Research Article

Singularities of Null Hypersurfaces of Pseudonull Curves

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The main goal of this paper is to study the singularities of null hypersurfaces of pseudonull curves. To do this we construct a null frame and a Lorentz distance-squared function of the pseudonull curve. The relations are shown between singularities of the null hypersurfaces and those, of the Lorentz distance-squared function. And we reveal the geometric meanings of the singularities of such hypersurfaces. In addition, we also introduce some properties of the nullcone Gaussian surface of the pseudonull curve.

1. Introduction

A pseudonull curve is not a null curve, but its tangent curve is a null curve. Mathematicians have got many good results on the aspect of differential geometry of pseudonull curves [1–7]. Pseudonull helices, pseudonull Mannheim curves, pseudonull rectifying curves, and pseudonull osculating curves are considered, respectively, in [1], [2], [3], and [4]. And the relations are gotten in [5] between pseudonormal and pseudorectifying curves in Minkowski space-time. Moreover, the involute-evolute of the pseudonull curve is studied in [6] and they prove that there is no involute of pseudonull curves in Minkowski 3-space.

However, there are few articles about pseudonull curves from the view point of singularities theory. In this paper, we focus on the singularities of null hypersurfaces of pseudonull curves. Here, we investigate the pseudonull curves with $k_1(s) \neq 0$. We get the relationship between the singularities of null hypersurfaces of pseudonull curves and the functions ρ and η . When $k_1(s) = 0$, the third normal vector equals $\gamma^{(3)}/\|\gamma^{(3)}\|$. This situation is considered in [7]. The authors, Petrović-Torgašev et al., make a contribution to the research of pseudonull curves. For example, they obtain the Frenet equations of pseudonull curves with only two curvatures and classify all such curves with constant curvatures. Besides curves in semi-Euclidean space, regarding singularity, have been studied extensively by Pei et al. and they make some achievements [8–11].

Here we discuss null hypersurfaces and nullcone Gaussian surfaces of pseudonull curves in $\mathbb{R}_2^4$. There are some

differences between Minkowski space-time (i.e., $\mathbb{R}_1^4$) and $\mathbb{R}_2^4$, for instance, the appearance of nullcone along a timelike pseudonull curve. Thus, we consider a one-parameter family of null Gauss indicatrices. We call it nullcone Gaussian surface. It can locally be a spacelike surface or a Lorentzian surface or a 1-lightlike surface depending on the characteristic of tangent plane of the surface. Moreover, we point that the nullcone Gaussian surface is also a regular surface. The null hypersurface is a surface bundle along a pseudonull curve whose fibres are the nullcone Gaussian surfaces. To allow a useful study of these singularities, we consider Lorentz distance-squared functions (which are denoted by $G : I \times \mathbb{R}_2^4 \rightarrow \mathbb{R}$, $G(s, \mathbf{v}) = \langle \gamma(s) - \mathbf{v}, \gamma(s) - \mathbf{v} \rangle$). These functions are the unfolding processes of these singularities in the local neighborhood of $(s_0, \mathbf{v}_0)$, and these functions only depend on the germ that they are unfolding. In this paper, we create these functions by varying a fixed point $\mathbf{v}$ in the Lorentz distance-squared function $G(s, \mathbf{v}) = \langle \gamma(s) - \mathbf{v}, \gamma(s) - \mathbf{v} \rangle$, to obtain a family of functions. We show that these singularities are versally unfolded by the family of Lorentz distance-squared functions. If the singularity of g_γ is A_k -type and if the corresponding four-parameter unfolding is versal, then, by applying Bruce's theory (cf. Bruce and Giblin [12]), we know that the discriminant set of the four-parameter unfolding processes is locally diffeomorphic to a cusp or a swallowtail or a butterfly under certain conditions. Thus, we have completed the classification of the singularities of a null hypersurface of a pseudonull curve.

The main result is Theorem 2 which is about the geometric meanings of the singularities of null hypersurfaces

of pseudonull curves. Our work must be conducive to the research of pseudonull curves and be an exploration to null submanifolds in this space. And we will consider the situation that γ'' is not a null vector in our other paper [13]. We assume throughout the paper that all manifolds and maps are C^∞ unless explicitly stated otherwise.

2. Preliminaries

In this section we introduce some basic notions for semi-Euclidean 4-space with index 2 and pseudonull curves. Let $\mathbb{R}^4$ be a 4-dimensional vector space. For $\mathbf{x} = (x^0, x^1, x^2, x^3)$ and $\mathbf{y} = (y^0, y^1, y^2, y^3)$ in $\mathbb{R}^4$, the *pseudoscalar product* of $\mathbf{x}$ and $\mathbf{y}$ is defined as $\langle \mathbf{x}, \mathbf{y} \rangle = -x^0 y^0 - x^1 y^1 + x^2 y^2 + x^3 y^3$. We call $(\mathbb{R}^4, \langle \cdot, \cdot \rangle)$ a *semi-Euclidean 4-space with index 2*. We write $\mathbb{R}_2^4$ instead of $(\mathbb{R}^4, \langle \cdot, \cdot \rangle)$. We say that a nonzero vector $\mathbf{x} \in \mathbb{R}_2^4$ is *spacelike*, *null*, or *timelike* if $\langle \mathbf{x}, \mathbf{x} \rangle > 0$, $\langle \mathbf{x}, \mathbf{x} \rangle = 0$, or $\langle \mathbf{x}, \mathbf{x} \rangle < 0$, respectively. The *norm* of the vector $\mathbf{x} \in \mathbb{R}_2^4 \setminus \{0\}$ is defined as $\|\mathbf{x}\| = \sqrt{|\langle \mathbf{x}, \mathbf{x} \rangle|}$. The signature of a vector $\mathbf{x} \in \mathbb{R}_2^4 \setminus \{0\}$ is defined as

$$\text{sign}(\mathbf{x}) = \begin{cases} 1 & \mathbf{x} \text{ is spacelike} \\ 0 & \mathbf{x} \text{ is null} \\ -1 & \mathbf{x} \text{ is timelike.} \end{cases} \quad (1)$$

We call $\text{NC}_\alpha = \{\mathbf{x} = (x^0, x^1, x^2, x^3) \in \mathbb{R}_2^4 \mid \langle \mathbf{x} - \alpha, \mathbf{x} - \alpha \rangle = 0\}$ a *nullcone* with vertex α and denote $\text{NC}^* = \text{NC}_0 \setminus \{0\}$.

