\eta [\rho + 3s(\beta-\alpha)(2-\beta-\alpha)] ds \\
 &\quad \left. + 3t(\beta-\alpha) \int_\eta^1 (1-s)^2 ds \right\} \\
 &= \frac{tf(\eta, u(\eta))}{2\rho} \\
 &\quad \times [-\rho(2\eta-t) - 3(\beta-\alpha)(2-\beta-\alpha)\eta^2 \\
 &\quad + 2(\beta-\alpha)(1-\eta)^3] \\
 &\leq \frac{tf(\eta, u(\eta))}{2\rho} \\
 &\quad \times [-3(\beta-\alpha)(2-\beta-\alpha)\eta^2 + 2(\beta-\alpha)(1-\eta)^2] \\
 &= \frac{(\beta-\alpha)tf(\eta, u(\eta))}{2\rho} \\
 &\quad \times \{[3(\beta+\alpha)-4]\eta^2 + 2(1-2\eta)\} \\
 &\leq 0, \quad t \in [0, \eta],
 \end{aligned}$$

$$\begin{aligned}
 (Au)''(t) &= \int_0^t \left[\frac{6}{\rho} \int_\alpha^\beta K(\tau, s) d\tau + 1 - \frac{6}{\rho} \right] f(s, u(s)) ds \\
 &\quad + \frac{6}{\rho} \int_t^\eta \left[\int_\alpha^\beta K(\tau, s) d\tau - 1 \right] f(s, u(s)) ds \\
 &\quad + \frac{6}{\rho} \int_\eta^1 \int_\alpha^\beta K(\tau, s) d\tau f(s, u(s)) ds \\
 &\leq \int_0^t \left[\frac{3}{\rho} \int_\alpha^\beta (1-\tau)(1+\tau-2s) d\tau + 1 - \frac{6}{\rho} \right] f(s, u(s)) ds \\
 &\quad + \frac{3}{\rho} \int_t^\eta \left[\int_\alpha^\beta (1-\tau)(1+\tau-2s) d\tau - 2 \right] f(s, u(s)) ds \\
 &\quad + \frac{3(\beta-\alpha)}{\rho} \int_\eta^1 (1-s)^2 f(s, u(s)) ds
 \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\rho} \left\{ -3(\beta - \alpha)(2 - \beta - \alpha) \int_0^t s f(s, u(s)) ds \right. \\
&\quad - \int_t^\eta [\rho + 3s(\beta - \alpha)(2 - \beta - \alpha)] f(s, u(s)) ds \\
&\quad \left. + 3(\beta - \alpha) \int_\eta^1 (1 - s)^2 f(s, u(s)) ds \right\} \\
&\leq \frac{f(\eta, u(\eta))}{\rho} \\
&\quad \times \left\{ -3(\beta - \alpha)(2 - \beta - \alpha) \int_0^t s ds \right. \\
&\quad - \int_t^\eta [\rho + 3s(\beta - \alpha)(2 - \beta - \alpha)] ds \\
&\quad \left. + 3(\beta - \alpha) \int_\eta^1 (1 - s)^2 ds \right\} \\
&= \frac{f(\eta, u(\eta))}{2\rho} \\
&\quad \times \left[-2\rho(\eta - t) - 3(\beta - \alpha)(2 - \beta - \alpha)\eta^2 \right. \\
&\quad \left. + 2(\beta - \alpha)(1 - \eta)^3 \right] \\
&\leq \frac{(\beta - \alpha)f(\eta, u(\eta))}{2\rho} \\
&\quad \times \left[-3(2 - \beta - \alpha)\eta^2 + 2(1 - \eta)^2 \right] \\
&= \frac{(\beta - \alpha)f(\eta, u(\eta))}{2\rho} \\
&\quad \times \left\{ [3(\beta + \alpha) - 4]\eta^2 + 2(1 - 2\eta) \right\} \\
&\leq 0, \quad t \in [0, \eta].
\end{aligned}$$

(17)