For any $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in \mathbb{R}_2^4$, we define the vector $\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{x}_3$ as

$$\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{x}_3 = \begin{vmatrix} -\mathbf{e}_0 & -\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ x_1^0 & x_1^1 & x_1^2 & x_1^3 \\ x_2^0 & x_2^1 & x_2^2 & x_2^3 \\ x_3^0 & x_3^1 & x_3^2 & x_3^3 \end{vmatrix}, \quad (2)$$

where $\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is the canonical basis of $\mathbb{R}_2^4$ and $\mathbf{x}_i = (x_i^0, x_i^1, x_i^2, x_i^3)$. We can easily show that

$$\langle \mathbf{x}, \mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{x}_3 \rangle = \det(\mathbf{x}, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3), \quad (3)$$

so that $\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{x}_3$ is pseudoorthogonal to any $\mathbf{x}_i$ ($i = 1, 2, 3$).

Let $\gamma : I \rightarrow \mathbb{R}_2^4$ be a smooth regular curve (i.e., $\gamma'(t) \neq 0$), where I is an open interval. For any $t \in I$, the curve γ is called *spacelike*, *null* (lightlike), or *timelike* if the velocity of the curve is $\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle > 0$, $\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle = 0$, or $\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle < 0$, respectively.

Let $\gamma : I \rightarrow \mathbb{R}_2^4$ be a unit speed timelike curve, parameterized by the arc length parameter s ; that is, $\langle \gamma'(s), \gamma'(s) \rangle = -1$. For any $s \in I$, the curve γ is called a *pseudonull curve* if $\gamma''(s)$ is a null vector (see [7]).

One can define the null frame $E = \{\mathbf{t}, \mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}$ of $\mathbb{R}_2^4$ which is positively oriented 4-tuple of vectors satisfying

$$\begin{aligned} \langle \mathbf{n}_1, \mathbf{n}_1 \rangle &= \langle \mathbf{n}_2, \mathbf{n}_2 \rangle = 0, & \langle \mathbf{n}_1, \mathbf{n}_2 \rangle &= \frac{1}{2}, \\ \langle \mathbf{t}, \mathbf{t} \rangle &= -1, & \langle \mathbf{n}_3, \mathbf{n}_3 \rangle &= 1, \\ \langle \mathbf{t}, \mathbf{n}_1 \rangle &= \langle \mathbf{t}, \mathbf{n}_2 \rangle = \langle \mathbf{t}, \mathbf{n}_3 \rangle = \langle \mathbf{n}_1, \mathbf{n}_3 \rangle = \langle \mathbf{n}_2, \mathbf{n}_3 \rangle = 0. \end{aligned} \quad (4)$$

For the sake of the simple forms and the convenience of our research objects, we take $\langle \mathbf{n}_1, \mathbf{n}_2 \rangle = 1/2$. The Frenet formula of γ with respect to the frame E is as follows:

$$\begin{aligned} \mathbf{t}'(s) &= \mathbf{n}_1(s) \\ \mathbf{n}_1'(s) &= k_1(s) \mathbf{n}_1(s) + k_2(s) \mathbf{n}_3(s) \\ \mathbf{n}_2'(s) &= \frac{1}{2} \mathbf{t}(s) - k_1(s) \mathbf{n}_2(s) + k_3(s) \mathbf{n}_3(s) \\ \mathbf{n}_3'(s) &= -2k_3(s) \mathbf{n}_1(s) - 2k_2(s) \mathbf{n}_2(s). \end{aligned} \quad (5)$$

When $k_1(s) = 0$, $\mathbf{n}_3(s) = \gamma^{(3)}(s)/\|\gamma^{(3)}(s)\|$ which is the situation that the authors consider in [7]. We focus on $k_1(s) \neq 0$. And we construct the null frame of γ in a different way. In [7], they also investigate W-curves, that is, the pseudonull curves with constant curvatures, and obtain the classifications of W-curves. Here, we do not discuss W-curves. We focus on the fact that $k_2(s)$ is not a constant. In particular $k_2(s) \neq 0$; otherwise $\gamma'', \gamma^{(3)}, \gamma^{(4)}, \dots$, are all null vectors with the same direction. It means γ'' is a lightlike straight line.

Without loss of generality, we suppose that $\gamma^{(3)}$ and $\gamma^{(4)}$ are not null vectors. And we have $\langle \gamma^{(3)}, \gamma^{(4)} \rangle \neq 0$ under the condition that $k_2(s)$ is not a constant. Thus, we take

$$\begin{aligned} \mathbf{t}(s) &= \frac{d\gamma}{ds}, \\ \mathbf{n}_1(s) &= \gamma''(s), \\ \mathbf{n}_2(s) &= -\frac{\langle \gamma^{(4)}, \gamma^{(4)} \rangle}{4 \langle \gamma^{(3)}, \gamma^{(3)} \rangle^2} \left(\gamma'' + 2 \frac{\langle \gamma^{(3)}, \gamma^{(3)} \rangle}{\langle \gamma^{(4)}, \gamma^{(4)} \rangle} \gamma^{(4)} \right)(s), \\ \mathbf{n}_3(s) &= \frac{\langle \gamma^{(3)}, \gamma^{(4)} \rangle}{\langle \gamma^{(3)}, \gamma^{(3)} \rangle^{3/2}} \left(\gamma'' + \frac{\langle \gamma^{(3)}, \gamma^{(3)} \rangle}{\langle \gamma^{(3)}, \gamma^{(4)} \rangle} \gamma^{(3)} \right)(s), \end{aligned} \quad (6)$$

which are appropriate for conditions (4). Thus, we have

$$\begin{aligned} k_1(s) &= -\frac{\langle \gamma^{(4)}, \gamma^{(4)} \rangle}{2 \langle \gamma^{(3)}, \gamma^{(3)} \rangle^2} \cdot \left\langle \gamma^{(3)}, \left(\gamma'' + 2 \frac{\langle \gamma^{(3)}, \gamma^{(3)} \rangle}{\langle \gamma^{(4)}, \gamma^{(4)} \rangle} \gamma^{(4)} \right) \right\rangle(s), \\ k_2(s) &= \frac{\langle \gamma^{(3)}, \gamma^{(4)} \rangle}{\langle \gamma^{(3)}, \gamma^{(3)} \rangle^{3/2}} \left\langle \gamma^{(3)}, \left(\gamma'' + \frac{\langle \gamma^{(3)}, \gamma^{(3)} \rangle}{\langle \gamma^{(3)}, \gamma^{(4)} \rangle} \gamma^{(3)} \right) \right\rangle(s), \\ k_3(s) &= \frac{\langle \gamma^{(3)}, \gamma^{(4)} \rangle}{\langle \gamma^{(3)}, \gamma^{(3)} \rangle^{3/2}} \left\langle \mathbf{n}_2', \left(\gamma'' + \frac{\langle \gamma^{(3)}, \gamma^{(3)} \rangle}{\langle \gamma^{(3)}, \gamma^{(4)} \rangle} \gamma^{(3)} \right) \right\rangle(s). \end{aligned} \quad (7)$$

We define the *family of null Gauss indicatrices* of γ

$$\mathbf{n}_{g_\gamma} : U \longrightarrow \text{NC}^* \quad (8)$$

by $\mathbf{n}\mathbf{g}_\gamma(s, \theta) = \tan \theta \mathbf{n}_1(s) - \cot \theta \mathbf{n}_2(s) + \mathbf{n}_3(s)$, where $U = I \times (0, \pi/2)$. It is also called the *nullcone Gaussian surface* of γ . By a straightforward calculation, we have

$$\begin{aligned} \frac{\partial \mathbf{n}\mathbf{g}_\gamma}{\partial \theta} &= (1 + \tan^2 \theta) \mathbf{n}_1 + (1 + \cot^2 \theta) \mathbf{n}_2, \\ \frac{\partial \mathbf{n}\mathbf{g}_\gamma}{\partial s} &= -\frac{1}{2} \cot \theta \mathbf{t} + (k_1 \tan \theta - 2k_3) \mathbf{n}_1 \\ &\quad + (k_1 \cot \theta - 2k_2) \mathbf{n}_2 + (k_2 \tan \theta - k_3 \cot \theta) \mathbf{n}_3. \end{aligned} \quad (9)$$

The above two vectors are definitely linearly independent, because $\cot \theta \neq 0$. Therefore, the nullcone Gaussian surface $\mathbf{n}\mathbf{g}_\gamma$ is a regular surface. Obviously, $\partial \mathbf{n}\mathbf{g}_\gamma / \partial \theta$ is a spacelike vector. Let

$$\begin{aligned} \iota &= (2 + \tan^2 \theta + \cot^2 \theta) \left(2k_2 \frac{\partial \mathbf{n}\mathbf{g}_\gamma}{\partial \theta} + (1 + \cot^2 \theta) \frac{\partial \mathbf{n}\mathbf{g}_\gamma}{\partial s} \right) \\ &\quad - \left\langle 2k_2 \frac{\partial \mathbf{n}\mathbf{g}_\gamma}{\partial \theta} + (1 + \cot^2 \theta) \frac{\partial \mathbf{n}\mathbf{g}_\gamma}{\partial s}, \frac{\partial \mathbf{n}\mathbf{g}_\gamma}{\partial \theta} \right\rangle \frac{\partial \mathbf{n}\mathbf{g}_\gamma}{\partial \theta}. \end{aligned} \quad (10)$$

We have $\langle \iota, \partial \mathbf{n}\mathbf{g}_\gamma / \partial \theta \rangle = 0$ and the vector ι might be spacelike, timelike, or null. Therefore, the nullcone Gaussian surface $\mathbf{n}\mathbf{g}_\gamma$ might locally be a spacelike surface or a Lorentzian surface or a 1-lightlike surface depending on the characteristic of tangent plane of the surface. In [14], the authors introduce the notion of r -lightlike surface. As a lightlike submanifold, its null space is not empty and its nullity degree is just r . In other words, there will always be some vectors simultaneously pseudoorthogonal to the tangent space as well as the normal space. And we will give some results associated with 1-lightlike nullcone Gaussian surface in our other paper [13]. On the other hand, when the nullcone Gaussian surface degenerates onto $\mathbf{n}_1(s)$ or $\mathbf{n}_2(s)$, it can be constructed by the sum or minus of the spacelike vector $\mathbf{n}_1(s) + \mathbf{n}_2(s)$ and the timelike vector $\mathbf{n}_1(s) - \mathbf{n}_2(s)$. We can use similar method in [13], which we do not consider here.

Remark 1. If we take $U = I \times (-(\pi/2), 0)$, we can have the same discussion. For the sake of the continuity of the surface, we only consider one situation.

We define the null hypersurface along γ

$$\mathbf{n}\mathbf{h}_\gamma : U \times \mathbb{R} \longrightarrow \mathbb{R}_2^4 \quad (11)$$

by $\mathbf{n}\mathbf{h}_\gamma(s, \theta, t) = \gamma(s) + t \mathbf{n}\mathbf{g}_\gamma(s, \theta)$. If we fix θ_0 , the null hypersurface turns into a null ruled surface along γ .

We also define new invariants of the pseudonull curve in $\mathbb{R}_2^4$ by

$$\eta(s) = \left(\frac{2k_2^2 k_3 - k_1' k_2}{k_1} - \frac{k_1 k_2}{2} + k_2' \right)(s). \quad (12)$$

Let $F : \mathbb{R}_2^4 \rightarrow \mathbb{R}$ be a submersion and $\gamma : I \rightarrow \mathbb{R}_2^4$ a timelike curve. We say that γ and $F^{-1}(0)$ have k -point contact for $t = t_0$ if the function $h(t) = F \circ \gamma(t)$ satisfies $h(t_0) =$

$h'(t_0) = \dots = h^{(k-1)}(t_0) = 0, h^{(k)}(t_0) \neq 0$. We also say that γ and $F^{-1}(0)$ have at least k -point contact for $t = t_0$ if the function $h(t) = F \circ \gamma(t)$ satisfies $h(t_0) = h'(t_0) = \dots = h^{(k-1)}(t_0) = 0$. For any fixed $\mathbf{v}_0 \in \mathbb{R}_2^4$, we have a model surface $\text{NC}_{\mathbf{v}_0}$. It is a light cone with vertex $\mathbf{v}_0$. We now consider the following conditions.