For $t \in (\eta, 1]$, we have

$$\begin{aligned}
&(Au)(t) \\
&= \int_0^\eta \left[-\frac{3(1 - t^2)}{\rho} \int_\alpha^\beta K(\tau, s) d\tau - \frac{1}{2}(1 - t)(1 + t - 2s) \right. \\
&\quad \left. + \frac{3(1 - t^2)}{\rho} \right] f(s, u(s)) ds \\
&\quad + \int_\eta^t \left[-\frac{3(1 - t^2)}{\rho} \int_\alpha^\beta K(\tau, s) d\tau \right. \\
&\quad \left. - \frac{1}{2}(1 - t)(1 + t - 2s) \right] f(s, u(s)) ds
\end{aligned}$$

$$\begin{aligned}
&+ \int_t^1 \left[-\frac{3(1 - t^2)}{\rho} \int_\alpha^\beta K(\tau, s) d\tau \right. \\
&\quad \left. - \frac{1}{2}(1 - s)^2 \right] f(s, u(s)) ds.
\end{aligned}$$

(18)

In view of $u \in P$, $1/2 < \eta \leq \alpha, \beta + \alpha \leq 4/3$, Remark 5, and (C1) and (C2), we get

$$\begin{aligned}
&(Au)'(t) \\
&= \int_0^\eta \left[\frac{6t}{\rho} \int_\alpha^\beta K(\tau, s) d\tau + t - s - \frac{6t}{\rho} \right] f(s, u(s)) ds \\
&\quad + \int_\eta^t \left[\frac{6t}{\rho} \int_\alpha^\beta K(\tau, s) d\tau + t - s \right] f(s, u(s)) ds \\
&\quad + \frac{6t}{\rho} \int_t^1 \int_\alpha^\beta K(\tau, s) d\tau f(s, u(s)) ds \\
&\leq \int_0^\eta \left[\frac{3t}{\rho} \int_\alpha^\beta (1 - \tau)(1 + \tau - 2s) d\tau \right. \\
&\quad \left. + t - s - \frac{6t}{\rho} \right] f(s, u(s)) ds \\
&\quad + \int_\eta^t \left[\frac{3t(\beta - \alpha)}{\rho} (1 - s)^2 + t - s \right] f(s, u(s)) ds \\
&\quad + \frac{3t(\beta - \alpha)}{\rho} \int_t^1 (1 - s)^2 f(s, u(s)) ds \\
&= \frac{1}{\rho} \left\{ -[3t(\beta - \alpha)(2 - \beta - \alpha) + \rho] \int_0^\eta s f(s, u(s)) ds \right. \\
&\quad + 3t(\beta - \alpha) \int_\eta^1 (1 - s)^2 f(s, u(s)) ds \\
&\quad \left. + \rho \int_\eta^t (t - s) f(s, u(s)) ds \right\} \\
&\leq \frac{f(\eta, u(\eta))}{\rho} \\
&\quad \times \left\{ -[3t(\beta - \alpha)(2 - \beta - \alpha) + \rho] \int_0^\eta s ds \right. \\
&\quad + 3t(\beta - \alpha) \int_\eta^1 (1 - s)^2 ds \\
&\quad \left. + \rho \int_\eta^t (t - s) ds \right\}
\end{aligned}$$

$$\begin{aligned}
&= \frac{tf(\eta, u(\eta))}{2\rho} \\
&\quad \times \left[-3(\beta - \alpha)(2 - \beta - \alpha)\eta^2 + 2(\beta - \alpha)(1 - \eta)^3 \right. \\
&\quad \left. + \rho(t - 2\eta) \right] \\
&\leq \frac{tf(\eta, u(\eta))}{2\rho} \\
&\quad \times \left[-3(\beta - \alpha)(2 - \beta - \alpha)\eta^2 + 2(\beta - \alpha)(1 - \eta)^2 \right. \\
&\quad \left. + \rho(1 - 2\eta) \right] \\
&= \frac{tf(\eta, u(\eta))}{2\rho} \\
&\quad \times \left\{ (\beta - \alpha)[3(\beta + \alpha) - 4]\eta^2 + (1 - 2\eta)[2(\beta - \alpha) + \rho] \right\} \\
&\leq 0, \quad t \in (\eta, 1].
\end{aligned} \tag{19}$$