- (A 1) The number of points p of $\gamma(I)$ where the model surface at p having five-point contact with the curve γ is finite.
- (A 2) There is no point p of $\gamma(I)$ where the model surface at p has greater than or equal to six-point contact with the curve γ .

Here, we present the main results in this paper.

Theorem 2. Let $\gamma : I \rightarrow \mathbb{R}_2^4$ be a timelike pseudonull curve. Let $\mathbf{v}_0 = \mathbf{n}\mathbf{h}_\gamma(s_0, \theta_0, t_0)$; one has the following.

- (1) $\text{NC}_{\mathbf{v}_0}$ and γ have at least 2-point contact at s_0 .
- (2) $\text{NC}_{\mathbf{v}_0}$ and γ have 3-point contact at s_0 if and only if there exists $\theta_0 \in (0, \pi/2)$ such that

$$\rho(s_0, \theta_0) \neq 0, \quad \gamma(s_0) - \mathbf{v}_0 = -\frac{2}{\cot \theta_0} \mathbf{n}\mathbf{g}_\gamma(s_0, \theta_0), \quad (13)$$

where $\rho(s, \theta) = -(1/2)k_1(s) \cot \theta + k_2(s)$. Under this condition, the null hypersurface $\mathbf{n}\mathbf{h}_\gamma$ at $\mathbf{v}_0$ is locally diffeomorphic to $C(2, 3) \times \mathbb{R}^2$ and the null focal set $\mathbf{n}\mathbf{f}_\gamma$ is nonsingular.

- (3) $\text{NC}_{\mathbf{v}_0}$ and γ have 4-point contact at s_0 if and only if there exists $\theta_0 = \theta(s_0) \in (0, \pi/2)$ such that

$$\begin{aligned} \rho(s_0, \theta(s_0)) &= 0, \quad \eta(s_0) \neq 0, \\ \gamma(s_0) - \mathbf{v}_0 &= -\frac{2}{\cot \theta(s_0)} \mathbf{n}\mathbf{g}_\gamma(s_0, \theta(s_0)), \end{aligned} \quad (14)$$

where $\rho(s, \theta) = -(1/2)k_1(s) \cot \theta + k_2(s)$. Under this condition, the null hypersurface $\mathbf{n}\mathbf{h}_\gamma$ at $\mathbf{v}_0$ is locally diffeomorphic to $\text{SW} \times \mathbb{R}$, the null focal set $\mathbf{n}\mathbf{f}_\gamma$ is locally diffeomorphic to $C(2, 3, 4) \times \mathbb{R}$, and the singular value set of $\mathbf{n}\mathbf{f}_\gamma$ is a regular curve.

- (4) $\text{NC}_{\mathbf{v}_0}$ and γ have 5-point contact at s_0 if and only if there exists $\theta(s_0) \in (0, \pi/2)$ such that

$$\begin{aligned} \rho(s_0, \theta(s_0)) &= \eta(s_0) = 0, \quad \eta'(s_0) \neq 0, \\ \gamma(s_0) - \mathbf{v}_0 &= -\frac{2}{\cot \theta(s_0)} \mathbf{n}\mathbf{g}_\gamma(s_0, \theta(s_0)), \end{aligned} \quad (15)$$

where $\rho(s, \theta) = -(1/2)k_1(s) \cot \theta + k_2(s)$. Under this condition, the null hypersurface $\mathbf{n}\mathbf{h}_\gamma$ at $\mathbf{v}_0$ is locally diffeomorphic to BF , the null focal set $\mathbf{n}\mathbf{f}_\gamma$ is locally diffeomorphic to $C(\text{BF}) \times \mathbb{R}$, and the singular value set of $\mathbf{n}\mathbf{f}_\gamma$ is locally diffeomorphic to the $C(2, 3, 4, 5)$ -cusp.

One calls $C(2, 3) = \{(x^1, x^2) \mid x^1 = u^2, x^2 = u^3\}$ a $(2, 3)$ -cusp, $C(2, 3, 4) = \{(x^1, x^2, x^3) \mid x^1 = u^2, x^2 = u^3, x^3 = u^4\}$ a $(2, 3, 4)$ -cusp (Figure 1), $C(2, 3, 4, 5) = \{(x^1, x^2, x^3, x^4) \mid x^1 = u^2, x^2 = u^3, x^3 = u^4, x^4 = u^5\}$ a $(2, 3, 4, 5)$ -cusp (Figure 2), $\text{SW} = \{(x^1, x^2, x^3) \mid x^1 = 3u^4 + u^2v, x^2 = 4u^3 + 2uv, x^3 = v\}$ a swallowtail, $\text{BF} =$

FIGURE 1: (2,3)-cusp and swallowtail.

$\{(x^1, x^2, x^3, x^4) \mid x^1 = 5u^4 + 3vu^2 + 2wu, x^2 = 4u^5 + 2vu^3 + wu^2, x^3 = u, x^4 = v\}$ a *butterfly* (Figure 3), and $C(BF) = \{(x^1, x^2, x^3, x^4) \mid x^1 = 6u^5 + u^3v, x^2 = 25u^4 + 9u^2v, x^3 = 10u^3 + 3uv, x^4 = v\}$ a *c-butterfly* (i.e., the singular value set of the butterflies).

We will give the proof of Theorem 2 in Section 4.