So, it follows from $(Au)'(t) \leq 0$ for $t \in [0, 1]$ that $(Au)(t)$ is decreasing on $[0, 1]$, which together with $(Au)(1) = 0$ implies that $(Au)(t) \geq 0$, $t \in [0, 1]$. At the same time, since $(Au)''(t) \leq 0$ for $t \in [0, \eta]$, we know that $(Au)(t)$ is concave on $[0, \eta]$. Therefore,

$$\begin{aligned}
(Au)(t) &\geq \left(1 - \frac{t}{\eta}\right)(Au)(0) + \frac{t}{\eta}(Au)(\eta) \\
&\geq \frac{\eta - t}{\eta} \|Au\|, \quad t \in [0, \eta].
\end{aligned} \tag{20}$$

Consequently,

$$\min_{t \in [1-\theta, \theta]} (Au)(t) = (Au)(\theta) \geq \frac{\eta - \theta}{\eta} \|Au\| = \theta^* \|Au\|. \tag{21}$$

This shows that $A : P \rightarrow P$. Furthermore, although $G(t, s)$ is not continuous, it follows from known textbook results, for example, [26], that $A : P \rightarrow P$ is completely continuous. $\square$

Lemma 8. For any $u \in P$ and $t \in [1 - \theta, \theta]$,

$$\begin{aligned}
&\int_0^{1-\theta} G(t, s) f(s, u(s)) ds + \int_\theta^\eta G(t, s) f(s, u(s)) ds \\
&\quad + \int_\eta^1 G(t, s) f(s, u(s)) ds \geq 0.
\end{aligned} \tag{22}$$

Proof. Let $u \in P$. Then it follows from $\theta < \eta \leq \alpha$, $\beta + \alpha \leq 4/3$, Remarks 5 and 6, and (C1) and (C2) that, for any $t \in [1 - \theta, \theta]$,

$$\begin{aligned}
&\int_0^{1-\theta} G(t, s) f(s, u(s)) ds \\
&\quad + \int_\theta^\eta G(t, s) f(s, u(s)) ds + \int_\eta^1 G(t, s) f(s, u(s)) ds \\
&\geq \int_0^{1-\theta} G(t, s) f(s, u(s)) ds + \int_\eta^1 G(t, s) f(s, u(s)) ds \\
&\geq f(\eta, u(\eta)) \left[\int_0^{1-\theta} G(t, s) ds + \int_\eta^1 G(t, s) ds \right] \\
&= f(\eta, u(\eta)) \\
&\quad \times \left\{ \int_0^{1-\theta} \left[-\frac{3(1-t^2)}{\rho} \int_\alpha^\beta K(\tau, s) d\tau \right. \right. \\
&\quad \left. \left. - \frac{1}{2}(1-t)(1+t-2s) + \frac{3(1-t^2)}{\rho} \right] ds \right. \\
&\quad \left. - \int_\eta^1 \left[\frac{3(1-t^2)}{\rho} \int_\alpha^\beta K(\tau, s) d\tau + \frac{1}{2}(1-s)^2 \right] ds \right\} \\
&\geq f(\eta, u(\eta)) \\
&\quad \times \left\{ \int_0^{1-\theta} \left[-\frac{3(1-t^2)}{2\rho} \int_\alpha^\beta (1-\tau)(1+\tau-2s) d\tau \right. \right. \\
&\quad \left. \left. - \frac{1}{2}(1-t)(1+t-2s) + \frac{3(1-t^2)}{\rho} \right] ds \right. \\
&\quad \left. - \frac{3(1-t^2)(\beta - \alpha) + \rho}{2\rho} \int_\eta^1 (1-s)^2 ds \right\} \\
&= \frac{f(\eta, u(\eta))}{2\rho} \\
&\quad \times \left\{ \frac{3}{2}(1-t^2)(\beta - \alpha)(2 - \beta - \alpha)(1 - \theta)^2 \right. \\
&\quad \left. + \rho(1-t)(1 - \theta)^2 \right. \\
&\quad \left. - [3(1-t^2)(\beta - \alpha) + \rho] \frac{(1-\eta)^3}{3} \right\} \\
&\geq \frac{f(\eta, u(\eta))}{2\rho} \\
&\quad \times \left\{ \frac{3}{2}(1-t^2)(\beta - \alpha)(2 - \beta - \alpha)(1 - \theta)^2 \right. \\
&\quad \left. + \rho(1-t)(1 - \theta)^2 \right.
\end{aligned}$$