3. A Family of Lorentz Distance-Squared Functions of Pseudonull Curve $\gamma(s)$

In this section we introduce one very useful family of functions on a pseudonull curve. For a pseudonull curve γ , we define the function:

$$G : I \times \mathbb{R}_2^4 \longrightarrow \mathbb{R}, \quad G(s, \mathbf{v}) = \langle \gamma(s) - \mathbf{v}, \gamma(s) - \mathbf{v} \rangle. \quad (16)$$

This function is called the *Lorentz distance-squared function* of the pseudonull curve γ . We use the notation $g_{\mathbf{v}}(s) = G(s, \mathbf{v})$ for any fixed vector $\mathbf{v}$ in $\mathbb{R}_2^4$. They describe the contact between $\gamma(s)$ and a nullcone. As we study this family of functions, it will become clear how singularities and the corresponding catastrophes arise.

By using (5) and by making tedious calculations, we can state Proposition 3.

Proposition 3. Let $\gamma : I \rightarrow \mathbb{R}_2^4$ be a timelike pseudonull curve. Supposing that $\gamma(s_0) \neq \mathbf{v}_0$, then one has the following:

- (1) $g_{\mathbf{v}_0}(s_0) = g'_{\mathbf{v}_0}(s_0) = 0$ if and only if there exist $\theta_0 \in (0, \pi/2)$ and $\mu \in \mathbb{R} \setminus \{0\}$ such that $\gamma(s_0) - \mathbf{v}_0 = \mu \mathbf{n}\mathbf{g}_{\gamma}(s_0, \theta_0)$;
- (2) $g_{\mathbf{v}_0}(s_0) = g'_{\mathbf{v}_0}(s_0) = g''_{\mathbf{v}_0}(s_0) = 0$ if and only if there exists $\theta_0 \in (0, \pi/2)$ such that $\gamma(s_0) - \mathbf{v}_0 = -(2/\cot \theta_0) \mathbf{n}\mathbf{g}_{\gamma}(s_0, \theta_0)$;
- (3) $g_{\mathbf{v}_0}(s_0) = g'_{\mathbf{v}_0}(s_0) = g''_{\mathbf{v}_0}(s_0) = g'''_{\mathbf{v}_0}(s_0) = 0$ if and only if there exists $\theta_0 \in (0, \pi/2)$ such that $\gamma(s_0) -$

$\mathbf{v}_0 = -(2/\cot \theta_0) \mathbf{n}\mathbf{g}_{\gamma}(s_0, \theta_0)$ and $-(1/2)k_1(s_0) \cot \theta_0 + k_2(s_0) = 0$, so one can write $\theta_0 = \theta(s_0)$;

- (4) $g_{\mathbf{v}_0}(s_0) = g'_{\mathbf{v}_0}(s_0) = g''_{\mathbf{v}_0}(s_0) = g'''_{\mathbf{v}_0}(s_0) = g^{(4)}_{\mathbf{v}_0}(s_0) = 0$ if and only if there exists $\theta(s_0) \in (0, \pi/2)$ such that $\gamma(s_0) - \mathbf{v}_0 = -(2/\cot \theta(s_0)) \mathbf{n}\mathbf{g}_{\gamma}(s_0, \theta(s_0))$, $-(1/2)k_1(s_0) \cot \theta(s_0) + k_2(s_0) = 0$, and $\eta(s_0) = 0$;
- (5) $g_{\mathbf{v}_0}(s_0) = g'_{\mathbf{v}_0}(s_0) = g''_{\mathbf{v}_0}(s_0) = g'''_{\mathbf{v}_0}(s_0) = g^{(4)}_{\mathbf{v}_0}(s_0) = g^{(5)}_{\mathbf{v}_0}(s_0) = 0$ if and only if there exists $\theta(s_0) \in (0, \pi/2)$ such that $\gamma(s_0) - \mathbf{v}_0 = -(1/\cot \theta(s_0)) \mathbf{n}\mathbf{g}_{\gamma}(s_0, \theta(s_0))$ and $-(1/2)k_1(s_0) \cot \theta(s_0) + k_2(s_0) = \eta(s_0) = \eta'(s_0) = 0$.

The above proposition also states that the discriminant set of the Lorentz distance-squared function G is given by

$$D_G = \mathbf{n}\mathbf{f}_{\gamma}(U \times \mathbb{R}) = \{\mathbf{v} = \gamma(s) + \mu \mathbf{n}\mathbf{g}_{\gamma}(s, \theta) \mid (s, \theta) \in U, \mu \in \mathbb{R}\}, \quad (17)$$

which is the image of the null hypersurface along γ . Therefore, a singular point of the null hypersurface is the point $\mathbf{v}_0 = \gamma(s_0) + \mu_0 \mathbf{n}\mathbf{g}_{\gamma}(s_0, \theta_0)$, where $\mu_0 = -(2/\cot \theta_0)$.

We define $\mathbf{n}\mathbf{f}_{\gamma} : U \rightarrow \mathbb{R}_2^4$ as

$$\mathbf{n}\mathbf{f}_{\gamma}(s, \theta) = \gamma(s) + \mu_0 \mathbf{n}\mathbf{g}_{\gamma}(s, \theta); \quad (18)$$

we call it the *null focal set* of γ . By definition, the null focal set is the singular value set of the null hypersurface $\mathbf{n}\mathbf{h}_{\gamma}$.

4. Classifications of Singularities

In this section we classify singularities of the null hypersurface along γ as an application of the unfolding theory of functions. Detailed descriptions could be found in [12]. Let $F : (\mathbb{R} \times \mathbb{R}^r, (s_0, \mathbf{x}_0)) \rightarrow \mathbb{R}$ be a function germ. We call F

FIGURE 2: Projection of a (2,3,4,5)-cusp, respectively, on $x^1 x^2 x^3$ -space, $x^1 x^2 x^4$ -space, $x^1 x^3 x^4$ -space, and $x^2 x^3 x^4$ -space.

an r -parameter unfolding of f , if $f(s) = F_{\mathbf{x}_0}(s, \mathbf{x}_0)$. We say f has A_k -singularity at s_0 , if $f^{(p)}(s_0) = 0$ for all $1 \leq p \leq k$ and $f^{(k+1)}(s_0) \neq 0$. Let F be an r -parameter unfolding of f and f has A_k -singularity ($k \geq 1$) at s_0 . We denote the $(k-1)$ -jet of the partial derivative $\partial F / \partial x^i$ at s_0 by

$$j^{(k-1)} \left(\frac{\partial F}{\partial x^i}(s, \mathbf{x}_0) \right) (s_0) = \sum_{j=1}^{k-1} \alpha_{ji} (s - s_0)^j, \quad (19)$$

$$(i = 1, \dots, r).$$

If the rank of $k \times r$ matrix $(\alpha_{0i}, \alpha_{ji})$ is k ($k \leq r$), then F is called a *versal unfolding* of f , where $\alpha_{0i} = \partial F / \partial x^i(s_0, \mathbf{x}_0)$.

Inspired by the proposition in the previous section, we have

$$D_F^l = \left\{ \mathbf{x} \in \mathbb{R}^r \mid \exists s \in \mathbb{R}, \right.$$

$$\left. F(s, \mathbf{x}) = \frac{\partial F}{\partial s}(s, \mathbf{x}) = \dots = \frac{\partial^l F}{\partial s^l}(s, \mathbf{x}) = 0 \right\}, \quad (20)$$

which is called a *discriminant set with order l* . Therefore, we have the following proposition.

Proposition 4. For a pseudonull curve γ , one has

$$D_G = D_G^1 = \mathbf{n}\mathfrak{h}_\gamma(U \times \mathbb{R}),$$

$$D_G^2 = \mathbf{n}\mathfrak{f}_\gamma(U), \quad (21)$$

$$D_G^3 \text{ is the singular value set of } \mathbf{n}\mathfrak{f}_\gamma.$$

FIGURE 3: When $w = 0$, projection of a butterfly, respectively, on $x^1 x^2 x^3$ -space, $x^1 x^2 x^4$ -space, $x^1 x^3 x^4$ -space, and $x^2 x^3 x^4$ -space.

Then we have the following classification theorem as a Corollary 6.6 in [12].

Theorem 5. Let $F : (\mathbb{R} \times \mathbb{R}^r, (s_0, \mathbf{x}_0)) \rightarrow \mathbb{R}$ be an r -parameter unfolding of f with A_k -singularity at s_0 . Suppose F is a versal unfolding of f ; then one has the following assertions.

- (a) If $k = 1$, then D_F is locally diffeomorphic to $\{0\} \times \mathbb{R}^{r-1}$ and $D_F^2 = \emptyset$.
- (b) If $k = 2$, then D_F is locally diffeomorphic to $C(2, 3) \times \mathbb{R}^{r-2}$ and D_F^2 is locally diffeomorphic to $\{0\} \times \mathbb{R}^{r-2}$ and $D_F^3 = \emptyset$.
- (c) If $k = 3$, then D_F is locally diffeomorphic to $SW \times \mathbb{R}^{r-3}$, D_F^2 is locally diffeomorphic to $C(2, 3, 4) \times \mathbb{R}^{r-3}$ and D_F^3 is locally diffeomorphic to $\{0\} \times \mathbb{R}^{r-3}$ and $D_F^4 = \emptyset$.
- (d) If $k = 4$, then D_F is locally diffeomorphic to $BF \times \mathbb{R}^{r-4}$, D_F^2 is locally diffeomorphic to $C(BF) \times \mathbb{R}^{r-4}$, D_F^3 is locally diffeomorphic to $C(2, 3, 4, 5) \times \mathbb{R}^{r-4}$, and D_F^4 is locally diffeomorphic to $\{0\} \times \mathbb{R}^{r-4}$ and $D_F^5 = \emptyset$.

For the proof of Theorem 2 we have the following fundamental proposition in this paper.

Proposition 6. If $g(s)$ has A_k -singularity ($k = 1, 2, 3, 4$) at s_0 , then G is a versal unfolding of g .

Proof. By definition, we have

$$G(s, \mathbf{v}) = -\left(x^0(s) - v^0\right)^2 - \left(x^1(s) - v^1\right)^2 + \left(x^2(s) - v^2\right)^2 + \left(x^3(s) - v^3\right)^2, \quad (22)$$

where $\mathbf{v}(s) = (x^0(s), x^1(s), x^2(s), x^3(s))$ and $\mathbf{v} = (v^0, v^1, v^2, v^3)$.

Thus, we get

$$\begin{aligned} \frac{\partial G}{\partial v^i}(s, \mathbf{v}) &= \begin{cases} 2(x^i(s) - v^i) & i = 0, 1 \\ -2(x^i(s) - v^i) & i = 2, 3, \end{cases} \\ \frac{\partial^2 G}{\partial s \partial v^i}(s, \mathbf{v}) &= \begin{cases} 2(x^i)'(s) & i = 0, 1 \\ -2(x^i)'(s) & i = 2, 3. \end{cases} \end{aligned} \quad (23)$$

For a fixed $\mathbf{v}_0 = (v_0^0, v_0^1, v_0^2, v_0^3)$, the 3-jet of $\partial G/\partial v^i(s, \mathbf{v}_0)$ at s_0 is

$$j^{(3)} \frac{\partial G}{\partial v^i}(s, \mathbf{v}_0)(s_0) = \begin{cases} 2(x^i)'(s_0)(s-s_0) + (x^i)''(s_0)(s-s_0)^2 \\ \quad + \frac{1}{3}(x^i)'''(s_0)(s-s_0)^3 & i = 0, 1 \\ -2(x^i)'(s_0)(s-s_0) - (x^i)''(s_0)(s-s_0)^2 \\ \quad - \frac{1}{3}(x^i)'''(s_0)(s-s_0)^3 & i = 2, 3. \end{cases} \quad (24)$$

to have rank 2. Otherwise, if $\text{rank } A_2 = 1$, it means that $\boldsymbol{\gamma}(s_0) - \mathbf{v}_0$ and $\mathbf{t}(s_0)$ are linearly dependent. This contradicts with the fact that $\{\mathbf{t}(s), \mathbf{n}_1(s), \mathbf{n}_2(s), \mathbf{n}_3(s)\}$ is the pseudoorthogonal frame of $\boldsymbol{\gamma}$.

$$A_2 = \begin{pmatrix} 2(x^0(s_0) - v^0) & 2(x^1(s_0) - v^1) & -2(x^2(s_0) - v^2) & -2(x^3(s_0) - v^3) \\ 2(x^0)'(s_0) & 2(x^1)'(s_0) & -2(x^2)'(s_0) & -2(x^3)'(s_0) \end{pmatrix} \quad (25)$$

The condition for versatility can be checked as follows.