$$\begin{aligned}
& - \left[3(1-t^2)(\beta - \alpha) + \rho \right] \frac{(1-\theta)^3}{3} \Big\} \\
& = \frac{(1-\theta)^2 f(\eta, u(\eta))}{2\rho} \\
& \times \left\{ (1-t^2)(\beta - \alpha) \left[3 - \frac{3}{2}(\beta + \alpha) - (1-\theta) \right] \right. \\
& \quad \left. + \rho(1-t) - \frac{\rho}{3}(1-\theta) \right\} \\
& \geq \frac{(1-\theta)^2 f(\eta, u(\eta))}{2\rho} \\
& \times \left\{ (1-t^2)(\beta - \alpha) \left[2 - \frac{3}{2}(\beta + \alpha) \right] + \frac{2\rho}{3}(1-\theta) \right\} \\
& \geq 0.
\end{aligned} \tag{23}$$

□

In what follows, we impose conditions on f such that the BVP (3) has at least two positive and decreasing solutions. For convenience, we denote

$$H_1 = \int_{1-\theta}^{\theta} G(\theta, s) ds, \quad H_2 = \int_0^{\eta} G(\theta, s) ds, \tag{24}$$

$$H_3 = \int_{1-\theta}^{\theta} G(1-\theta, s) ds.$$

Theorem 9. Suppose that there exist numbers a, b , and c with $0 < a < b < \theta^* c$ such that

$$f(\theta, c) > \frac{c}{H_1}, \tag{25}$$

$$f\left(0, \frac{b}{\theta^*}\right) < \frac{b}{H_2}, \tag{26}$$

$$f(\theta, \theta^* a) > \frac{a}{H_3}. \tag{27}$$

Then the BVP (3) has at least two positive and decreasing solutions.

Proof. First, we define the increasing, nonnegative, and continuous functionals γ, ω , and ψ on P as follows:

$$\begin{aligned}
\gamma(u) &= \min_{t \in [1-\theta, \theta]} u(t) = u(\theta), \\
\omega(u) &= \max_{t \in [\theta, 1]} u(t) = u(\theta), \\
\psi(u) &= \max_{t \in [1-\theta, 1]} u(t) = u(1-\theta).
\end{aligned} \tag{28}$$

Obviously, for any $u \in P$, $\gamma(u) = \omega(u) \leq \psi(u)$. At the same time, for each $u \in P$, in view of $\gamma(u) = \min_{t \in [1-\theta, \theta]} u(t) \geq \theta^* \|u\|$, we have

$$\|u\| \leq \frac{1}{\theta^*} \gamma(u) \quad \text{for } u \in P. \tag{29}$$

In addition, we also note that

$$\omega(\lambda u) = \lambda \omega(u) \quad \text{for } 0 \leq \lambda \leq 1, \quad u \in P. \tag{30}$$

Next, we assert that $\gamma(Au) > c$ for all $u \in \partial P(\gamma, c)$.

To prove this, let $u \in \partial P(\gamma, c)$; that is, $u \in P$ and $\gamma(u) = u(\theta) = c$. Then

$$u(t) \geq u(\theta) = c, \quad t \in [1-\theta, \theta]. \tag{31}$$

In view of Lemma 8, (25), (31), and (C1) and (C2), we have

$$\begin{aligned}
\gamma(Au) &= (Au)(\theta) \\
&= \int_0^1 G(\theta, s) f(s, u(s)) ds \\
&\geq \int_{1-\theta}^{\theta} G(\theta, s) f(s, u(s)) ds \\
&\geq \int_{1-\theta}^{\theta} G(\theta, s) f(\theta, c) ds \\
&> \frac{c}{H_1} \int_{1-\theta}^{\theta} G(\theta, s) ds \\
&= c.
\end{aligned} \tag{32}$$

Then, we assert that $\omega(Au) < b$ for all $u \in \partial P(\omega, b)$.