(1) By Proposition 3, g has A_1 -singularity at s_0 if and only if there exist $\theta_0 \in (0, \pi/2)$ and $\mu \in \mathbb{R} \setminus \{0\}$ such that $\boldsymbol{\gamma}(s_0) - \mathbf{v}_0 = \mu \mathbf{n}_{\boldsymbol{\gamma}}(s_0, \theta_0)$ and $\mu \neq -(2/\cot \theta_0)$. When g has A_1 -singularity at s_0 , we require the 1×4 matrix $A_1 = (2(x^0(s_0) - v^0), 2(x^1(s_0) - v^1), -2(x^2(s_0) - v^2), -2(x^3(s_0) - v^3))$ to have rank 1, which it always does since $\mathbf{v}_0 \neq \boldsymbol{\gamma}(s_0)$.

(2) It also follows from Proposition 3 that g has A_2 -singularity at s_0 if and only if there exists $\theta_0 \in (0, \pi/2)$ such that $\boldsymbol{\gamma}(s_0) - \mathbf{v}_0 = -(2/\cot \theta_0) \mathbf{n}_{\boldsymbol{\gamma}}(s_0, \theta_0)$ and $\rho(s_0, \theta(s_0)) \neq 0$. We require 2×4 matrix

(3) By Proposition 3, g has A_3 -singularity at s_0 if and only if there exists $\theta(s_0) \in (0, \pi/2)$ such that $\boldsymbol{\gamma}(s_0) - \mathbf{v}_0 = -(2/\cot \theta(s_0)) \mathbf{n}_{\boldsymbol{\gamma}}(s_0, \theta(s_0))$, $\rho(s_0, \theta(s_0)) = 0$, and $\eta(s_0) \neq 0$. We require 3×4 matrix

$$A_3 = \begin{pmatrix} 2(x^0(s_0) - v^0) & 2(x^1(s_0) - v^1) & -2(x^2(s_0) - v^2) & -2(x^3(s_0) - v^3) \\ 2(x^0)'(s_0) & 2(x^1)'(s_0) & -2(x^2)'(s_0) & -2(x^3)'(s_0) \\ (x^0)''(s_0) & (x^1)''(s_0) & -(x^2)''(s_0) & -(x^3)''(s_0) \end{pmatrix} \quad (26)$$

to have rank 3. Otherwise, if $\text{rank } A_3 = 2$, it means that $\boldsymbol{\gamma}''(s_0)$ can be generated by $\boldsymbol{\gamma}(s_0) - \mathbf{v}_0$ and $\mathbf{t}(s_0)$. Through a straightforward calculation, we can easily show that it is a contradiction.

(4) By Proposition 3, g has A_4 -singularity at s_0 if and only if there exists $\theta(s_0) \in (0, \pi/2)$ such that $\boldsymbol{\gamma}(s_0) - \mathbf{v}_0 = -(2/\cot \theta(s_0)) \mathbf{n}_{\boldsymbol{\gamma}}(s_0, \theta(s_0))$, $\rho(s_0, \theta(s_0)) = \eta(s_0) = 0$, and $\eta'(s_0) \neq 0$. We require 4×4 matrix

$$A_4 = \begin{pmatrix} 2(x^0(s_0) - v^0) & 2(x^1(s_0) - v^1) & -2(x^2(s_0) - v^2) & -2(x^3(s_0) - v^3) \\ 2(x^0)'(s_0) & 2(x^1)'(s_0) & -2(x^2)'(s_0) & -2(x^3)'(s_0) \\ (x^0)''(s_0) & (x^1)''(s_0) & -(x^2)''(s_0) & -(x^3)''(s_0) \\ \frac{1}{3}(x^0)'''(s_0) & \frac{1}{3}(x^1)'''(s_0) & -\frac{1}{3}(x^2)'''(s_0) & -\frac{1}{3}(x^3)'''(s_0) \end{pmatrix} \quad (27)$$

to have rank 4.

In fact,

$$\begin{aligned} \det A_4 &= \frac{4}{3} \det(\boldsymbol{\gamma}(s_0) - \mathbf{v}_0, \boldsymbol{\gamma}'(s_0), \boldsymbol{\gamma}''(s_0), \boldsymbol{\gamma}'''(s_0)) \\ &= \frac{8}{3} k_2(s_0) \det(\mathbf{n}_2, \mathbf{t}, \mathbf{n}_1, \mathbf{n}_3) \neq 0. \end{aligned} \quad (28)$$

Therefore, $\text{rank } A_4 = 4$.

In summary, G is a versal unfolding of g . This completes the proof. $\square$

We now give the proof of Theorem 2.

Proof of Theorem 2. Let $\boldsymbol{\gamma} : I \rightarrow \mathbb{R}_2^4$ be a pseudonull curve. As $\mathbf{v}_0 = \mathbf{n}_{\boldsymbol{\gamma}}(s_0, \theta_0, t_0)$, we give a function $\mathfrak{G} : \mathbb{R}_2^4 \rightarrow \mathbb{R}$, by $\mathfrak{G}(\mathbf{u}) = \langle \mathbf{u} - \mathbf{v}_0, \mathbf{u} - \mathbf{v}_0 \rangle$; then we assume that $g_{\mathbf{v}_0}(s) =$

$\mathfrak{G}(\gamma(s))$. Because $\mathfrak{G}^{-1}(0) = \text{NC}_{\mathbf{v}_0}$ and 0 is a regular value of $\mathfrak{G}, \gamma$ and $\text{NC}_{\mathbf{v}_0}$ have $(k+1)$ -point contact for s_0 if and only if $g_{\mathbf{v}_0}(s)$ has A_k -singularity at s_0 . Thus γ and $\text{NC}_{\mathbf{v}_0}$ have at least 2-point contact for s_0 if and only if $g_{\mathbf{v}_0}(s_0) = g'_{\mathbf{v}_0}(s_0) = 0$. By Proposition 4, we have

$$\begin{aligned} D_G^1 &= \mathfrak{nh}_\gamma(U \times \mathbb{R}), & D_G^2 &= \mathfrak{nf}_\gamma(U), \\ D_G^3 &\text{ is the singular value set of } \mathfrak{nf}_\gamma. \end{aligned} \quad (29)$$