To see this, suppose $u \in \partial P(\omega, b)$; that is, $u \in P$ and $\omega(u) = b$. Since $\|u\| \leq (1/\theta^*)\gamma(u) = (1/\theta^*)\omega(u)$, we have

$$0 \leq u(t) \leq \|u\| \leq \frac{b}{\theta^*}, \quad t \in [0, \eta]. \tag{33}$$

In view of Remark 6, (26), (33), and (C1) and (C2), we get

$$\begin{aligned}
\omega(Au) &= (Au)(\theta) \\
&= \int_0^1 G(\theta, s) f(s, u(s)) ds \\
&\leq \int_0^{\eta} G(\theta, s) f(s, u(s)) ds \\
&\leq \int_0^{\eta} G(\theta, s) f\left(0, \frac{b}{\theta^*}\right) ds \\
&< \frac{b}{H_2} \int_0^{\eta} G(\theta, s) ds \\
&= b.
\end{aligned} \tag{34}$$

Finally, we assert that $P(\psi, a) \neq \emptyset$, and $\psi(Au) > a$ for all $u \in \partial P(\psi, a)$.

In fact, the constant function $a/2 \in P(\psi, a)$. Moreover, for $u \in \partial P(\psi, a)$, that is, $u \in P$ and $\psi(u) = u(1-\theta) = a$. Then

$$u(t) \geq \theta^* \|u\| \geq \theta^* u(1-\theta) = \theta^* a, \quad t \in [1-\theta, \theta]. \tag{35}$$

In view of Lemma 8, (27), (35), and (C1) and (C2), we get

$$\begin{aligned}
 \psi(Au) &= (Au)(1 - \theta) \\
 &= \int_0^1 G(1 - \theta, s) f(s, u(s)) ds \\
 &\geq \int_{1-\theta}^\theta G(1 - \theta, s) f(s, u(s)) ds \\
 &\geq \int_{1-\theta}^\theta G(1 - \theta, s) f(\theta, \theta^* a) ds \\
 &> \frac{a}{H_3} \int_{1-\theta}^\theta G(1 - \theta, s) ds \\
 &= a.
 \end{aligned} \tag{36}$$

To sum up, all the hypotheses of Theorem 3 are satisfied. Hence A has at least two fixed points; that is, the BVP (3) has at least two positive and decreasing solutions u_1 and u_2 such that

$$\begin{aligned}
 a &< \max_{t \in [1-\theta, 1]} u_1(t) \quad \text{with} \quad \max_{t \in [\theta, 1]} u_1(t) < b, \\
 b &< \max_{t \in [\theta, 1]} u_2(t) \quad \text{with} \quad \min_{t \in [1-\theta, \theta]} u_2(t) < c.
 \end{aligned} \tag{37}$$

□

3. An Example

Example 1. Consider the BVP

$$\begin{aligned}
 u'''(t) &= f(t, u(t)), \quad t \in [0, 1], \\
 u'(0) &= u(1) = 0, \quad u''\left(\frac{3}{5}\right) + \int_{3/5}^{11/15} u(t) dt = 0,
 \end{aligned} \tag{38}$$

where

$$f(t, u) = \begin{cases} \sqrt{u} + \frac{1-t^2}{2} + 85, & (t, u) \in [0, 1] \times [0, 144], \\ 4748u + \frac{1-t^2}{2} - 683615, & (t, u) \in [0, 1] \times [144, 145], \\ \frac{969u^2}{4205} + \frac{1-t^2}{2}, & (t, u) \in [0, 1] \times [145, +\infty). \end{cases} \tag{39}$$

Since $\eta = \alpha = 3/5$ and $\beta = 11/15$, if we choose $\theta = 17/30$, then a simple calculation shows that

$$\begin{aligned}
 \rho &= \frac{19502}{3375}, \quad \theta^* = \frac{1}{18}, \quad H_1 = \frac{262691}{8775900}, \\
 H_2 &= \frac{5106323}{63186480}, \quad H_3 = \frac{15256439}{394915500}.
 \end{aligned} \tag{40}$$

Thus, if we let $a = 3$, $b = 8$, and $c = 145$, then it is easy to verify that all the conditions of Theorem 9 are satisfied. So, it follows from Theorem 9 that the BVP (38) has at least two positive and decreasing solutions.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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