By combining Proposition 3, Theorem 5, and Proposition 6, we get the results. $\square$

5. Generic Properties

In this section we consider the generic properties of pseudonull curves in $\mathbb{R}_2^4$. The main tool is transversality theorem. Assume that $\text{Emb}_t(I, \mathbb{R}_2^4)$ is the space of timelike embedding $\gamma : I \rightarrow \mathbb{R}_2^4$ equipped with Whitney C^∞ -topology. We also consider the function $\mathcal{G} : \mathbb{R}_2^4 \times \mathbb{R}_2^4 \rightarrow \mathbb{R}$ defined by $\mathcal{G}(\mathbf{u}, \mathbf{v}) = \langle \mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle$. We claim that $\mathcal{G}_\mathbf{v}$ is a submersion for any $\mathbf{v} \in \mathbb{R}_2^4$, where $\mathcal{G}_\mathbf{v}(\mathbf{u}) = \langle \mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle$. For any $\gamma \in \text{Emb}_t(I, \mathbb{R}_2^4)$, we have $G = \mathcal{G} \circ (\gamma \times \text{id}_{\mathbb{R}_2^4})$. We also have the l -jet extension

$$j_1^l G : I \times \mathbb{R}_2^4 \longrightarrow J^l(I, \mathbb{R}) \quad (30)$$

defined by $j_1^l G(s, \mathbf{v}) = j^l g_\mathbf{v}(s)$; we consider the trivialization $J^l(I, \mathbb{R}) \equiv I \times \mathbb{R} \times J^l(1, 1)$. For any submanifold $Q \subset J^l(1, 1)$, we denote that $\widetilde{Q} = I \times \{0\} \times Q$. It is obvious that $j_1^l G$ is a submersion and $\widetilde{Q}$ is an immersed submanifold of $J^l(I, \mathbb{R})$. Then $j_1^l G$ is transversal to $\widetilde{Q}$. We have the following proposition as a corollary of Lemma 6 in [15].

Proposition 7. *Let Q be a submanifold of $J^l(1, 1)$. Then the set*

$$T_Q = \{\gamma \in \text{Emb}_t(I, \mathbb{R}_2^4) \mid j_1^l G \text{ is transversal to } \widetilde{Q}\} \quad (31)$$

is a residual subset of $\text{Emb}_t(I, \mathbb{R}_2^4)$. If Q is a closed subset, then T_Q is open.

Let $f : (\mathbb{R}, 0) \rightarrow (\mathbb{R}, 0)$ be a function germ which has A_k -singularity at 0. It is well known that there exists a diffeomorphism germ $\phi : (\mathbb{R}, 0) \rightarrow (\mathbb{R}, 0)$ such that $f \circ \phi(s) = \pm s^{k+1}$. This is the classification of A_k -singularities. For any $z = j^l f(0) \in J^l(1, 1)$, we have the orbit $L^l(z)$ given by the action of the Lie group of l -jet diffeomorphism germs. If f has A_k -singularity, then the codimension of the orbit is k . There is another characterization of versal unfolding as follows [16].

Proposition 8. *Assume that $F : (\mathbb{R} \times \mathbb{R}^r, 0) \rightarrow (\mathbb{R}, 0)$ is an r -parameter unfolding of $f : (\mathbb{R}, 0) \rightarrow (\mathbb{R}, 0)$ which has an A_k -singularity at 0. Then F is a versal unfolding if and only if $j_1^l G$ is transversal to the orbit $L^l(j^l f(0))$ for $l \geq k+1$. Here, $j_1^l G : (\mathbb{R} \times \mathbb{R}^r, 0) \rightarrow J^l(\mathbb{R}, \mathbb{R})$ is the l -jet extension of G given by $j_1^l G(s, \mathbf{x}) = j^l G_\mathbf{x}(s)$.*

The generic classification theorem is given as follows.

Theorem 9. *There exists a dense and open subset $T_{L_k^l} \subset \text{Emb}_t(I, \mathbb{R}_2^4)$ such that, for any $\gamma \in T_{L_k^l}$, the lightlike hypersurface of γ is locally diffeomorphic to the cuspidal edge, the swallowtail, or the butterfly at any singular point.*

Proof. For $l \geq 5$, we discuss the decomposition of the jet space $J^l(1, 1)$ into $L^l(1)$ orbits; we now define a semialgebraic set by

$$\Sigma^l = \{z = j^l f(0) \in J^l(1, 1) \mid f \text{ has an } A_{\geq 5}\text{-singularity}\}. \quad (32)$$

Then the codimension of Σ^l is 5. Therefore, the codimension of $\widetilde{\Sigma}^l = I \times \{0\} \times \Sigma^l$ is 6. Then the orbit decomposition of $J^l(1, 1) - \Sigma^l$ is

$$J^l(1, 1) - \Sigma^l = L_0^l \cup L_1^l \cup L_2^l \cup L_3^l \cup L_4^l, \quad (33)$$

where L_k^l is the orbit going through the A_k -singularity. Thus, the codimension of L_k^l is $k+1$. We consider the l -jet extension $j_1^l G$ of the function G . By Proposition 7, there exists a dense and open subset $T_{L_k^l} \subset \text{Emb}_t(I, \mathbb{R}_2^4)$ such that $j_1^l G$ is transversal to $\widetilde{L}_k^l$ ($k = 0, 1, 2, 3, 4$) and the orbit decomposition of $\widetilde{\Sigma}^l$. This means that $j_1^l G(I \times \mathbb{R}_2^4) \cap \widetilde{\Sigma}^l = \emptyset$ and G is a versal unfolding of g at any point $(s_0, \mathbf{v}_0)$. By Theorem 5, the discriminant set of G (i.e., the lightlike hypersurface of γ) is locally diffeomorphic to cuspidal edge, swallowtail, or butterfly if the point is singular. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests in this work.

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